Optimization of stepped shells

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Abstract: - Problems of analysis and optimization of axisymmetric shells of piece wise constant thickness are studied. The cases of quasistatic and dynamic loading are considered separately whereas the shells may be manufactured from both, elastic and inelastic materials. Minimum weight designs of inelastic shells of piece wise constant thickness are established under the condition that the limit load is fixed. Also the designs of maximum load carrying capacity are determined for given weight (material volume) of the shell. Necessary optimality conditions are derived with the aid of variational methods of the theory of optimal control. Particular cases of maximization of the load carrying capacity of spherical caps and conical shells are studied in a greater detail in the cases of von Mises and Hill's materials.

Key-Words: - optimization, optimal control, inelastic material, crack, conical shell, spherical shell, cylindrical shell

1 Introduction

Optimization of structural elements is a field of research which has theoretical and practical importance. There exist many monograph books and review papers devoted to optimal design of plates and shells made of elastic and inelastic materials (Banichuk [3]; Kruzelecki and Życzkowski [21]; Lellep and Lepik [23]; Bendsoe [4]; Kirsch [19]).

Inelastic spherical caps subjected to uniform pressure are investigated by Lellep and Tungel [29,30] in the cases of Tresca and Mises materials. Similar problems regarding to the optimal design of stepped conical shells are solved by Lellep and Puman [25,26,27]. In the last paper a conical shell with the rigid central boss is studied under the condition that the joint between the boss and shell wall is not ideal, it is weakened due to the circular crack of constant depth.

Lellep and Mürk [24] employed this concept in the case of inelastic annular plates with piece wise constant thickness. The plates subjected to the initial impulsive loading have weakened with cracks emanating at the re-entrant corners of steps.

Solid annular plates of piece wise constant thickness have been studied by Laplume, Datoussaid and Guerlement [22] in the case of the Tresca material. In this study the plates under consideration are subjected to a specified ring load whereas a set of technological constraints is taken into account.

The intensive use of composite materials and polymers in the structural and mechanical engineering generates the need for investigation of the load carrying capacity of composite structural elements. In recent years Pan and Seshary [31], also Capsoni, Corradi and Viena [8]; Corradi, Luzzi and Viena [12] have studied anisotropic behaviour of solids and structures in the case of an inelastic material obeying Hill's yield condition.

The concept of Hill's anisotropic material with lower and upper bound theorems of limit analysis (Hodge [15]; Chakrabarty [9]) will be employed in the present study, as well.

In the present paper an attempt is made to study various shells of revolution from the unique point of view. Both, elastic and inelastic shells including spherical caps, conical shells and circular tubes are considered.

2 Problem Formulation

Let us consider axisymmetric behaviour of a rotationally symmetric shell of piece wise constant thickness. Due to symmetry the current position of a cross section can be specified by the coordinate φ whereas $\varphi \in [\alpha, \beta]$. It is assumed that the shell wall thickness $h = h_j$ for $\varphi \in [\alpha_j, \alpha_{j+1}]$ where $j = 0, \dots, n$. Here $\varphi_0 = \alpha$ and $\varphi_{n+1} = \beta$ where α and β are given numbers. However, constants h_j $(j = 0, \dots, n)$ and α_j $(j = 1, \dots, n)$ are considered as unknown design parameters to be defined so that the cost function

$$J = \sum_{j=0}^{n} \left(G_j + \int_{D_j} F_j \, d\varphi \right) \quad (1)$$

attains the minimum value.

In (1) the functions F_j and G_j are given continuous and differentiable functions depending on the stress strain state of the shell and on design parameters. We assume that

$$\begin{aligned} F_j &= F_j(S,U,W,P) \\ G_j &= G_j(a_1,\ldots a_n,h_0,\ldots,h_n,P) \end{aligned} \quad (2)$$

where U and W stand for meridional and normal displacements of the middle surface of the shell and S is the area of cross section. For convenience sake segments $[a_j,a_{j+1}]$ are denoted by D_j in (1) and P is the load intensity.

We are looking for the design of the shell corresponding to minimum of the cost criterion (1) under constraints

$$R_j(S,U,W,P) \le 0 \quad (3)$$

for $\varphi \in D_j$ $(j = 0,\ldots,n)$ and

$$\sum_{j=0}^{n} \int_{D_j} T_j(S,U,W,P)\, d\varphi = A_0 . \quad (4)$$

In (3), (4) A_0 is a given constant, R_j and T_j stand for given continuous and continuously differentiable functions.

It is assumed that in addition to the global constraints (3), (4) certain restrictions are imposed on the stress strain state at singular cross sections only. Let the local constraints be presented as (here $j = 0,\ldots,n$)

$$g_j(h_0,\ldots,h_n,a_1,\ldots a_n,N_1(\varphi_j),M_1(\varphi_j),W(\varphi_j),U(\varphi_j)) \le 0 \quad (5)$$

where φ_j is a certain point of the set D_j. In (5) and henceworth M_1 and M_2 denote bending moments in the normal and meridional directions, respectively. The quantities N_1 and N_2 are membrane forces in these directions.

The statement of the problem (1) - (5) includes a number of particular cases which have deserved attention in the literature as special problems (see [3], [23]-[30]).

3 Governing equations

Equilibrium equations of a shell element have the form (see Hodge [15]; Soedel [34]) (prims denote the differentiation with respect to φ)

$$\begin{aligned} &(rN_1)' - N_2 r_1 \cos\varphi - rQ + r r_1 P_1 = 0 \\ &(rQ)' + N_1 r - N_2 r_1 \sin\varphi + r r_1 P_2 = 0 \\ &(rM_1)' - M_2 r_1 \cos\varphi - r r_1 Q = 0 \end{aligned} \quad (6)$$

where r is the distance between current element and the axis of symmetry and r_1 is the radius of curvature of the meridian at the current point.

The geometrical relations for a rotationally symmetric shell are (Ventsel, Krauthammer [35], Chakrabarty [9])

$$\begin{aligned} \varepsilon_1 &= \frac{1}{r_1}(U' - W) \\ \varepsilon_2 &= \frac{1}{r}(U\cos\varphi - W\sin\varphi) \\ \kappa_1 &= -\frac{1}{r_1}\left(\frac{1}{r_1}(U + W')\right)' \\ \kappa_2 &= -\frac{1}{r r_1}(U + W')\cos\varphi \end{aligned} \quad (7)$$

In (7) $\varepsilon_1, \varepsilon_2$ stand for linear deformation components whereas κ_1 and κ_2 are curvature components. In the present paper we consider both, elastic and inelastic shells. In the case of an elastic material the constitutive equations are presented by Hooke's law. The latter states that (see Soedel, [34])

$$\begin{aligned} N_1 &= K_j(\varepsilon_1 + \nu\varepsilon_2) \\ N_2 &= K_j(\varepsilon_2 + \nu\varepsilon_1) \\ M_1 &= D_{mj}(\kappa_1 + \nu\kappa_2) \\ M_2 &= D_{mj}(\kappa_2 + \nu\kappa_1) \end{aligned} \quad (8)$$

for $\varphi \in D_j$ $(j = 0,\ldots,n)$ where $K_j = Eh_j/(1-\nu^2)$, $D_{mj} = Eh_j^{\,3}/12(1-\nu^2)$. Here E is the Young modulus and ν stands for Poisson's modulus.

Substituting (7), (8) in (6) leads to differential equations with respect to displacements U and W which have to be taken into account when minimizing the cost function (1) in the case of an elastic material.

However, in the case of an ideal plastic material the yield condition and associated gradientality law must be satisfied at each point of the shell. The yield condition can be presented via bending moments and membrane forces as

$$\Phi_j(M_1,M_2,N_1,N_2,M_{0j},N_{0j}) \le 0 \quad (9)$$

for $\varphi \in D_j$ $(j = 0,\ldots,n)$. It is assumed herein that Φ_j is a continuous piece wise smooth function surrounding a convex region in the space of moments and membrane forces. Here M_{0j} and N_{0j} stand for the yield moment and yield force for $\varphi \in D_j$. Together with the yield condition (9) the gradientality law holds good. According to the latter one has

$$\varepsilon_i = \lambda_j \frac{\partial \Phi_j}{\partial N_i}$$
$$\kappa_i = \lambda_j \frac{\partial \Phi_j}{\partial M_i} \tag{10}$$

where λ_j stands for a non-negative scalar multiplier and $\varphi \in D_j$ $(j = 0,\ldots,n)$ whereas $i = 1,2$.

The yield surface $\Phi_j = 0$ has among others non-regular points where the normal vector to the surface changes its direction abruptly. At these points the deformation rate vector with components $(\varepsilon_1, \varepsilon_2, \kappa_1, \kappa_2)$ which are defined by right hand sides of (10) must be presented as linear combination with non-negative coefficients of corresponding gradients to adjacent faces of the surface $\Phi_j = 0$.

Note that at interior points of the region $\Phi_j \le 0$ multipliers λ_j vanish in (10). This reflects the matter that in rigid regions $\varepsilon_i = \kappa_i = 0 \quad (i = 1,2)$

4 Necessary optimality conditions

The posed problem will be considered as a particular problem of the theory of optimal control (see Bryson, Ho [7], Hull [16], Ahmed [1]). In the case of an inelastic material obeying the non-linear yield condition (9) governing equations (6), (7), (10) can be presented as

$y_i' = f_{ij}$

for $\varphi \in D_j$ where $i = 1,\ldots,6$ and $j = 0,\ldots,n$. Here

$y_1 = M_1, y_2 = N_1, y_3 = Q, y_4 = W, y_5 = Z, y_6 = U$

are state variables and N_2, M_2, S are the controls.

It appears that the optimality conditions can be presented as

$$\frac{\partial H_j}{\partial N_2} = 0, \frac{\partial H_j}{\partial M_2} = 0, \frac{\partial H_j}{\partial S} = 0$$

and

$$\psi_i' = -\frac{\partial H_j}{\partial y_i}$$

for $\varphi \in D_j$ $(j = 0,\ldots,n)$ and $i = 1,\ldots,6$. Here H_j stands for the Hamiltonian function defined as

$$H_j = -F_j + \sum_{i=1}^{6} \psi_i f_{ij} - \varphi_j R_j - \psi_{oj} T_j - \lambda_j \Phi_j - \nu_j \chi_j,$$

where ψ_i $(i = 1,\ldots,6)$ are so-called adjoint (conjugate) variables and $\varphi_j, \psi_{0j}, \lambda_j, \nu_j$ - preliminarily unknown Lagrangeian multipliers.

For determination of design parameters $h_j, (j = 0,\ldots,n)$ and $\alpha_i, (i = 0,\ldots,n)$ one obtains the equations

$$\sum_{j=0}^{n} \left\{ \frac{\partial}{\partial h_i}(G_j + \mu_j g_j) - \int_{D_j} \frac{\partial H_j}{\partial h_i} d\varphi \right\} = 0$$

for i=0,...,n and

$$\sum_{j=0}^{k} \left\{ \frac{\partial}{\partial \alpha_i}(G_j + \mu_j g_j) - \int_{D_j} \frac{\partial H_j}{\partial \alpha_i} d\varphi \right\} + H_i(\alpha_i - 0) -$$
$$- H_i(\alpha_i + 0) = 0$$

for $i = 1,\ldots,n$.

Detailed analysis shows that similar equations can be obtained in the case of elastic shells, as well.

Particular problems of optimization of elastic shells and plates are presented in [1].

5 Stepped spherical shells

Consider now stepped spherical shells (Fig.1) of radius A subjected to the uniform external pressure loading. In this case $r_1 = A, r = A \sin \varphi$ and the equilibrium equations have the form

$$(N_1 \sin \varphi)' - N_2 \cos \varphi = Q \sin \varphi$$
$$(N_1 + N_2 + P) \sin \varphi = -(Q \sin \varphi)' \tag{11}$$
$$(M_1 \sin \varphi)' - M_2 \cos \varphi = Q \sin \varphi$$

For spherical shells the strain components (7) can be written as

$$\varepsilon_1 = \frac{1}{A}(U' - W)$$
$$\varepsilon_2 = \frac{1}{A}(U \cot \varphi - W)$$
$$\kappa_1 = -\frac{1}{A^2}(U + W')' \tag{12}$$
$$\kappa_2 = -\frac{\cot \varphi}{A^2}(U + W')$$

Assume that the material of the shell obeys von Mises yield condition. In this paper we use the approximation of the exact yield surface corresponding to the Mises condition in the form (see Lellep, Tungel [29])

$$\Phi_j = \frac{1}{M_{oj}^2}\left(M_1^2 - M_1 M_2 + M_2^2\right) +$$
$$+ \frac{1}{N_{oj}^2}\left(N_1^2 - N_1 N_2 + N_2^2\right) - 1 = 0 \tag{13}$$

for $\varphi \in D_j \ (j = 0,\ldots,n)$. It is known from the literature that (12) presents a good approximation to the exact yield surface.

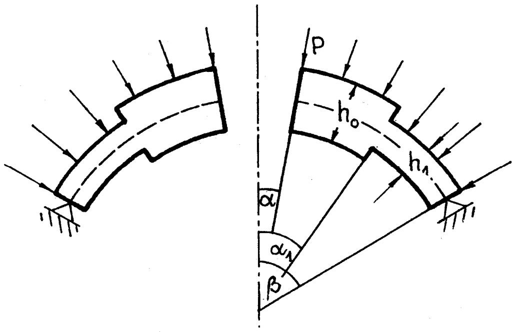

Fig.1. Spherical shell

According to the associated flow law and the yield surface (13) one has

$$\begin{aligned}
\varepsilon_1 &= \frac{\lambda_j(2N_1 - N_2)}{N_{0j}^2} \\
\varepsilon_2 &= \frac{\lambda_j(2N_2 - N_1)}{N_{0j}^2} \\
\kappa_1 &= \frac{\lambda_j(2M_1 - M_2)}{M_{0j}^2} \\
\kappa_2 &= \frac{\lambda_j(2M_2 - M_1)}{M_{0j}^2}
\end{aligned} \tag{14}$$

for $\varphi \in D_j \ (j = 0,\ldots,n)$. Here λ_j stands for an unknown non-negative scalar multiplier. Evidently, equations (11) - (14) admit the presentation

$$\begin{aligned}
N_1' &= (N_2 - N_1)\cot\varphi + Q\cot\varphi \\
Q' &= -N_1 - N_2 - P - Q\cot\varphi \\
M_1' &= (M_2 - M_1)\cot\varphi + Q \\
W' &= Z \\
U' &= W + \frac{A\lambda_j}{N_{0j}^2}(2N_1 - N_2) \\
Z' &= -W - \frac{A\lambda_j}{N_{0j}^2}(2N_1 - N_2) - \frac{A^2\lambda_j}{M_{0j}^2}(2M_1 - M_2)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&Z + U + \frac{A^2\lambda_j}{\cot\varphi\, M_{0j}^2}(2M_2 - M_1) = 0 \\
&\frac{1}{A}(U\cot\varphi - W) - \frac{\lambda_j}{N_{0j}^2}(2N_2 - N_1) = 0
\end{aligned} \tag{16}$$

for $\varphi \in D_j \ (j = 0,\ldots,n)$.

From the system (16) one can define

$$\lambda_j = \frac{N_{oj}^2}{A(2N_2 - N_1)}(U\cot\varphi - W), \tag{17}$$

provided $2N_2 \neq N_1$, and

$$\chi_j = Z + U + \frac{A\,N_{oj}^2}{M_{0j}^2\cot\varphi}\,\frac{2M_2 - M_1}{2N_2 - N_1}(U\cot\varphi - W) = 0 \tag{18}$$

for $j = 0,\ldots,n$.

The material volume of the shell can be presented as

$$V = \sum_{j=0}^{n} h_j(\cos\alpha_j - \cos\alpha_{j+1}) \tag{19}$$

The minimum weight problem of the spherical cap is thus a particular case of (1) - (5) with $F_j = 0$, $G_j = h_j(\cos\alpha_j - \cos\alpha_{j+1})$ for $j = 0,\ldots,n$. In this case $R_j = T_j = 0, A_0 = 0$ and $g_j = 0$. If the problem consists in the maximization of the limit load for given material volume (weight) of the shell then one has $G_0 = -P, \quad G_j = 0 \ (j = 1,\ldots,n); F_j = 0$ and $T_j = h_j(\cos\alpha_j - \cos\alpha_{j+1})/(\alpha_{j+1} - \alpha_1)$ for $j = 0,\ldots,n$ and $A_0 = V$.

6 Stepped conical shells

In the case of conical shells equilibrium equations have the form

$$\begin{aligned}
&(rN_1)' - N_2 = 0 \\
&\left((rM_1)' - M_2\right)' - N_2\frac{\sin\varphi}{\cos^2\varphi} + \frac{Pr}{\cos^2\varphi} = 0
\end{aligned} \tag{20}$$

where prims denote differentiation with respect to the current radius and φ is the angle of inclination of a generator of the mid surface (Fig.2). Note that equations (20) can be obtained from (6) bearing in mind that φ = *const* and r_1 tends to infinity so that $dr = r_1\cos\varphi\, d\varphi$ remains finite.

The strain rate components corresponding to equations (20) are

$$\begin{aligned}
\varepsilon_1 &= \frac{dU}{dr}\cos\varphi \\
\varepsilon_2 &= \frac{1}{r}(U\cos\varphi + W\sin\varphi) \\
\kappa_1 &= -\frac{d^2W}{dr^2}\cos^2\varphi \\
\kappa_2 &= -\frac{1}{r}\frac{dW}{dr}\cos^2\varphi
\end{aligned} \tag{21}$$

It is assumed that the yield condition can be presented in this form (9) whereas the equation $\Phi_j = 0$ admits to determine

$$M_2 = M\left(M_1, N_1, N_2, M_{0j}, N_{0j}\right) \quad \text{for } r \in \left(a_j, a_{j+1}\right);\ j = 0,\ldots,n. \tag{22}$$

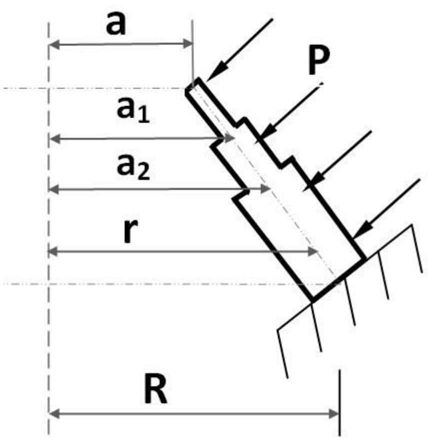

Fig. 2 Conical shell

It appears that the set of governing equations can be simplified, if one edge of the shell is absolutely free. Let us consider this case in the greater detail.
Substituting (22) in (20) and integrating the second equation with respect to *r* leads to the system

$$\begin{aligned} N_1' &= \frac{1}{r}\left(N_2 - N_1\right) \\ M_1' &= -\frac{1}{r}\left(M_1 - M\right) - N_1 \frac{\sin\varphi}{\cos^2\varphi} - \frac{P\left(r^2 - a^2\right)}{2\cos^2\varphi \cdot r} \end{aligned} \tag{23}$$

where prims denote the differentiation with respect to *r*.

It can be easily rechecked that the associated flow law (10) with (21) yields

$$\begin{aligned} W' &= Z \\ Z' &= \frac{Z}{r\dfrac{\partial \Phi_j}{\partial M_2}} \cdot \frac{\partial \Phi_j}{\partial M_1} \\ U' &= \frac{\lambda_j}{\cos\varphi} \cdot \frac{\partial \Phi_j}{\partial N_1} \end{aligned} \tag{24}$$

and

$$W \sin\varphi + U \cos\varphi - r\lambda_j \frac{\partial \Phi_j}{\partial N_2} = 0 \tag{25}$$

whereas

$$\lambda_j = -\frac{Z\cos^2\varphi}{r} \frac{1}{\dfrac{\partial \Phi_j}{\partial M_2}} \tag{26}$$

for $r \in \left(a_j, a_{j+1}\right);\ j = 0,\ldots,n.$

Thus in the present case according to (23)-(26) one has five state variables $\left(N_1, M_1, W, Z, U\right)$ whereas in the general case the number of these is equal to six.

The material volume of a conical shell may be presented as

$$V = \frac{\pi}{\cos\varphi} \sum_{j=0}^{n} h_j \left(a_{j+1}^2 - a_j^2\right) \tag{27}$$

In (27) a_j stands for the radius of the circle separating parts of shells with thicknesses h_{j-1} and h_j, whereas $a_0 = a,\ a_{n+1} = R$

7 Stepped cylindrical shells

Let us consider small deflections of circular cylindrical shells of length *l* and radius *R* (Fig. 3). We shall confine our attention to axisymmetric free vibrations of the shell caused by an initial excitation.

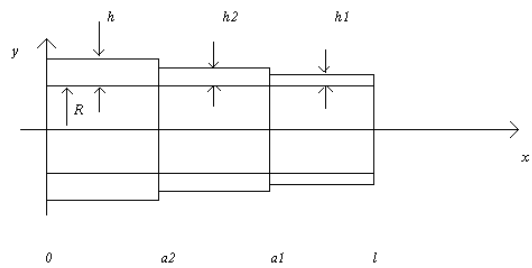

Fig.3: Stepped cylindrical shell

Let the origin of the axis *0x* be at the left end of the tube. Assume that the left end of the shell is clamped whereas the right hand end is absolutely free or simply supported. It is assumed that the thickness h of the shell is piece wise constant, e.g. $h(x) = h_j$ for $x \in \left(a_j, a_{j+1}\right)$, where $j = 0,\ldots,n$. Here the quantities $h_j\ \left(j = 0,\ldots,n\right)$ stand for fixed constants. Similarly, $a_j\ \left(j = 0,\ldots,n+1\right)$ are given constants whereas it is reasonable to use notations $a_0 = 0, a_{n+1} = 1$. It is known in the linear elastic fracture mechanics (see Anderson [2], Bronberg [5], Broek [6]) that repeated loading and stress concentration at sharp corners entail cracks. Thus it is reasonable to assume that at the re-entrant corners of steps e.g. at $x = a_j\ \left(j = 1,\ldots,n\right)$ cracks of depth c_j are located. For the simplicity sake we assume that these flaws are stable circular surface cracks. In the present study like Rizos *et al.* [33], Chondros *et al.* [11], Dimarogonos [13], Kukla [20] no attention will be paid to the crack extension during operation of the structure.

In the case of small deflections of axisymmetric cylindrical shells the stress resultants contributing to the strain energy are membrane forces N_1 and N_2 in axial and hoop direction, respectively, bending moment M and shear force Q. Equilibrium conditions of a shell element have the form (see Reddy [32], Soedel [34], Ventsel and Krauthammer [35])

$$\frac{\partial N_1}{\partial x} = \rho\, h_j \frac{\partial^2 U}{\partial t^2}$$
$$\frac{\partial M}{\partial x} = Q \tag{28}$$
$$\frac{\partial Q}{\partial x} = \frac{N_2}{R} - p + \rho\, h_j \frac{\partial^2 W}{\partial t^2}$$

$x \in (a_j, a_{j+1})$, where $j = 0,\ldots,n$. Here U and W stand for displacements in the axial and transverse direction whereas p is the intensity of distributed transverse pressure, ρ is the material density and t stands for time. Neglecting the axial force N_1 and the axial displacement U one can present the equilibrium equations (28) as

$$\frac{\partial^2 M}{\partial x^2} - \frac{N_2}{R} + p - \rho h_j \frac{\partial^2 W}{\partial t^2} = 0 \tag{29}$$

for $x \in (a_j, a_{j+1})$, $j = 0,\ldots,n$.

Strain components corresponding to the obtained equation of motion

$$\varepsilon_1 = \frac{\partial U}{\partial x}$$
$$\varepsilon_2 = \frac{W}{R} \tag{30}$$
$$\kappa = -\frac{\partial^2 W}{\partial x^2}$$

It is assumed that in the case of shells made of isotropic elastic material generalized Hooke's law reads as (see Reddy [32])

$$\begin{Bmatrix} N_1 \\ N_2 \\ M \end{Bmatrix} = \frac{\mathrm{Eh}}{1\text{-}\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & h^2/12 \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \kappa \end{Bmatrix} \tag{31}$$

Substituting (31) with (30) in (29) yields the equation

$$\frac{\partial^4 W}{\partial x^4} + \frac{12(1-\nu^2)}{R^2 h_j^2} W = -\frac{12 \rho h_j (1-\nu^2)}{E h_j^3} \frac{\partial^2 W}{\partial t^2} \tag{32}$$

which must be satisfied for $x \in (a_j, a_{j+1})$, $j = 0,\ldots,n$.

When deriving (32) it was taken into account that according to the Hooke's law

$$N_1 = \frac{Eh}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2)$$
$$N_2 = \frac{Eh}{1-\nu^2}(\nu\varepsilon_1 + \varepsilon_2)$$

Since $N_1 = 0$ and $\varepsilon_1 = -\nu\,\varepsilon_2$ one has

$$N_2 = Eh\frac{W}{R} \tag{33}$$

and

$$\frac{\partial U}{\partial x} = -\nu \frac{W}{R} \tag{34}$$

The presence of flaws or cracks in a structural member involves considerable local flexibilities. Additional local flexibility due to a crack depends on the crack geometry as well as on the geometry of the structural element and its loading. Probably the first attempt to prescribe the local flexibility of a cracked beam was undertaken by Irwin who recognized the relationship between the compliance C of the beam and stress intensity factor K. Later Dimarogonas [13]; Chondros *et al.* [10], Rizos *et al.* [33], Kukla [20] introduced so called massless rotating spring model which reveals the relationship between the stress intensity factor and local compliance of the beam. In the present paper we are following the papers by Chondros *et al.* [10] and assume that

$$W'(a_{j+},t) - W'(a_{j-},t) = \overline{C}_j\, M(a_j,t) \tag{35}$$

where $\overline{C}_j$ stands for the additional compliance of the shell due to the crack.

The additional compliance $\overline{C}_j$ can be calculated making use of the methods of the linear elastic fracture mechanics (see Anderson [2], Lellep and Sakkov [28]).

It is easy to show that the equation of motion has a solution

$$W = \sin \omega t \cdot X_j(x)$$

for $x \in (a_j, a_{j+1})$, $j = 0,\ldots,n$. It appears that

$$X_j(x) = A_j \sin(r_j x) + B_j \cos(r_j x) + C_j sh(r_j x) + D_j ch(r_j x)$$

where A_j, B_j, C_j, D_j $(j = 0,\ldots,n)$ are arbitrary constants. The constants of integrations are defined from the system of requirements including boundary and intermediate conditions (35). This leads to a linear homogeneous system of algebraic equations with the vanishing determinant Δ. From the equation $\Delta = 0$ one can calculate the characteristic numbers k_j versus crack lengths c_j.

8 Numerical results

The results of calculations are presented in Fig. 4 – Fig.11 and Table 1- 5.

In Table 1 optimal values of the design parameters are presented for the case of a spherical shell with a single step of the thickness. Table 1 corresponds to the maximization of the limit load for given weight of the spherical shell. Material of the shell obeys the yield condition (13) with $j = 0$ and 1. Here γ stands for the ratio of thicknesses and $p = PA / N_*$, $N_* = 2\sigma_0 h_*$, σ_0 being the yield stress of the material. Here h_* is the thickness of the reference shell of constant thickness whereas p_* is the limit load for the reference shell. The coefficient of effectivity e is calculated as

$$e = \left(\frac{p}{p_*} - 1 \right) 100 .$$

Table 1: Optimal design of a spherical cap ($k=0,02$; $\alpha_0=0,8$; $\alpha_2=0,6$)

γ_0	α_1	γ_1	p	e
1,2	0,964	0,155	1,278	23,7%
1,4	0,938	0,176	1,398	35,4%
1,6	0,919	0,185	1,484	43,6%
1,8	0,904	0,199	1,545	49,5%
2,0	0,893	0,198	1,586	53,5%

It can be seen from Table 1 that the effectivity of the design is quite high even in the case of the design with single step of the thickness. When maximizing the limit load it is assumed that the optimized shell and the reference shell of constant thickness have the common weight.

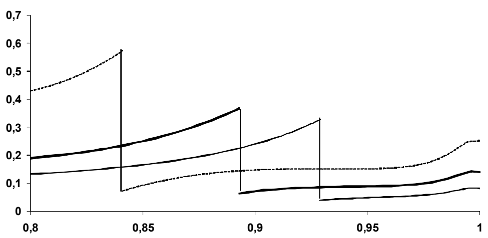

Fig. 4: Circumferential moment of spherical cap.

Calculations carried out showed that radial stress resultants N_1, M_1 are continuous at each point. However, N_2 and M_2 have jumps at these cross sections where the thickness has jumps. The distributions of the moment M_2 are presented in Fig.4. Here the bold, solid and dashed lines correspond to the ratios of thicknesses equal to 2.0, 4.0 and 1.5, respectively.

Calculations for stepped conical shells are implemented in the cases of the Mises' and Hill's yield conditions. Corresponding results are presented in Fig.5 and Table 2.

Fig.5 corresponds to the shell material which obeys the yield condition (13) and the associated flow law. In Fig.5 the nondimensional limit load of the conical shell versus α_1 is shown for different values of the ratio of thicknesses $h_1 / h_0 = \gamma_1$.

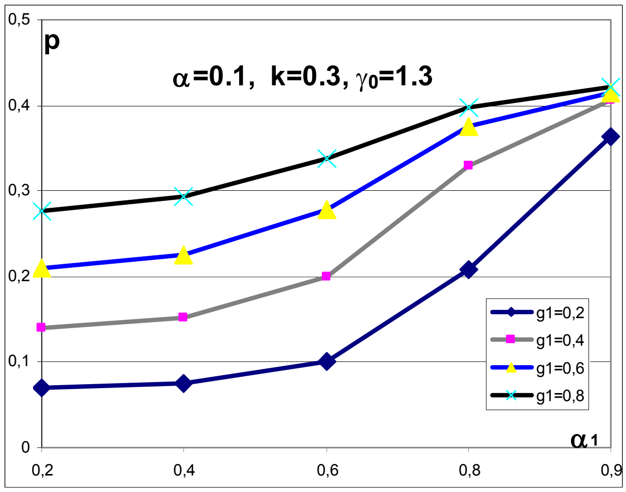

Fig. 5: Limit loads of conical shells.

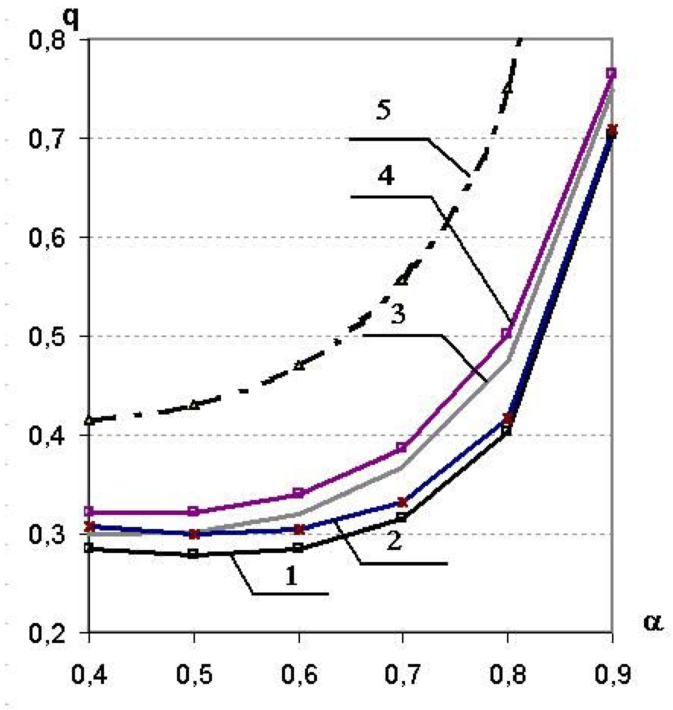

Fig. 6: Comparison with FEM.

Calculations have been carried out for the conical shell with the internal radius $a = 0{,}1R$ and $k = 0{,}3$ where the parameter

$$k = \frac{M_*}{RN_*} \frac{\cos^2 \varphi}{\sin^2 \varphi},$$

M_* being the limit moment of the reference shell with thickness h_*.

In Fig. 6 the solution procedure prescribed above is compared with the finite element method and the upper bound solution. Here $k = 0.3$ and $q = P\sin\varphi$. The solutions obtained numerically by the finite element method are presented by curves 3 and 4. The finite element tehnique resorting to the beam element was implemented in Mathcad. In calculations about 100 elements were used. Curves 1 and 2 present the optimal solution of the problem with given upper bound of the thickness (here $h_0 / h_* = 1$ and $h_0 / h_* = 1.2$ respectively. The dashed line presents the upper bound solution for the shell of constant thickness.

The upper bound corresponds to the simpliest kinematically admissible distribution of velocities, whereas

$$W = W_0 \frac{r - R}{a - R}$$

Table 2: Optimal parameters of conical shells.

α_0	p	α_1	γ_0	γ_1	V_*	e
0,4	3,118	0,853	1,2	0,579	0,84	1,171
0,5	3,119	0,886	1,2	0,501	0,75	1,200
0,6	3,203	0,913	1,2	0,428	0,64	1,199
0,7	3,544	0,942	1,2	0,292	0,51	1,195
0,8	4,533	0,963	1,2	0,192	0,36	1,199

In Table 2 the first row shows the value of the internal radius of the shell whereas V_* is the material volume of the reference shell of constant thickness. Here $k = 0{,}3$.

Table 3: Optimal parameters of conical shells for $k = 0{,}5$ and $\gamma_0 = 1{,}3$.

α_0	P	P_0	γ_1	α_1	e
0,4	4,2489	3,8836	0,7519	0,7350	1,094
0,5	4,2803	3,9450	0,7039	0,7890	1,085
0,6	4,5114	4,2270	0,7184	0,8185	1,067
0,7	5,0983	4,9000	0,8194	0,8256	1,040
0,8	6,5570	6,4900	0,9524	0,8302	1,010
0,9	11,906	11,670	0,1409	0,9751	1,020

Optimal values of design parameters of conical shells made of a Hill's material are presented in Table 3. Table 3 corresponds to the problem of maximization of of the limit load. It can be seen from Tables 2 and 3 that in the case of conical shells the design with stepped thickness is more effective for shells with smaller values of *k*.

Results of calculations for elastic cylindrical shells clamped at the left end (Fig. 3) are presented in Fig.7-11 and Table 4-5. Free vibrations of stepped shells with cracks emanating at the re-entrant corners of steps are considered. Here $s = c / h$ stands for the ratio of the crack depth and shell wall thickness, respectively. In Fig.7 k is a characteristic number proportional to the natural frequency of the shell.

Curves 1-4 in Fig.7 correspond to different locations of steps indicated in Tables 4-5. The results presented herein correspond to shells with two steps located at $a_1 = \alpha_1 l$ and $a_2 = \alpha_1 l$, l being the half of length of the tube.

The tube under consideration has thicknesses $h_1 = 0{,}3h_0$ and $h_2 = 0{,}6h_0$.

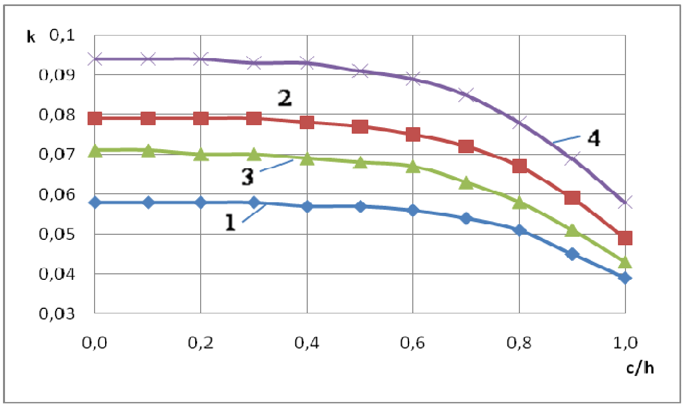

Fig. 7: Eigenfrequencies of the cracked shell.

Table 4: Eigenfrequencies of a cylindrical shell.

N	α_2	α_1	s=0,0	s=0,1	s=0,2	s=0,3	s=0,4	s=0,5
1	0,1	0,3	0,058	0,058	0,058	0,058	0,057	0,057
2	0,1	0,9	0,079	0,079	0,079	0,079	0,078	0,077
3	0,3	0,6	0,071	0,071	0,07	0,07	0,069	0,068
4	0,6	0,9	0,094	0,094	0,094	0,093	0,093	0,091

Table 5. Eigenfrequencies of a cylindrical shell.

N	α_2	α_1	s=0,6	s=0,7	s=0,8	s=0,9	s=1,0
1	0,1	0,3	0,056	0,054	0,051	0,045	0,039
2	0,1	0,9	0,075	0,072	0,067	0,059	0,049
3	0,3	0,6	0,067	0,063	0,058	0,051	0,043
4	0,6	0,9	0,089	0,085	0,078	0,069	0,058

It can be seen from Fig. 7 that the deeper the crack the lower the natural frequency of cracked shell. Fig. 8-9 correspond to the shell clamped at the left end and simply supported at the right hand end whereas results presented in Fig. 10-11 are associated with the tube with absolutely free right hand end. In Fig. 8 and Fig. 11 relationships between the characteristic number k and the crack length c/h are depicted for fixed locations of the step. Different curves in Fig.8 and Fig.11 correspond to different values of the ratio $\gamma = h_1 / h_0$. In Fig. 9 and Fig. 10 the ratio of thickness is fixed; different curves are associated with different locations of the step.

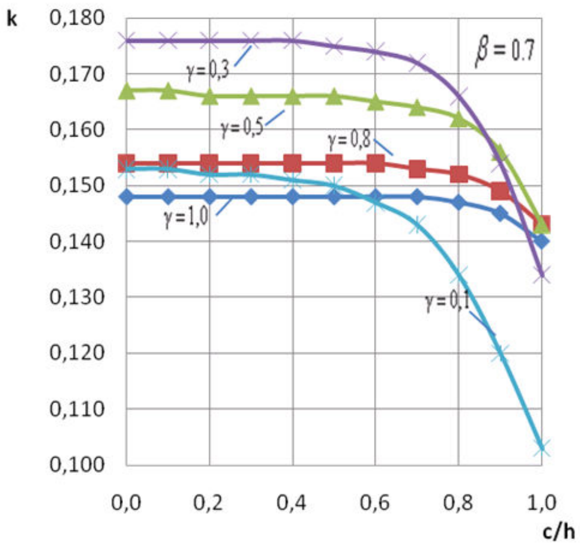

Fig. 8: Cantilever tube ($\beta = 0{,}7$)

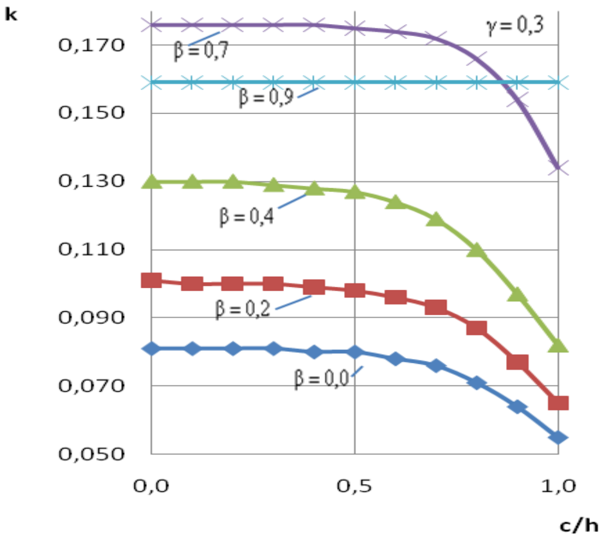

Fig. 9: Cantilever tube ($h_1 / h_0 = 0{,}3$)

9 Conclusion

Methods of optimization of both, elastic and inelastic shells were developed. Axisymmetric shells of piece wise constant thickness were considered. It was shown that the optimized shells were able to carry higher loads than corresponding reference shells of constant thickness. The cases of spherical, conical and circular cylindrical shells were studied in a greater detail. For spherical and conical shells made of materials which obey the Mises' and Hill's yield conditions the problems of minimum weight for constrained limit load and problems of maximum load carrying capacity for given weight have solved.

Calculations carried out showed that the load carrying capacity of spherical caps can be increased significantly even in the case of designs with one step of the thickness. When the number of steps is increased then the efficiency also increases. Similar conclusions can be drawn in the case of conical shells, as well.

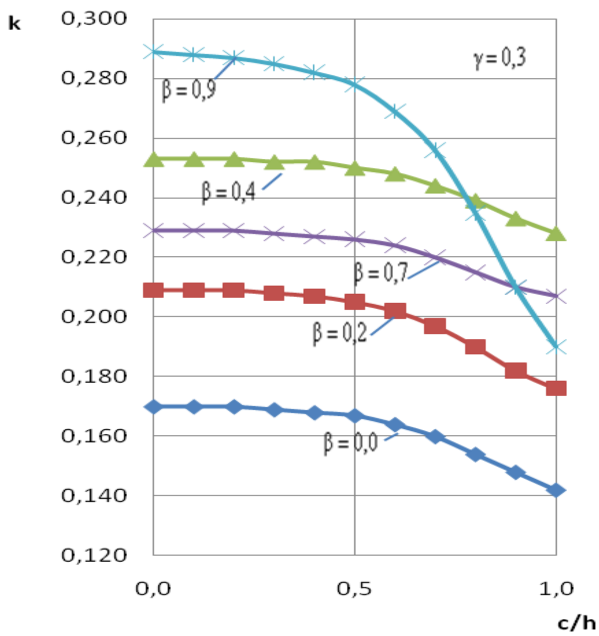

Fig. 10: Cracked tube ($h_1 / h_0 = 0{,}3$)

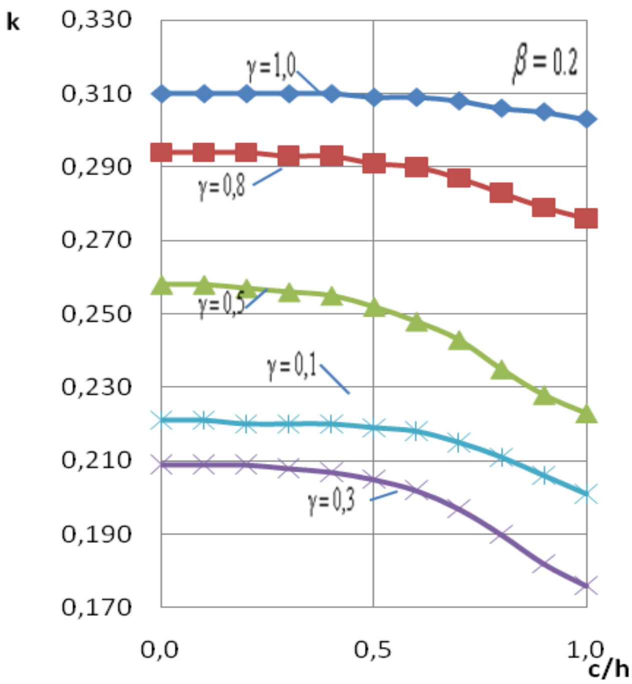

Fig. 11: Cracked tube ($a_1 / l = 0{,}2$)

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