Quintic B-spline Curve Generation using Given Points and Gradients and Modification based on Specified Radius of Curvature

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Abstract: - A method to generate a quintic B-spline curve which passes through the given points is described. In this case, there are four more unknown control point positions than there are equations. To overcome this problem three methods are described. First, solve the underdetermined system as it stands. Secondly, decrease the number of unknown control point positions in an underdetermined system in order to convert it to a determined system. Third, a method to increase the number of equations is employed to change an underdetermined system to a determined system. In addition to this, another method to generate a quintic B-spline curve using given points with gradients in sequence is described. In this case, a linear system will be either overdetermined, determined or underdetermined. This depends on the number of given points with gradients in sequence. Additionally, a method to modify a quintic B-spline curve is described. The objective is to change an aesthetically unpleasing curve to an aesthetically pleasing curve. Algebraic functions are used to determine an aesthetically pleasing radius of curvature distribution. This is accomplished by minimizing the difference between the quintic B-spline curves radius of curvature and the specified radius of curvature using the least-squares method. Examples of curve generation are given.

Key-Words: - B-spline curve generation, curvature vector, given points, given points with gradients, underdetermined system, overdetermined system, curve shape modification

1 Introduction

A NURBS curve, which is commonly used in the field of CAD・CAM and Computer Graphics, is used as an expression of a freeform curve. Particularly, cubic NURBS curves are widely used. In the smoothing of curves, it may be desirable to interpolate second derivative information at the knots. This is not possible with cubic splines, and so splines of higher degree have to be used. For symmetric boundary conditions, it is more convenient to work with quintic splines [1].

In this study, radius of curvature ranging over multi segments of a NURBS curve is modified based on the specified radius of curvature. Therefore, a quintic NURBS curve is needed in this study. In addition to this, the weights of the NURBS curve are set to one. For this reason, the curve used in this study is a quintic B-spline curve.

The objective of this study is to develop a B-spline curve generation method using given points in sequence for practical use.

Positions and gradients are given to the B-spline curve equations and first derivative equations of the B-spline curve respectively. Then, a B-spline curve is generated. Afterwards, if necessary, the shape of this B-spline curve is modified according to the target radius of curvature that has been smoothed.

Using the least-squares method, the shape of the designed curve is modified based on the target radius of curvature specified.

There are many related works for generation of a curve. There are data fitting by interpolation [2, 3], and data fitting with B-spline [4], and data fitting by principal component analysis following the statistical method [5-10].

There are many related works for generation of a fair curve dealing with knots. These are knot insertion [11, 12], knot removal algorithm for B-spline curves [13], knot removal algorithm for NURBS curves [14], and fair curve generation by knot value and weight modification [15].

There are many related works for generation of fair curvature distribution. Fair curvature distribution algorithms by modifying knot spacing [16, 17], and by removing and reinserting knots [18-22] have been published.

Curve generation algorithms related to curvature by modifying the control points have been published. These use a clothoidal curve for specifying the curvature [23], and modify the shape of the curve based on the target radius of curvature specified [24-28]. Curve generation algorithms related to curvature by specifying curvature distribution have also been published [29].

Section 2 of this paper describes a quintic B-spline curve, the derivatives of a quintic B-spline curve, curvature vector, curvature, and radius of curvature. Section 3 describes the generation of a quintic B-spline curve which passes through the given points in sequence and the generation of a quintic B-spline curve using the given points with gradients in sequence. In section 4, B-spline curve shape modification based on the specified radius of curvature is described. Section 5 gives the engineering examples of curve generation.

2 Freeform Curve Expression

The objective of freeform curve design is to design the framework of surface patches. Surface patches are defined as tensor products, which are bi-variate and normally defined by u and v. In other words, one knot sequence in u direction, and another knot sequence in v direction are defined despite the complexity of the surface patches. Therefore, knot spacing is fixed in this study.

2.1 Quintic B-spline Curve Expression

An $n-5$ segments $(n \geq 6)$ quintic B-spline curve is composed of n control points such as $\boldsymbol{q}_0, \boldsymbol{q}_1, \cdots, \boldsymbol{q}_{n-1}$ as

$$\boldsymbol{R}(t) = \sum_{i=0}^{n-1} N_{i,6}(t) \cdot \boldsymbol{q}_i \ , \tag{1}$$

where $N_{i,6}(t)$ $(i = 0,1,\cdots,n-1)$ are B-spline basis functions.

These functions are defined by the de Boor-Cox [30] recursion formulas, and are recursively defined by knot sequence $t_0, t_1, t_2, t_3, \cdots, t_{n+5}$ as

$$\left.\begin{aligned} N_{j,1}(t) &= \begin{cases} 1 & (t_j \leq t < t_{j+1}) \\ 0 & \text{otherwise} \end{cases} \\ N_{j,M}(t) &= \frac{t - t_j}{t_{j+M-1} - t_j} N_{j,M-1}(t) + \frac{t_{j+M} - t}{t_{j+M} - t_{j+1}} N_{j+1,M-1}(t) \end{aligned}\right\}, \tag{2}$$

where $j = 0,1,\cdots,n+4$ and $M = 2,3,\cdots,6$, j corresponds to the knot number.

If the knot vector contains repeated knot values called multiple knots, then a division of the form $N_{j,M-1}(t)/(t_{j+M-1} - t_j) = 0/0$ (for some j) may be encountered during the execution of the recursion. Whenever this occurs, it is assumed that $0/0 = 0$ [31].

A one segment quintic B-spline curve with the knot vector $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$ is

$$\begin{aligned} \boldsymbol{R}(t) = \frac{1}{120}\Big\{ & (1-t)^5 \boldsymbol{q}_0 \\ & + (5t^5 - 20t^4 + 20t^3 + 20t^2 - 50t + 26)\boldsymbol{q}_1 \\ & + (-10t^5 + 30t^4 - 60t^2 + 66)\boldsymbol{q}_2 \\ & + (10t^5 - 20t^4 - 20t^3 + 20t^2 + 50t + 26)\boldsymbol{q}_3 \\ & + (-5t^5 + 5t^4 + 10t^3 + 10t^2 + 5t + 1)\boldsymbol{q}_4 \\ & + t^5 \boldsymbol{q}_5 \Big\}. \end{aligned} \tag{3}$$

The first derivative of a quintic B-spline curve shown in Eq.(3) is expressed as

$$\begin{aligned} \frac{d\boldsymbol{R}(t)}{dt} = \frac{1}{24}\Big\{ & -(1-t)^4 \boldsymbol{q}_0 \\ & + (5t^4 - 16t^3 + 12t^2 + 8t - 10)\boldsymbol{q}_1 \\ & + (-10t^4 + 24t^3 - 24t)\boldsymbol{q}_2 \\ & + (10t^4 - 16t^3 - 12t^2 + 8t + 10)\boldsymbol{q}_3 \\ & + (-5t^4 + 4t^3 + 6t^2 + 4t + 1)\boldsymbol{q}_4 \\ & + t^4 \boldsymbol{q}_5 \Big\}. \end{aligned} \tag{4}$$

The second derivative of a quintic B-spline curve shown in Eq.(3) is expressed as

$$\begin{aligned} \frac{d^2\boldsymbol{R}(t)}{dt^2} = \frac{1}{6}\Big\{ & (1-t)^3 \boldsymbol{q}_0 \\ & + (5t^3 - 12t^2 + 6t + 2)\boldsymbol{q}_1 \\ & + (-10t^3 + 18t^2 - 6)\boldsymbol{q}_2 \\ & + (10t^3 - 12t^2 - 6t + 2)\boldsymbol{q}_3 \\ & + (-5t^3 + 3t^2 + 3t + 1)\boldsymbol{q}_4 \\ & + t^3 \boldsymbol{q}_5 \Big\}. \end{aligned} \tag{5}$$

The third derivative of a quintic B-spline curve shown in Eq.(3) is expressed as

$$\begin{aligned} \frac{d^3\boldsymbol{R}(t)}{dt^3} = \frac{1}{2}\Big\{ & -(1-t)^2 \boldsymbol{q}_0 \\ & + (5t^2 - 8t + 2)\boldsymbol{q}_1 \\ & + (-10t^2 + 12t)\boldsymbol{q}_2 \\ & + (10t^2 - 8t - 2)\boldsymbol{q}_3 \\ & + (-5t^2 + 2t + 1)\boldsymbol{q}_4 \\ & + t^2 \boldsymbol{q}_5 \Big\}. \end{aligned} \tag{6}$$

The fourth derivative of a quintic B-spline curve shown in Eq.(3) is expressed as

$$\begin{aligned} \frac{d^4\boldsymbol{R}(t)}{dt^4} = & (1-t)\boldsymbol{q}_0 \\ & + (5t - 4)\boldsymbol{q}_1 \\ & + (-10t + 6)\boldsymbol{q}_2 \\ & + (10t - 4)\boldsymbol{q}_3 \\ & + (-5t + 1)\boldsymbol{q}_4 \\ & + t\boldsymbol{q}_5 \ . \end{aligned} \tag{7}$$

Curvature vector is expressed as

$$\boldsymbol{\kappa}(t)=\frac{\left(\dot{\boldsymbol{R}}(t)\times\ddot{\boldsymbol{R}}(t)\right)\times\dot{\boldsymbol{R}}(t)}{\left(\dot{\boldsymbol{R}}(t)\right)^{4}}, \tag{8}$$

where $\dot{\boldsymbol{R}}(t)$ is the first derivative of a B-spline curve, and $\ddot{\boldsymbol{R}}(t)$ is the second derivative of a B-spline curve.

Curvature is the magnitude of the curvature vector, therefore, curvature is expressed as

$$\kappa(t)=\left|\boldsymbol{\kappa}(t)\right|. \tag{9}$$

By definition, the curvature of a curve is nonnegative. However, in many cases it is useful to ascribe a sign to the curvature [32]. The choosing of the sign is commonly connected with the tangent rotation by moving along the curve in the direction of the increasing parameter. The curvature of the curve is positive when its tangent rotates counter-clockwise, the curvature of the curve is negative when its tangent rotates clockwise.

The radius of curvature is the reciprocal number of curvature, therefore, the radius of curvature is expressed as

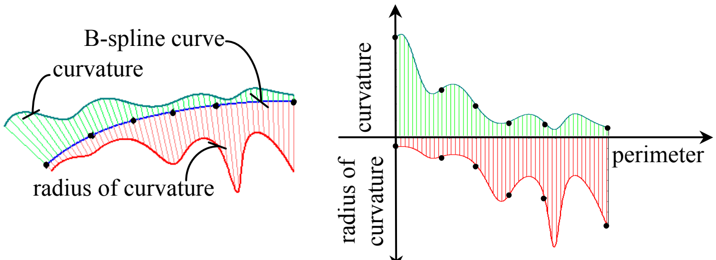
$$\rho(t)=\frac{1}{\kappa(t)}. \tag{10}$$

2.2 B-spline Curve Display

The curvature of a curve is the most significant descriptor of its shape [33]. To check the shape of a curve by displaying its curvature or radius of curvature plots is widely known. This is simply the graph of $\kappa(t)$ or $\rho(t)$.

Curvature information is plotted using straight lines drawn outward from and perpendicular to the curve, with the line length proportional to the amount of curvature at that spot. Radius of curvature information is plotted using straight lines drawn inward from and perpendicular to the curve, with the line length proportional to the amount of radius of curvature at that spot.

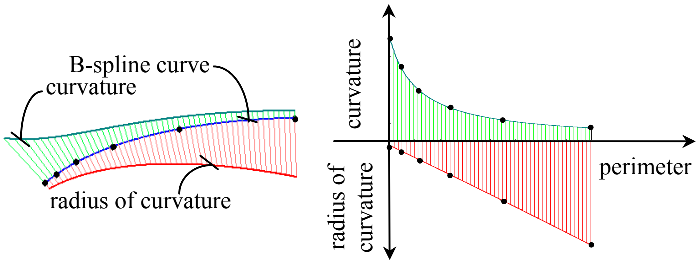
It is hard to distinguish the shape of the two curves by just looking at their graphs. If the radius of curvature plots are drawn for both, the difference between the two curves is recognized immediately. Curve shape is judged by looking at the lines coming out from the curve and seeing how their lengths change along the path, not along the parameter. Therefore, curvature or radius of curvature distribution must be drawn to the perimeter of the curve.



(a) B-spline curve with curvature and radius of curvature plots
(b) curvature and radius of curvature distribution

• Point marks indicate knot position.

Fig.1 B-spline curve, curvature and radius of curvature distribution



(a) modified B-spline curve with curvature and radius of curvature plots
(b) curvature and radius of curvature distribution

• Point marks indicate knot position.

Fig.2 Modified B-spline curve, curvature and radius of curvature distribution

A B-spline curve with curvature and radius of curvature plots, and curvature and radius of curvature distribution are shown in Fig.1(a), (b) respectively.

A shape modified B-spline curve with curvature and radius of curvature plots, and curvature and radius of curvature distribution are shown in Fig.2(a), (b) respectively.

While both curves shown in Fig.1(a) and Fig.2(a) can hardly be distinguished by just looking at their curves, the curve with curvature and radius of curvature plots tell the two curves apart immediately.

The curvature distribution of a quintic B-spline curve is not monotone as is shown in Fig.1(b). But curvature and radius of curvature distribution of a shape modified B-spline curve is monotone as is shown in Fig.2(b). This display technique provides designers with the ability to evaluate the quality of a designed curve.

Seeing a B-spline curve with curvature and radius of curvature plots gives designers a deeper understanding of their design.

The relation of curvature to radius of curvature is inverted. Therefore, it can be seen in Fig.1(b) and Fig.2(b) that in a portion of a curve where the curvature is small, at this portion the radius of the curvature will be large. As shown in Fig.2(b), if the radius of curvature to the perimeter is linear, the curvature distribution will be parabolic. On the contrary, if the curvature to the perimeter is linear, radius of curvature distribution will be parabolic.

In case the curve shape is close to a straight line, the radius of curvature becomes infinity. Therefore, a limit value should be assigned to the radius of curvature. Both curvature distribution and radius of curvature distribution displays are effective to examine the shape of a curve.

3 Quintic B-spline Curve Generation

In this section, methods to generate a quintic B-spline curve which passes through the given points in sequence and to generate a quintic B-spline curve using the given points with gradients in sequence are described. One widely used form of data fitting is interpolation. Sometimes the problem is to interpolate to positional data alone, and sometimes it is necessary to interpolate to derivatives as well as positions, at least at some of the points [34].

3.1 Generation of a Quintic B-spline Curve which Passes through Given Points in Sequence

A method to generate a quintic B-spline curve which passes through the given points in sequence is described.

To simplify further description, notations shown in equations hereafter are summarized. i is the ordinal number of the given points in sequence, m serves as both the last ordinal number of the given points in sequence and the total number of B-spline curve segments, $m+1$ is the total number of given points, and $m+5$ is the total number of control points. j is the ordinal number of the given gradients assigned to the given points, n is the last ordinal number of the given gradients, and $n+1$ is the total number of given gradients.

The parameter of Eq.(3) is set to zero by defining the geometrical knot position corresponding to the knot of the knot vector. Then, Eq.(3) is expressed as

$$\boldsymbol{R}_i = \frac{1}{120}(\boldsymbol{q}_i + 26\boldsymbol{q}_{i+1} + 66\boldsymbol{q}_{i+2} + 26\boldsymbol{q}_{i+3} + \boldsymbol{q}_{i+4}), \quad (i = 0, 1, 2, 3, \cdots, m) \tag{11}$$

The positional vectors $\boldsymbol{P}_i (i = 0,1,2,3,\cdots,m)$ of the given points in sequence are assigned to $\boldsymbol{R}_i (i = 0,1,2,3,\cdots,m)$ in Eq.(11), and $\boldsymbol{q}_i, \boldsymbol{q}_{i+1}, \boldsymbol{q}_{i+2}, \boldsymbol{q}_{i+3}, \boldsymbol{q}_{i+4}$ $(i = 0,1,2,3,\cdots,m)$ are the control points of a quintic B-spline curve.

When the control points of a quintic B-spline curve are calculated using Eq.(11), the number of unknowns, which are the positions of the control points, are four more than the number of equations which are expressed by Eq.(11). That is, this linear system to be solved to obtain the control points is underdetermined. Three methods are used as follows. One, a method to calculate the control point positions in an underdetermined system as it stands. Two, a method to convert an underdetermined system to a determined one by expressing the unknown control points by the linear combination of known control points. Three, convert an underdetermined system to a determined system by assigning four gradients to given points to increase the number of equations.

For an underdetermined system, while setting auxiliary function, the linear system is solved under the constraint condition by selecting one solution from an infinite number of exact solutions using Lagrange's method of indeterminate multipliers. In another method, the curvatures at both ends of the quintic B-spline curve are assumed to be zero. Then, the second derivatives at both ends of the quintic B-spline curve are set to zero, while setting the parameter of Eq.(3) at zero by defining the geometrical knot position corresponding to the knot of the knot vector. The following equations are then obtained.

$$\left.\begin{array}{l} \boldsymbol{q}_0 + 2\boldsymbol{q}_1 - 6\boldsymbol{q}_2 + 2\boldsymbol{q}_3 + \boldsymbol{q}_4 = \boldsymbol{0} \\ \boldsymbol{q}_m + 2\boldsymbol{q}_{m+1} - 6\boldsymbol{q}_{m+2} + 2\boldsymbol{q}_{m+3} + \boldsymbol{q}_{m+4} = \boldsymbol{0} \end{array}\right\}, \tag{12}$$

Accordingly, the fourth derivatives at both ends of the quintic B-spline curve are set to zero. Then, in the same manner, the following equations are obtained.

$$\left.\begin{array}{l} \boldsymbol{q}_0 - 4\boldsymbol{q}_1 + 6\boldsymbol{q}_2 - 4\boldsymbol{q}_3 + \boldsymbol{q}_4 = \boldsymbol{0} \\ \boldsymbol{q}_m - 4\boldsymbol{q}_{m+1} + 6\boldsymbol{q}_{m+2} - 4\boldsymbol{q}_{m+3} + \boldsymbol{q}_{m+4} = \boldsymbol{0} \end{array}\right\}, \tag{13}$$

Using Eq.(12) and Eq.(13), Eq.(14) is obtained.

$$\left.\begin{array}{l} \boldsymbol{q}_0 = 2\boldsymbol{q}_2 - \boldsymbol{q}_4 \\ \boldsymbol{q}_1 = 2\boldsymbol{q}_2 - \boldsymbol{q}_3 \\ \boldsymbol{q}_{m+3} = 2\boldsymbol{q}_{m+2} - \boldsymbol{q}_{m+1} \\ \boldsymbol{q}_{m+4} = 2\boldsymbol{q}_{m+2} - \boldsymbol{q}_m \end{array}\right\} \tag{14}$$

In Eq.(14), unknown control points $\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_{m+3}$, and $\boldsymbol{q}_{m+4}$ are expressed by the linear combination of known control points. Thus, $\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_{m+3}$, and $\boldsymbol{q}_{m+4}$ are made known. Therefore, the number of equations will be equal to the number of unknowns. That is, this linear system is determined.

Using different boundary derivatives has a strong influence on the shape and approximation quality of the curve [35, 36].

A quintic B-spline curve which passes through the given points in sequence, assuming the curvatures at both ends of the quintic B-spline curve are zero, and q_0, q_1, q_{m+3}, and q_{m+4} in Eq.(14) are illustrated in Fig.3.

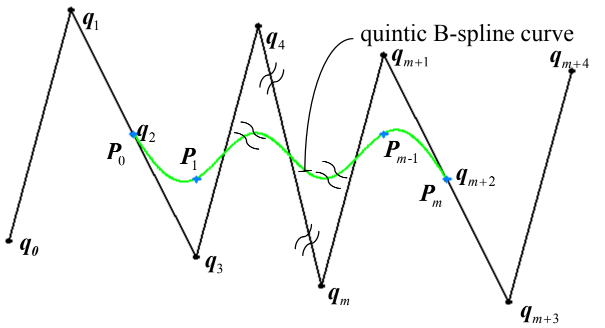


Fig.3 Illustration for a quintic B-spline curve which passes through the given points in sequence. $P_0, P_1, \cdots, P_{m-1}, P_m$ are given points. $q_0, q_1, \cdots, q_{m+3}, q_{m+4}$ are control points of a quintic B-spline curve.

$P_i (i = 0,1,2,3,\cdots,m)$ in Fig.3 are the positional vectors of the given points in sequence, and are assigned to $R_i (i = 0,1,2,3,\cdots,m)$ in Eq.(11). As shown in Fig.3, both ends of a quintic B-spline P_0 and P_m, which are also the given points, have the same position of the control point q_2 and q_{m+2}. This can be derived from Eq.(3) and Eq.(14). P_0 is the midpoint of q_1 and q_3. Also, it is midpoint of q_0 and q_4. Another end point P_m is the midpoint of q_{m+1} and q_{m+3}. Also, it is the midpoint of q_m and q_{m+4}. This can be derived from Eq.(14).

As another method, in addition to the given points in sequence, gradients are assigned to the given points to compensate for the difference between the number of unknowns and that of the equations. For this, the location of the four gradients is assigned situationally.

Eq.(15) is applied to the gradients $d_j (j = 0,1,2,3,\cdots,n)$ by setting the parameter of Eq.(4) to zero by defining the geometrical knot position corresponding to the knot of the knot vector.

$$\frac{dR_j}{dt} = \frac{1}{24}(-q_j - 10q_{j+1} + 10q_{j+3} + q_{j+4}), \quad (15)$$

$(j = 0, 1, 2, 3, \cdots, n)$

The j shown in Eq.(15) corresponds to the i in Eq.(11) and is assigned to the given points situationally.

The defined gradients are located at the beginning given point and it's adjacent point, and at the end given point and its adjacent point in general.

As a magnitude of the first derivative, the value of the distance between the two adjacent given points is assigned as a default value. As for further adjustment, the magnitude of the first derivative is determined interactively [37].

Using the given points in sequence and four location specified gradients, the linear system becomes determined. That is, the number of unknowns is equal to the number of equations.

The concept of a quintic B-spline curve generation using the given points in sequence and four location specified gradients is illustrated in Fig.4. The defined gradients are located at the beginning given point and it's adjacent point, and at the end given point and it's adjacent point. That is, the j of Eq.(15) are determined as $0, 1, n-1,$ and n respectively. $P_0, P_1, P_2, P_3, P_4, \cdots, P_{m-2}, P_{m-1}$, and P_m are the positional vectors of the given points in sequence, and d_0, d_1, d_{n-1}, and d_n are the four location specified gradients.

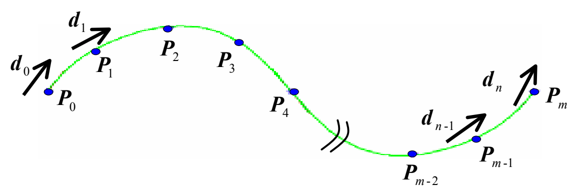


Fig.4 Illustration for a quintic B-spline curve using the given points in sequence and four location specified gradients

As examples of a quintic B-spline curve which passes through the given points in sequence, three quintic B-spline curves which simulate filleting curves are shown with their curvature plots in Fig.5. In Fig.5(a), and (b), the j of Eq.(15) are determined situationally as 0, 2, 4, and 6. In Fig.5(c), the j of Eq.(15) are determined situationally as 0, 1, 3, and 4. In the case of a curve including a filleting segment, note that two gradients are placed at the start and end points of the filleting segment as well as the quintic B-spline curve beginning and end points.

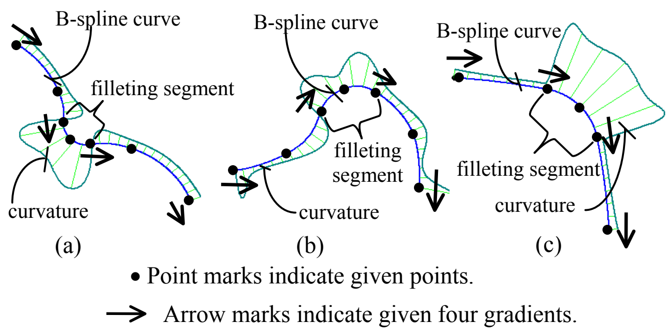


Fig.5 Examples of a quintic B-spline curve which passes through given points in sequence and four location specified gradients

3.2 Generation of a Quintic B-spline Curve using Given Points with Gradients

In this sub-section, a quintic B-spline curve generation using given points with gradients in sequence is described.

The concept of generating a quintic B-spline curve using given points with gradients in sequence is illustrated in Fig.6.

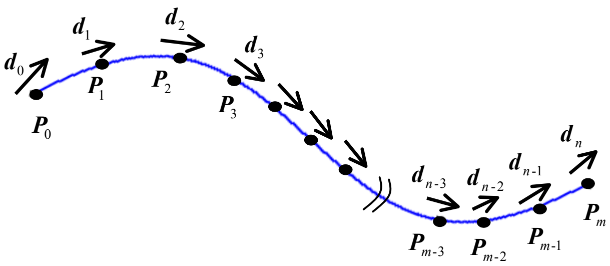


Fig.6 Concept of generating a quintic B-spline curve using given points with gradients in sequence

$\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \cdots, \boldsymbol{P}_{m-3}, \boldsymbol{P}_{m-2}, \boldsymbol{P}_{m-1},$ and $\boldsymbol{P}_m$ are the positional vectors of the given points in sequence, while $\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \cdots, \boldsymbol{d}_{n-3}, \boldsymbol{d}_{n-2}, \boldsymbol{d}_{n-1},$ and $\boldsymbol{d}_n$ are gradients assigned to the given points in sequence. A quintic B-spline curve which passes through the given points in sequence and which has the first derivatives at these given points is generated.

A quintic B-spline curve is generated by solving Eq.(11) and Eq.(15) simultaneously by making m in Eq.(11) equal to n in Eq.(15). In this case, the i in Eq.(11) corresponds to the j in Eq.(15). If the number of given points with gradients is 4, the number of B-spline curve equations (Eq.(11)) is 4 and the number of first derivative equations (Eq.(15)) is 4. As a linear system, the total number of equations is 8, whereas the total number of control points of a quintic B-spline curve is 8. Therefore, this linear system is determined. That is, the rank of a coefficient matrix of a linear system is equal to the number of unknowns. The solution to this linear system is exact.

But, in case the number of given points with gradients is 3, the number of equations (Eq.(11)) which pass through the given points is 3, and the number of equations of the first derivative (Eq.(15)) is 3. In this case, as a linear system, the total number of equations is 6, whereas the number of control points of the quintic B-spline curve is 7. That is, the number of equations is less than the number of unknowns. Therefore, this linear system is underdetermined [38].

For an underdetermined system, while setting auxiliary function, the linear system is solved under the constraint condition by selecting one solution from an infinite number of exact solutions using Lagrange's method of indeterminate multipliers.

In case the number of given points with gradients is 5, the number of equations (Eq.(11)) is 5, and the number of equations of the first derivative (Eq.(15)) is 5. In this case, as a linear system, the total number of equations is 10, whereas the number of control points of the quintic B-spline curve is 9. That is, the number of equations exceeds the number of unknowns. Therefore, this linear system is overdetermined [39].

For an overdetermined system, the differences between the right and left sides of all the equations of the system are minimized. The control points calculated are therefore an approximation.

For a system where the number of given points with gradients is 5 or more, the linear system is overdetermined. For these systems, in accordance with the increments of the differences between the number of equations and the number of unknowns, the status of the approximation worsens.

The above mentioned situations are summarized in Table 1. A determined linear system is shown by the cross hatching.

Table 1. Linear system condition

(I)	(II)	(III)	(IV)
2	6	underdetermined	exact
3	7	underdetermined	exact
4	8	determined	exact
5	9	overdetermined	approximation
6	10	overdetermined	approximation
7	11	overdetermined	approximation
8	12	overdetermined	approximation

(I) number of given points with gradients
(II) number of control points of a quintic B-spline curve
(III) system condition (underdetermined, determined, overdetermined)
(IV) solution status

As an example, in case the number of given points with gradients is 3, that is, m in Eq.(11) and n in Eq.(15) are 3, the quintic B-spline curve generated as an underdetermined system is shown in Fig.7 with its first derivative vectors, which are drawn outward perpendicular to the curve by the straight lines. The solution to this linear system is exact. The length of the line is proportional to the first derivative at that spot. This is an unusual way of displaying the first derivative vectors. Nevertheless, this helps visual recognition of the quintic B-spline curve shape and the magnitude variation of its first derivative.

Therefore, it is visually recognized that in Fig.7 the first derivative vectors shown are perpendicular to the assigned gradients $\boldsymbol{d}_i$ at $\boldsymbol{P}_i\left(i = 0,1,2\right)$.

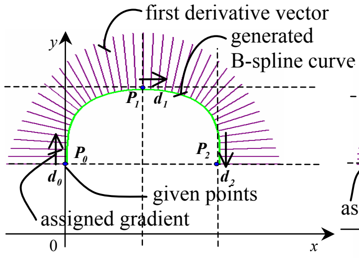


Fig.7 Quintic B-spline curve and its first derivative vectors, in case of underdetermined

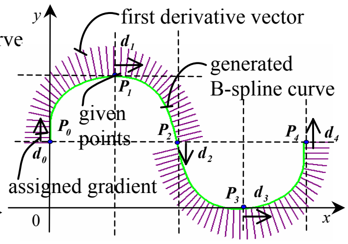


Fig.8 Quintic B-spline curve and its first derivative vectors, in case of overdetermined

In case the number of given points with gradients is 4, that is, m in Eq.(11) and n in Eq.(15) is 4, the quintic B-spline curve is generated as a determined system. The solution to this linear system is exact.

Furthermore, in case the number of given points with gradients is 5, that is, m in Eq.(11) and n in Eq.(15) are 5, the quintic B-spline curve is generated as an overdetermined system and is shown with its first derivative vectors in Fig.8. The solution to this linear system is an approximation. Therefore, in Fig.8, it is visually recognized that the first derivative vector shown is not perpendicular to the assigned gradient $\boldsymbol{d}_2$ at $\boldsymbol{P}_2$. In this manner, a quintic B-spline curve is generated based on the given points with gradients in sequence.

4 Curve Shape Modification based on Specified Radius of Curvature Distribution

In this section, a method to modify a quintic B-spline curve shape according to the specified radius of curvature distribution to realize an aesthetically pleasing freeform curve is described.

Radius of curvature is suitable, because it conforms to our visual recognition of the shape of the curve. In a case where curve shape is very close to a straight line, the radius of curvature becomes infinity. Also, at the point of inflection, curvature value becomes zero. Therefore, radius of curvature value becomes infinite. For these reasons, radius of curvature value is converted to curvature value for computation.

The concept of radius of curvature specification and quintic B-spline curve shape modification based on the specified radius of curvature distribution is shown in Fig.9. A quintic B-spline curve and its radius of curvature plots are shown in Fig.9(a).

A method to modify the shape of the quintic B-spline curve shown in Fig.9(a) to the curve shown in Fig.9(b) is examined.

Radius of curvature plots shown in Fig.9(a) are drawn inward from and perpendicular to the curve using straight lines. The length of the line is proportional to the radius of curvature at that spot on the curve. However, the straight lines are not parallel to each other and the beginning points of the individual straight lines are different. Therefore, a curve with a radius of curvature display is suitable to examine the variation of radius of curvature as a whole. But, it is not suitable to examine the length of the straight lines and variation of radius of curvature in detail.

Therefore, considering the parameter of the quintic B-spline curve is different from the perimeter of the curve, the perimeter of a quintic B-spline curve as a straight line is set to the horizontal axis, and the radius of curvature is set to the vertical axis as shown in Fig.9(c). Then, the radius of curvature distribution to the perimeter is drawn. After this, the target radius of curvature specified is superimposed on the current radius of curvature distribution. Linear, quadratic, and cubic algebraic function is applied as the target radius of curvature to the current radius of curvature distribution to modify the shape of the quintic B-spline curve. In detail, the coefficients of the algebraic function are calculated by the least-squares method using the current radius of curvature distribution. Then, the radius of curvature is specified by the determined algebraic functions. To determine the target radius of curvature distribution by using algebraic functions has been published [24-28].

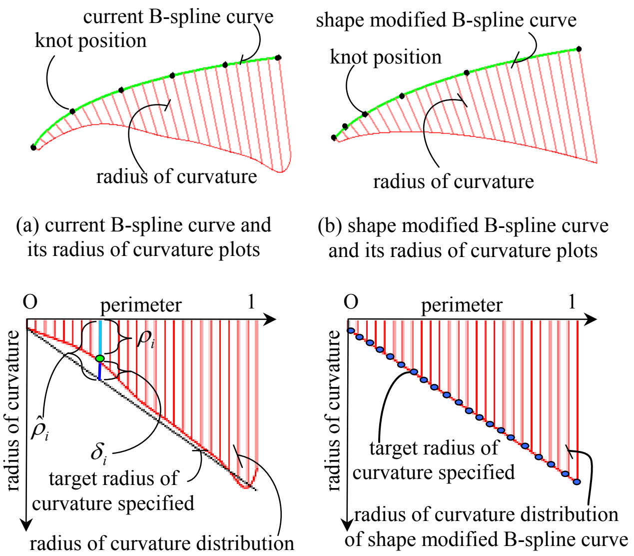
In addition to this, the target radius of curvature is specified by smoothing the radius of curvature distribution using neighborhood values as in image processing. Assuming $f(i)$ is the radius of curvature value of perimeter position i of the current curve, the value $g(i)$ of the smoothed radius of curvature corresponding to $f(i)$ of the current curve is calculated by using one of the smoothing operators shown in Eq.(16).

$$\left.\begin{aligned}
g_1(i) &= \frac{1}{3}\left[f(i-1)+f(i)+f(i+1)\right] \\
g_2(i) &= \frac{1}{5}\left[f(i-2)+f(i-1)+f(i)+f(i+1)+f(i+2)\right] \\
g_3(i) &= \frac{1}{7}\left[f(i-3)+f(i-2)+f(i-1)+f(i)+f(i+1)\right. \\
&\qquad \left.+f(i+2)+f(i+3)\right] \\
g_4(i) &= \frac{1}{9}\left[f(i-4)+f(i-3)+f(i-2)+f(i-1)+f(i)\right. \\
&\qquad \left.+f(i+1)+f(i+2)+f(i+3)+f(i+4)\right]
\end{aligned}\right\} \quad (16)$$

The selection of an operator shown in Eq.(16) and the iteration number of operations are determined interactively by looking at the profile of the radius of curvature distribution.

While increasing the number of iteration, the profile of the radius of curvature distribution becomes a straight line. For an infinite number of iteration, it will be a straight line with no gradient. This means that the shape of the curve will be an arc.

After the profile of the radius of curvature distribution is determined, the specified radius of curvature is superimposed on the current radius of curvature distribution.



(a) current B-spline curve and its radius of curvature plots

(b) shape modified B-spline curve and its radius of curvature plots

(c) difference between current radius of curvature and target radius of curvature specified

(d) radius of curvature of shape modified B-spline curve and target radius of curvature specified (same as in (c))

Fig.9 Concept of radius of curvature specification and quintic B-spline curve shape modification based on the target radius of curvature distribution specified

An example is shown in Fig.9(c). The i th of radius of curvature distribution of a perimetrically represented quintic B-spline curve is denoted as ρ_i, the specified radius of curvature at the same spot is denoted as $\hat{\rho}_i$, the difference δ_i is shown by Eq.(17) and is illustrated in Fig.9(c).

$$\delta_i = \rho_i(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,\ q_{n-2}^y) - \hat{\rho}_i \quad (17)$$

Where $i = 0,1,2,\cdots,m-1$, m is the number of specified radius of curvature, and n is the number of B-spline curve segments plus 5, which is the degree of the curve.

$S(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,q_{n-2}^y)$ which is the sum of the squared differences for all specified radius of curvatures in Eq.(18) is minimized by introducing the least-squares method. The radius of curvature expression is non-linear. Therefore, by Taylor's theorem, Eq.(18) is linearized as in Eq.(19).

$$\begin{aligned}
&S(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,\ q_{n-2}^y) \\
&= \sum_{i=0}^{m-1}\left[\rho_i(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,\ q_{n-2}^y) - \hat{\rho}_i\right]^2
\end{aligned} \quad (18)$$

$$\begin{aligned}
&S(q_1^x+\Delta q_1^x,\ \cdots,\ q_{n-2}^x+\Delta q_{n-2}^x,\ q_1^y+\Delta q_1^y,\ \cdots,\ q_{n-2}^y+\Delta q_{n-2}^y) \\
&= \sum_{i=0}^{m-1}\left[\rho_i(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,\ q_{n-2}^y) + \frac{\partial \rho_i}{\partial q_1^x}\Delta q_1^x + \right. \\
&\left. \cdots + \frac{\partial \rho_i}{\partial q_{n-2}^x}\Delta q_{n-2}^x + \frac{\partial \rho_i}{\partial q_1^y}\Delta q_1^y + \cdots + \frac{\partial \rho_i}{\partial q_{n-2}^y}\Delta q_{n-2}^y - \hat{\rho}_i\right]^2
\end{aligned} \quad (19)$$

Eq.(19) is minimized by equating to zero all the partial derivatives of $S(q_1^x+\Delta q_1^x,\cdots,q_{n-2}^x+\Delta q_{n-2}^x,\ q_1^y+\Delta q_1^y,\cdots,q_{n-2}^y +\Delta q_{n-2}^y)$ with respect to Δq_r^x and $\Delta q_r^y\,(r=1,2,\cdots,n-2)$ as

$$\left.\begin{aligned}
\frac{\partial S}{\partial \Delta q_r^x} &= 0 \quad (r=1,2,\cdots,n-2) \\
\frac{\partial S}{\partial \Delta q_r^y} &= 0 \quad (r=1,2,\cdots,n-2)
\end{aligned}\right\}. \quad (20)$$

Using these simultaneous linear equations, Δq_r^x and $\Delta q_r^y\,(r=1,2,\cdots,n-2)$ are calculated. Then, q_r^x and q_r^y are determined.

This kind of study on the radius of curvature, or the curvature to realize a fair curve is called a constrained non-linear minimization problem [40]. For computation, ρ_i and $\hat{\rho}_i$ are calculated based on the perimeter. Then, the perimeter used is converted to the parameter to calculate the position of the control points of the quintic B-spline curve. Thus, a quintic B-spline curve is generated. The total length of the curve, which is the perimeter, is calculated and rescaled as 1. Repeating these operations, positions of the control points of the quintic B-spline curve are determined while $\delta_i\,(i=0,1,\cdots,m-1)$ are minimized for the entire perimeter.

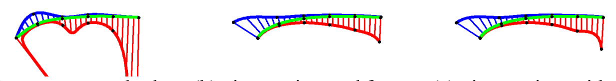
Smoothed radius of curvature distribution, which is specified as target radius of curvature distribution is shown in Fig.9(c). Using the above mentioned method, the shape of the curve is modified based on the radius of curvature distribution specified. The dotted line shown in Fig.9(d) is the smoothed radius of curvature distribution shown in Fig.9(c). It is visually recognized that the radius of curvature distribution of the shape modified curve shown in Fig.9(d) matches to the specified radius of curvature.

5 Examples of Curve Generation

In this section, examples of curve generation using the methods described in the previous section are given.

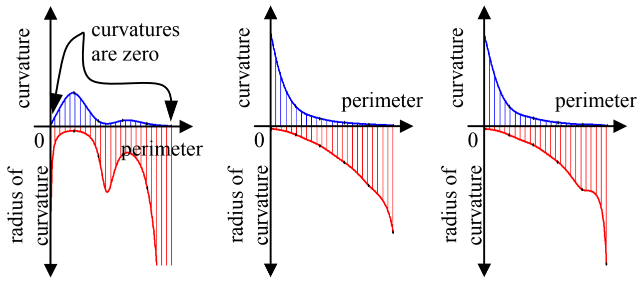
An example of a quintic B-spline curve generation which passes through some given points in sequence, while setting the curvatures at both ends to zero is shown in Fig.10(a) with its curvature and radius of curvature plots.

Also, an example of a quintic B-spline curve generation using some given points in sequence and four location specified gradients is shown in Fig.10(b) with its curvature and radius of curvature plots.



(a) curvatures at both ends are zero (b) given points and four gradients (c) given points with gradients

Fig.10 Quintic B-spline curves



(a) curvatures at both ends are zero (b) given points and four gradients (c) given points with gradients

Fig.11 Curvature and radius of curvature distribution

In adding to these examples, a quintic B-spline curve generation using some given points with gradients is shown in Fig.10(c) with its curvature and radius of curvature plots.

Furthermore, to make the variation of curvature and radius of curvature clear, curvature distribution and radius of curvature distribution of these sample curves are shown corresponding to the generation methods, in Fig.11(a), (b), (c). These correspond to Fig.10(a), (b), (c).

Notice that the curvature at both ends of the curve are zero in Fig.11(a).

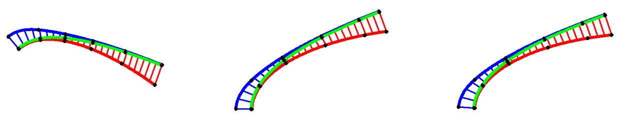
The quintic B-spline curves shown in Fig.10(a) and (b) are generated by solving the determined linear system. Therefore, these two curves pass through their given points in sequence.

In Fig.10(c), the number of given points with gradients is six. The number of control points of a quintic B-spline is ten. In this case, the total number of equations is twelve, whereas the number of control points of the quintic B-spline curve is ten. That is, the number of equations exceeds the number of unknowns. Therefore, this linear system is overdetermined.

The solution, which is the position of the control points for this system, is an approximation.

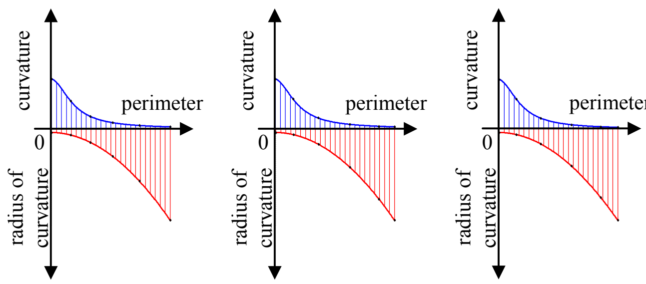
The position of the given points is compared with the geometrical knot positions of the generated quintic B-spline curve. The maximum positional difference is 2.7094321×10^{-1}. And, the mean positional difference is 1.080901×10^{-1}. The maximum directional difference is 2.813751×10^{-2} degree. And, the mean directional difference is 1.291814×10^{-2} degree. The perimeter of the curve, which is the arc length, is 244.398140.

Example of curve shape modification is shown. The coefficients of the quadratic algebraic function are calculated by the least-squares method using the radius of curvature distribution shown in Fig.11(b). The coefficients calculated are applied to the radius of curvature distributions shown in Fig.11 to modify the shapes of the curves. Then, the shapes of these three curves shown in Fig.10 are modified by using the shape modification algorithm described in the section four. These three shape modified curves are shown in Fig.12 with their curvature and radius of curvature plots. Fig.12(a) corresponds to Fig.10(a), Fig.12(b) corresponds to Fig.10(b), and Fig.12(c) corresponds to Fig.10(c).



(a) (b) (c)

Fig.12 Shape modified curves corresponding to Fig.10



(a) (b) (c)

Fig.13 Curvature and radius of curvature distribution corresponding to Fig.12

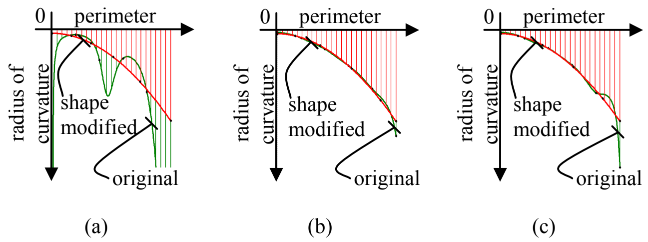


Fig.14 Radius of curvature distributions of original curves and those of shape modified curves
These correspond to Fig.13.

Curvature and radius of curvature distributions of these three shape modified curves are shown in Fig.13.

Radius of curvature distributions of these three shape modified curves are shown with radius of curvature distributions of their original curves in Fig.14.

It is visually clear that the shapes of the curves are modified significantly.

6 Conclusion

A method to generate a quintic B-spline curve which passes through the given points in sequence is described. In this case, the number of unknowns, which are the positions of the control points, are four more than the number of equations which express quintic B-spline curves. In other words, there are four more unknowns than there are equations.

Two methods have been developed to compensate for the difference between the number of unknowns and that of the equations. These are assuming that the curvatures at both ends of the curve are zero, and assigning four gradients to the given points situationally.

In addition to these methods, another method to generate a quintic B-spline curve which passes through the given points and which has the first derivatives at these given points has been developed.

In this case, a linear system will be underdetermined, determined or overdetermined depending on the number of given points with gradients.

For an underdetermined system, the linear system is solved under the constraint condition while setting auxiliary function by selecting one solution from an infinite number of exact solutions using Lagrange's method of indeterminate multipliers.

For an overdetermined system, the differences between the right and left sides of all the equations of this linear system are minimized.

A method to modify a quintic B-spline curve shape according to the specified radius of curvature distribution to realize an aesthetically pleasing freeform curve is described. The difference between the B-spline curve radius of curvature and the specified radius of curvature is minimized by introducing the least-squares method. A reverse computational technique is applied to solve this problem. This kind of study on the radius of curvature, or the curvature to realize a fair curve is called a constrained non-linear minimization problem.

Examples of curve generation are given.

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