Uniqueness of Positive Solutions for Degenerate Logistic Neumann Problems in a Half Space

WEI DONG
Hebei University of Engineering
Hebei Unversity of Engineering, Handan, Hebei056038
CHINA
wdongau@yahoo.com.cn

Tieguo Ji
Hebei University of Engineering
Hebei Unversity of Engineering, Handan, Hebei056038
CHINA
Rosemary1976@163.com

Abstract: In this paper, we consider the existence and uniqueness positive solutions of the following boundary Neumann problem in a half space

$$-\Delta u = a(x)u - b(x)f(u)\ x \in T, \frac{\partial u}{\partial n} = 0, \text{ on } \partial T,$$

where $T = \{x = (x_1, x_2, \cdots, x_N) : x_N > 0\}, (N \geq 2)$, $a(x)$ and $b(x)$ are continuous functions with $b(x)$ non-negative on R^N and n is outward pointing unit normal vector of ∂T, we show that under rather general conditions on $a(x)$ and $b(x)$ for large $|x|$ and $f(u)$ behaves like u^q, where constant $q > 1$, the above problems possesses a minimal positive solution and a maximal positive solution, respectively, Moreover, we establish a relationship between the above problem and the following problem

$$-\Delta u = a(x)u - b(x)f(u)\ x \in R^N,$$

We establish a comparison principal which our proof of the existence results rely essentially on and make use of a rather intuitive squeezing method to get the existence theorems. Furthermore, by analyzing the behavior of the positive solution for the problem in whole space, we show the boundary Neumann problem in half space has only one positive solution. Our results improve the previous works.

Keyword: Sub-super solution, Neumann problem, Comparison principle, degenerate logistic, positive solutions

1 Introduction

In this paper, we are concerned with positive solutions of the following boundary Neumann problem

$$\begin{cases} -\Delta u = a(x)u - b(x)f(u), \ x \in T \subset R^N \\ \dfrac{\partial u}{\partial n} = 0, \ \text{on} \partial T \end{cases} \quad (1)$$

where $T = \{x = (x_1, x_2, \cdots, x_N) : x_N > 0\}, (N \geq 2)$, q is a constant greater than 1, $a(x)$ and $b(x)$ are continuous functions with $b(x)$ non-negative on R^N and n is outward pointing unit normal vector of ∂T, Equations of this kind in bounded or unbounded region with different boundary values have attracted extensive study because of its interest to mathematical biology, Riemannian geometry and generalized reaction-diffusion and in non-Newtonian fluid theory. The existence of exact solution and the asymptotic and numerical solution of problem (1) for different nonlinearities have been attracted considerable interest in the last decades. We refer to [1,5,6,7,8,10,13,22]and the references therein for some of the previous research.

The Dirichlet problems with different types in the upper half space or rough boundary domains, under two measures on the boundary, have been thoroughly investigated (see [2,3,4,25,26,27,28]). In 2004, Du and Guo in [17] proved that any boundary positive solution of the following Dirichlet problem:

$$\begin{cases} -\Delta u = f(u), & x \in T \\ u = 0, & x \in \partial T \end{cases}$$

is unique and is a function of x_n only provide that f is locally quasi-monotone on $(0,\infty)$ and satisfies

(2): for some $a>0$,

$$f(s)>0 \text{ in } (0,a),\quad f(s)<0 \text{ in } (a,\infty),$$

(3): for some small $d>0$, there exists a constant $\delta>0$ such that

$$f(s)>\mathrm{d}\,s \quad \text{for all} \quad s\in(0,\delta)$$

We say that $f(s)$ is locally quasi-monotone on $(0,\infty)$ if for any bounded interval $[s_1,s_2]\subset[0,\infty)$, there exists a continuous increasing function $L(s)$ such that $f(s)+L(s)$ is non-decreasing in s for $s\in[s_1,s_2]$.

Clearly, this condition is less restrictive than requiring $f(s)$ to be locally Lipschitz continuous on $[0,\infty)$.

In 2005, for α is a positive constant (or ∞), Dong in [12] showed that the following problem

$$\begin{cases} -\Delta u=f(u), & x\in T\\ \quad u=\alpha, & x\in\partial T\end{cases}$$

has a unique positive solution if $f(s)$ is locally quasi-monotone on $(0,\infty)$ and satisfies (2).

In the present paper, we will consider the boundary Neumann problem in the upper half space for more general nonlinearity. We only consider the existence of positive solutions. By a positive solution to (1), we mean a function $u\in W^{1,2}\cap C(T)$ satisfying $u>0$ in T such that

$$\int_T Du\cdot D\psi dx=\int_T g(x,u)\psi dx,\ \forall\,\psi\in C_0^\infty(T\cup\partial T)$$

and

$$\frac{\partial u}{\partial n}=0, \text{ on } \partial T$$

where $g(x,u)=a(x)u-b(x)f(u)$.

Recently, when $b(x)$ is positive, called logistic case, it is shown in [15, 16] that problem (1) has a unique positive solution. In this paper, we are interested in the more challenge degenerate logistic case in unbounded domain, where $b(x)>0, b(x)\neq 0$, but the zero set of b is the closure of some suitable regular nonempty subdomain: Ω_0

$$\overline{\Omega}_0=\{x\in\overline{\Omega}: b(x)=0\ \}$$

and

$$\overline{\Omega}_0=R^N\setminus T$$

Through out this paper, we always assume that for some γ and ξ such that $\gamma>0$, there exist positive numbers α_1,α_2 and β_1,β_2 such that

$$\begin{aligned}\alpha_1&=\underline{\lim}_{|x|\to\infty}\frac{a(x)}{|x|^\gamma},\alpha_2=\overline{\lim}_{|x|\to\infty}\frac{a(x)}{|x|^\gamma}\\ \beta_1&=\underline{\lim}_{|x|\to\infty}\frac{b(x)}{|x|^\xi},\beta_2=\overline{\lim}_{|x|\to\infty}\frac{b(x)}{|x|^\xi}\end{aligned}\tag{4}$$

And $f(t)$ satisfies the conditions (5) and (6) listed below.

(5): $f(t)\geq 0$, $f(t)/t$ is increasing on $(0,\infty)$ and $\lim_{t\to 0} f(t)/t=0$;

(6): $\int_1^\infty F(t)^{-\frac{1}{2}}dt<\infty$, where $F(t)=\int_0^t f(s)ds$

It is easily shown that under these conditions, problem (1) has at least one (weak) positive solution. By standard regularity theory of elliptic equations, any $W_{loc}^{1,2}(R^N)$ solution of (1) belongs to $C^1(R^N)$.

Let us now describe our results in more details. In section 2, we establish a comparison principal which our proofs of the existence results rely essentially on. We make use of a rather intuitive squeezing method as follows to obtain the existence theorem as follows.

Let B_r be a ball on R^N with centered at origin with radius r, $\Omega_r=B_r\cap T$, $\Gamma_1=\partial B_r\cap T$ and $\Gamma_2=\partial T\cap\Omega_r$. Then for large $r>0$, the following problem:

$$\begin{cases} -\Delta u=a(x)u-b(x)f(u), & x\in\Omega_r\\ \qquad u=0, & x\in\Gamma\\ \qquad \dfrac{\partial u}{\partial n}=0, & x\in\Gamma_2\end{cases}$$

has a unique positive solution u_r. On the other hand, the mixed boundary problem

$$\begin{cases} -\Delta u=a(x)v-b(x)f(v), & x\in B_n\\ \qquad u=\infty, & x\in\Gamma_1\\ \qquad \dfrac{\partial v}{\partial n}=0, & x\in\Gamma_2\end{cases}$$

has a positive solution v_r. When r increases to infinity, u_r and v_r converges to a minimal positive solution and a maximal positive solution for (1), respectively, namely:

Theorem 1. If $\lambda_1(\Omega_0,\alpha)>0$, then, problem (1) possesses a minimal positive solution $\underline{u}$ and a maximal positive solution $\bar{u}$, respectively.

In order to obtain a complete understanding of problem (1), in section 3, we need to study the following problem:

$$-\Delta u = a(x)u - b(x)f(u),\ x \in R^N \qquad (7)$$

Under the assumptions on $a(x), b(x)$ and $f(t)$, furthermore, for some positive constants d_1, d_2 and $q > 1$, $f(t)$ satisfies

$$\begin{aligned} &\underline{\lim}_{t\to 0}\frac{f(t)}{t^q} \ge d_1 > 0 \\ &\overline{\lim}_{t\to\infty}\frac{f(t)}{t^q} \le d_2 < \infty \end{aligned}. \qquad (8)$$

We obtain the following asymptotic behavior of positive solutions for (7) as $|x| \to \infty$ first.

Theorem 2 . Suppose $u \in C^1(R^N)$ is a positive solution of (7). If (4) and (8) are satisfied, then for some positive constants c_1 and c_2 such that $0 < c_1 \le c_2 < \infty$, we have

$$\underline{\lim}_{|x|\to\infty}\frac{u^{q-1}(x)}{|x|^{\gamma-\tau}} \ge c_1 \qquad (9)$$

and

$$\overline{\lim}_{|x|\to\infty}\frac{u^{q-1}(x)}{|x|^{\gamma-\tau}} \le c_2 \qquad (10)$$

Next we combine the squeezing method in [18] with the iteration argument motivated by one attributed to Safonov (see also [14, 19]) to obtain the uniqueness result in whole space.

Theorem 3. Suppose $f(t)$ satisfies (8) and $\lambda_1(\Omega_0, \alpha) > 0$. Furthermore, if $f(u)$ satisfies:

$$\begin{cases} \text{when } \gamma > \tau, \lim\limits_{u\to\infty}\dfrac{f(u)}{u^q} = k_1 > 0 \\ \text{when } \gamma < \tau, \lim\limits_{u\to 0}\dfrac{f(u)}{u^q} = k_2 > 0 \\ \text{when } \gamma = \tau, f(u) = Cu^q, C > 0 \end{cases} \qquad (11)$$

Then problem (7) has a unique positive solution.

In section 4, we establish a relationship between the positive solutions of (1) and ones of (7), and utilizing the uniqueness result for problem (7), we obtain our main uniqueness result.

Theorem 4. Assume that $f(t)$ satisfies (8) and (11), moreover $\lambda_1(\Omega_0, \alpha) > 0$, then, problem (1) has a unique positive solution.

2 Existence of Positive Solutions of Problem (1)

In this section, we adapt the comparison principle in [18] and modify it, we obtain the following new comparison principle.

Lemma 5. (Comparison principle) Suppose that Ω is a bounded domain in R^N which $\partial\Omega$ splits into Γ_1 and Γ_2 . $\alpha(x)$ and $\beta(x)$ are continuous with $\beta(x) \ge 0$, $\beta(x) \ne 0$ on Ω and $\|\alpha\|_{L^\infty(\Omega)} < \infty$. Let $u_1, u_2 \in C^1(\Omega)$ be positive in Ω and satisfy (in the weak sense)

$$\Delta u_1 + \alpha(x)u_1 - \beta(x)f(u_1) \le 0 \le \Delta u_2 + \alpha(x)u_2 - \beta(x)f(u_2)$$

in Ω and

$$\overline{\lim}_{dist(x,\Gamma_1\to 0)}(u_2 - u_1) \le 0$$

$$\frac{\partial u_1}{\partial n} \ge \frac{\partial u_2}{\partial n},\ x \in \Gamma_2.$$

where $f(u)$ is a continuous function which for every $x \in \Omega$, $f(u)/u$ is strictly increasing for u in the range $\inf_\Omega\{u_1,u_2\} < u < \sup_\Omega\{u_1,u_2\}$. Then $u_2 \le u_1$ in Ω.

This Lemma can be easily derived from Lemma 2.1 in [18].

Lemma 6. Suppose that Ω is bounded domain in R^N and $\beta(x)$ are continuous with $\beta(x) \ge 0$. If $\lambda_1(\Omega_0, \alpha) > 0$ and f satisfies (5)-(6), then, the following problems

$$\begin{cases} -\Delta u = \alpha(x)u - \beta(x)f(u), & x \in \Omega \\ u = \infty, & x \in \partial\Omega \end{cases} \qquad (12)$$

has at least one positive solution.

Proof. we first consider the following problem

$$-\Delta u = \alpha(x)u - \left(\beta(x) + \frac{1}{k}\right)f(u),\quad \forall x \in \Omega, \qquad u|_{\partial\Omega} = k \qquad (13)$$

Since f satisfies (6), we can easily obtain $\lim_{t\to\infty}\frac{f(t)}{t} = \infty$. Hence there exists a large number $M_0 \ge k$, such that for all $M \ge M_0$, $\alpha(x)M - \left(\beta(x) + \frac{1}{k}\right)g(x,M) \le 0$. Thus M is a supersolution of (13). Obviously, $v \equiv 0$ is a subsolution of (13). A standard sub- and supersolution argument(see [10,21]) and Lemma 5 imply that problem (13) has a unique positive solution u_k and u_k is increasing with k , By standard regularity theory in [10,21], $u_k \in C^{1,\alpha}(K)$ for any compact $K \subset \Omega$, and some $\alpha \in (0,1)$. If we can also obtain an upper bound for the sequence $\{u_k\}$, then $\{u_k\}$ converges to a positive solution u of (12) in $C^1(\bar{\Omega})$.

Next we will look for the upper bound.

For any compact subset K of $\Omega \setminus \bar{D}_0$, there exists an open set Ω_1 such that $K \subset \Omega_1 \subset\subset \Omega \setminus \bar{D}_0$. Since $\beta(x) > 0, \quad \forall x \in \bar{\Omega}_1$, Theorem 1.1 in [17] implies the following boundary blow-up problem

$$-\Delta v = \max_{x\in\bar{\Omega}_1} \alpha(x)v - \inf_{x\in\bar{\Omega}_1} \beta(x) f(v), \forall x \in \Omega_1, v|_{\partial\Omega_1} = \infty$$

has a positive solution v. For every positive integer k,

$$\begin{aligned} -\Delta u_k &= \alpha(x)u_k - \beta(x)f(u_k) \\ &\le \max_{x\in\bar{\Omega}_1} \alpha(x)u_k - \inf_{x\in\bar{\Omega}_1} \beta(x) f(u_k), \quad \forall x \in \Omega_1, \end{aligned}$$

and $u_k|_{\partial\Omega_1} < \infty$. Thus by Lemma 5 we obtain $u_k \le v$. So $u_k \le M^*$ for some $M^* > 0$ and all $x \in K \subset \Omega \setminus \bar{D}_0$.

If we can also find an upper bound for u_k on a small neighborhood of D_0 then we can use the monotone method in [10,19] to see $u = \lim_{k\to\infty} u_k$ is a positive solution of (12).

Let N_η denote the $\eta - neighborhood$ of D_0 such that $\bar{N}_\eta \subset \Omega$. By the properties of the first eigenvalue(see [11,23]), $\lambda_1(N_\eta, \alpha) > 0$) if η is sufficiently small. By what we have already proved, we can find a positive constant $\bar{M}$ such that $u_k \le \bar{M}$ for all $k \ge 1$ and $x \in \partial N_{\frac{\eta}{2}}$. Let ϕ be a positive eigenfunction corresponding to $\lambda_1(N_\eta, \alpha)$, we can find a large positive constant L such that $L\phi > \bar{M}$ for $x \in \partial N_{\frac{\eta}{2}}$. Thus

$$-\Delta(L\phi) - \alpha(x)(L\phi) = \lambda_1(N_\eta, \alpha)(L\phi) > 0, \quad \forall x \in N_{\frac{\eta}{2}}$$

and

$$-\Delta u_k - \alpha(x)u_k = -\beta(x)f(u_k) \le 0, \quad \forall x \in N_{\frac{\eta}{2}}.$$

By Corollary 2.4 in [17] and $L\phi \ge u_k$ for all $k \ge 1, \forall x \in \partial N_{\frac{\eta}{2}}$, we obtain $u_k \le L\phi$ for all $x \in \bar{N}_{\frac{\eta}{2}}$. So we find an upper bound for the sequence $\{u_k\}$ on any compact subset $K \subset \Omega$. Thus (12) has at least one positive solution.

Next we will show the existence result Theorem 1.

Let B_r be a ball on R^N with centered at origin with radius r, $\Omega_r = B_r \cap T$, $\Gamma_1 = \partial B_r \cap T$ and $\Gamma_2 = \partial T \cap \Omega_r$.

Now we first consider the following problem:

$$\begin{cases} -\Delta u = \alpha(x)u - \beta(x)f(u), x \in \Omega \\ \qquad u = 0, x \in \partial\Omega \end{cases} \tag{14}$$

Since condition (4) holds, by the properties of the first eigenvalue (see [11,21]), there exists a large $r_0 > 0$ such that for all $r \ge r_0$, $\bar{\Omega}_0 \subset \Omega_{r_0}$ $\lambda(\Omega_r, a) < 0$. Let ϕ be a positive eigenfunction corresponding to $\lambda_1(\Omega, \alpha)$. Since $\lim_{t\to 0} f(t)/t = 0$, then for all small positive constant ε, it easily checked that $\varepsilon\phi$ is a subsolution of problem (14) with $\Omega = \Omega_r$. By Lemma 6, the problem (12) with $\Omega = \Omega_r$ has a unique positive solution u_0. Obviously it is the supersolution of (14). A standard sub-and super solution argument (see [10,19]) and Lemma 5 imply that problem (14) has a unique positive solution v. Next we consider the following problem:

$$\begin{cases} -\Delta u = a(x)u - b(x)f(u), \quad x \in \Omega_r \\ \qquad u = 0, \quad x \in \Gamma_1 \\ \qquad \dfrac{\partial u}{\partial n} = 0, \quad x \in \Gamma_2 \end{cases} \tag{15}$$

It is very clear v and u is the sub- and supersolution of above problem. By standard sub-supersolution method for elliptic equation, the problem (15) has at least one positive solution u_r in the order interval $[v, u]$. It follows from Lemma 5 that it has a unique positive solution.

By standard sub-supersolution method for elliptic equation, the problem (15) has at least one positive solution u_r in the order interval $[v, u]$. It follows from Lemma 5 that it has a unique positive solution. Let us choose an increasing sequence of positive real numbers $r_n > r_0$ and $r_n \to \infty$ as $n \to \infty$. By the discussion above, problem (15) with $\Omega_r = \Omega_{r_n}$ has a unique positive solution u_n .It follows from Lemma 5 that $u_n \le u_{n+1}$. If we can find an upper bound for $u_n(x)$ on any fixed Ω_R, then by a standard regularity argument, $u(x) = \lim_{n\to\infty} u_n(x)$ is well-defined in T and it would be a positive solution of problem (1). To find such an upper bound, we consider the problem

$$-\Delta v=\alpha(x)v-\beta(x)f(v),\ x\in\Omega_R,\ \ v|_{\partial\Omega_R}=\infty .$$

By Lemma 6, the above problem has a positive solution $v(x)$. Then clearly by the comparison principle Lemma 5, we obtain

$$u_n(x)\le v(x),\quad \forall x\in\Omega_R .$$

for all large n such that $r_n>R$. This is the bound we are looking for, and hence the existence of a solution for (1) is proved.

From $u_n\le u_{n+1}$ we find

$$\underline{u}(x)\ge u_n(x)>0$$

for each n, and hence $\underline{u}$ is a positive solution of (1). For an arbitrary positive solution u of (1), we can see that u satisfies

$$-\Delta u=\alpha(x)u-\beta(x)f(u),\qquad u|_{\Gamma_1}>0 .$$

By Lemma 5 $u\ge u_n$ on Ω_{r_r} for each n, and hence

$$u\ge \underline{u}=\lim_{n\to\infty}u_n$$

So $\underline{u}$ is the minimal positive solution of (1).

Next we will show the existence of a maximal positive solution of (1). To this end, we choose an increasing sequence of real number r_n such that $r_n\to\infty$ as $n\to\infty$ and denote $B_n=\Omega_{r_n}$ .We consider the following mixed boundary problem

$$\begin{cases}-\Delta u=a(x)u-b(x)f(u), & x\in B_n\\ u=\infty, & x\in\Gamma_1\\ \dfrac{\partial u}{\partial n}=0, & x\in\Gamma_2\end{cases}\tag{16}$$

Obviously $u=0$ is a subsolution of problem (16). By Lemma 6, the following equation

$$\begin{cases}-\Delta v=\alpha(x)v-\beta(x)f(v), & x\in B_n\\ v=\infty, & x\in\partial B_n\end{cases}$$

has a positive solution and we denote it as v_n. It is easy to show $\dfrac{\partial v_n}{\partial n}\ge 0$ and v_n is a supersolution of (16). Thus problem (16) has at least one positive solution u_n.

Applying Lemma 5, we see

$$u_n\ge u_{n+1}>\underline{u},x\in B_n\quad\text{for all } n.$$

So $\bar{u}=\lim_{n\to\infty}u_n$ is well-defined on T . Furthermore, by standard regularity considerations, we know $\bar{u}$ satisfies (1) on T and $\bar{u}\ge\underline{u}$, so $\bar{u}$ is a positive solution of (1).

Clearly any positive solution u of (1) satisfies, for each n,

$$-\Delta u=a(x)u-b(x)f(u),$$

$$u|_{\Gamma 1}<\infty,\ \frac{\partial u}{\partial n}=0.$$

It follows from Lemma 5 that we see

$$u_n\ge u\quad\text{on}\quad B_n\ \text{for all}\ n,$$

and hence

$$\bar{u}=\lim_{n\to\infty}\omega_n\ge u$$

The proof is now finished.

3 The Whole Space Problem

In this section, we will prove the asymptotic behavior of the positive solution of problem (7), and then make use of this result to prove the uniqueness result in Theorem 3.

Before we start to prove our uniqueness result Theorem 3, we need the following existence Lemma.

Lemma 7. If $\lambda_1(\Omega_0,\alpha)>0$ and conditions (4)-(6) are satisfied, then problem (7) possesses a minimal positive solution and a maximal positive solution $\bar{u}$.

Proof. By condition (4), there exists a large $r>0$, such that $\bar{\Omega}_0\subset B_r$, $\lambda_1(B_r,a)<0$, and it follows from the proof of Theorem 1 that the following problem

$$\begin{cases}-\Delta u=\alpha(x)u-\beta(x)f(u), & x\in B_r\\ u=0, & x\in\partial B_r\end{cases}$$

has a unique positive solution u_r.

Let us choose an increasing sequence of positive real numbers r_n with $r_1>r$ and $r_n\to\infty$ as $n\to\infty$ .By the properties of the first eigenvalue in [11, 23], and by the proof of Theorem 1, the above problem with $r=r_n$ has a unique positive solution u_n. By the comparison principle Lemma 5, we deduce

$$u_n\le u_{n+1}.$$

If we can find an upper bound for u_n on any mixed B_R , then by a standard regularity argument, $\underline{u}=\lim_{n\to\infty}u_n$ is well-defined in R^N and it would be a positive solution of (7).

To find such an upper bound, we consider the problem

$$\begin{cases}-\Delta u=\alpha(x)u-\beta(x)f(u), & x\in B_R\\ u=\infty, & x\in\partial B_R\end{cases}$$

Lemma 6 implies that this problem has a positive solution v. Then by Lemma 5,

$$u_n(x)\le v(x),\ \forall x\in B_R$$

for all large n such that $r_n > R$. This is the bound we are looking for, and hence the existence of a solution for (7) is proved.

From $u_n \le u_{n+1}$ we find $\underline{u} \ge u_n(x) > 0$ for each n, and hence $\underline{u}$ is a positive solution of (7). For an arbitrary positive solution u of (7), we can see that u satisfies

$$-\Delta u = \alpha(x)u - \beta(x)f(u),\ \ u|_{\partial B_{r_n}} > 0.$$

So $\underline{u}$ is the minimal positive solution of (7).

Next we will show the existence of a maximal positive solution of (7). To this end, we choose an increasing sequence of real number r_n such that $r_n \to \infty$ as $n \to \infty$ and denote $B_n = B_{r_n}$. We consider the boundary blow-up problem:

$$-\Delta\omega = \alpha(x)\omega - \beta(x)f(\omega) \ \text{ in } B_n \ \ u|_{\partial B_n} = \infty \quad (17)$$

It follows from Lemma 6 that (17) has a positive solution and we denote it as ω_n. Applying Lemma 5, we see

$$\omega_n \ge \omega_{n+1} \ge \underline{u},\ \ x \in B_n \text{ for all } n.$$

Thus

$$\bar{u} = \lim_{n\to\infty} \omega_n$$

is well-defined on R^N. Furthermore, by standard regularity considerations, we know $\bar{u}$ satisfies (7) on R^N and $\bar{u} \ge \underline{u}$, so $\bar{u}$ is a positive solution of (7).

Clearly any positive solution u of (7) satisfies, for each n,

$$-\Delta u = \alpha(x)u - \beta(x)f(u),\ \ u|_{\partial B_{r_n}} < \infty.$$

It follows from Lemma 5 that we see

$$\omega_n \ge u \ \text{ on } B_n \text{ for all } n.$$

And hence

$$\bar{u} = \lim_{n\to\infty} \omega_n > u.$$

This finishes the proof.

Next we will show the asymptotic behavior of positive solutions for (7) as $|x| \to \infty$ and use the result to prove Theorem 3.

Proof of Theorem 2. Because $f(u)$ satisfies (8), then there exist two positive constant $0 < h_1 \le h_2$ such that

$$h_1 t^q \le f(t) \le h_2 t^q \quad (18)$$

By Proposition 3.2 in [10], the following problem:

$$-\Delta v = a(x)v - h_2 b(x)v^q,\ x \in R^N \quad (19)$$

possesses a minimal positive v.

By the constructions of the minimal positive solutions v, on any fixed B_R, we have

$$-\Delta v > a(x)v - b(x)f(v)$$

By Lemma 5, we can easily obtain $v \le \underline{u}$, where $\underline{u}$ is the minimal positive solution of (7). By Lemma 3.1 in [11], we have

$$\underline{\lim}_{|x|\to\infty} \frac{v^{q-1}(x)}{|x|^{\gamma-\tau}} \ge \frac{\alpha_1}{h_2\beta_2}$$

Thus there exists $c_1 > 0$ such that

$$\underline{\lim}_{|x|\to\infty} \frac{\underline{u}^{q-1}(x)}{|x|^{\gamma-\tau}} \ge c_1 \quad (20)$$

By the same method as above, the following problem:

$$-\Delta v = a(x)\omega - h_1 b(x)\omega^q,\ x \in R^N \quad (21)$$

has a maximal positive solution ω such that $\bar{u} \le \omega$, where $\bar{u}$ is the maximal positive solution of (7). By the Lemma 3.1 in [14], we have

$$\overline{\lim}_{|x|\to\infty} \frac{\omega^{q-1}(x)}{|x|^{\gamma-\tau}} \le \frac{\alpha_2}{h_1\beta_1} \quad (22)$$

Thus

$$\overline{\lim}_{|x|\to\infty} \frac{\bar{u}^{q-1}(x)}{|x|^{\gamma-\tau}} \le \frac{\alpha_2}{h_1\beta_1} \quad (23)$$

It follows from (20) and (23) that the Theorem 3 is complete.

What remains is to show the uniqueness result for problem (7). The following technical lemma is the core of our iteration argument to be used in the uniqueness proof.

Lemma 8. Suppose that (4), (5), (6), (8) and (11) hold, u_1, u_2 are positive solutions of (7). Then there exists $R > 1$ large so that, if $x_0 \in R^N$ satisfies, for some $k_* \ge k > 1$,

$$|x_0| > R,\ u_2(x_0) > k_* u_1(x_0),$$

thus we can find $y_0 \in R^N$, and positive constants $c_0 = c_0(R,k)$ and $r_0 = r_0(R,k)$ independent of x_0 and k_* such that

$$\begin{aligned} &|y_0 - x_0| = r_0 |x_0|^{-\gamma/2} \\ &u_2(y_0) > (1+c_0)k_* u_1(y_0) \end{aligned} \quad (24)$$

Proof. By (4), (9) and (10), for all large $R_1 > 1$ and $|x| > R_1$, we have

$$\begin{aligned} &(1/2)\alpha_1 |x|^\gamma < a(x) < 2\alpha_2 |x|^\gamma \\ &(1/2)\beta_1 |x|^\gamma < a(x) < 2\beta_2 |x|^\gamma \end{aligned} \quad (25)$$

and, for $i = 1,2$,

$$\mu_1 |x|^{(\gamma-\tau)/(q-1)} < u_i(x) < \mu_2 |x|^{(\gamma-\tau)/(q-1)} \quad (26)$$

where

$$\mu_1 = (1/2)\left(\frac{\alpha_1}{h_2\beta_2}\right)^{1/(q-1)},\ \mu_2 = 2\left(\frac{\alpha_2}{h_1\beta_1}\right)^{1/(q-1)}$$

We now fixed $R_1 > 1$ large enough so that $R^{-1-(\gamma/2)} < 1/2$ and (25), (26) hold for all x satisfying $|x| > R_1/2$. Then we define

$$\Omega_0 = \{x \in R^N : u_2(x) > k_* u_1(x)\} \bigcap B_r(x_0),$$

where

$$r = r_0 |x_0|^{-\gamma/2}, B_r(x_0) = \{x \in R^N : |x - x_0| < r\},$$

and $r_0 \in (0,1)$ is to be determined below.

Clearly $x \in \Omega_0$ implies

$$|x_0| - r \le |x| \le |x_0| + r,$$

which in turn implies, due to $|x_0| > R_1$, $|x| > R$ and our choice of R_1,

$$(1/2)|x_0| < |x| < (3/2)|x_0| \tag{27}$$

Using (25)-(27) and the assumption that $u_2 - k_* u_1 > 0$ in Ω_0, we now consider $\Delta(u_2 - k_* u_1)$ in Ω_0 in three cases.

Case 1. $\gamma > \tau$ **.**

By Theorem 2, if $\gamma > \tau$, then $u(x) \to \infty$ as $|x| \to \infty$. Then, it follows from (11) that

$$\lim_{|x|\to\infty} \frac{f(u(x))}{u(x)^q} = k_1$$

So for some $\varepsilon > 0$ small enough, there exists a large $R_2 > R_1$ such that if $|x| > R_2$, we have

$$(k_1 - \varepsilon)u^q \le f(u) \le (k_1 + \varepsilon)u^q$$

and

$$(k_1 - \varepsilon)k_*^{\,q-1} - (k_1 + \varepsilon) > 0$$

Then we deduce, for $x \in \Omega_0$

$$\begin{aligned}
&\Delta(u_2(x) - k_* u_1(x)) \\
&= -a(x)(u_2(x) - k_* u_1(x)) + b(x)(f(u_2) - k_* f(u_1)) \\
&\ge -a(x)(u_2(x) - k_* u_1(x)) + b(x)((k_1 - \varepsilon)u_2{}_2^q \\
&\quad - k_*(k_1 + \varepsilon)u_1^q) \\
&\ge -a(x)(u_2(x) - k_* u_1(x)) + b(x)((k_1 - \varepsilon)u_2{}_1^q \\
&\quad - k_*(k_1 + \varepsilon)u_1^q) \\
&\ge -2\alpha_2 |x|^\gamma (u_2(x) - k_* u_1(x)) \\
&\qquad + b(x)k_* u_1^q((k_1 - \varepsilon)k_*^{q-1} - (k_1 + \varepsilon)) \\
&\ge -M|x_*|^\gamma (u_2(x) - k_* u_1(x)) + \frac{1}{2}k_* |x|^\tau A_1^q \beta_1 \\
&\quad |x|^{\frac{(\gamma-\tau)^q}{q-1}} ((k_1 - \varepsilon)k_*^{q-1} - (k_1 + \varepsilon)) \\
&\ge -M|x_*|^\gamma (u_2(x) - k_* u_1(x)) + m_1 k_* |x_*|^\sigma
\end{aligned}$$

where

$$M = 2\alpha_2 \max((\frac{1}{2})^\gamma, (\frac{3}{2})^\gamma), \sigma = \tau + \frac{(\gamma - \tau)q}{q-1}$$

$$m_1 = \frac{1}{2}A_1^q \beta_1((k_1 - \varepsilon)k_*^{q-1} - (k_1 + \varepsilon))\min((\frac{1}{2})^\sigma, (\frac{3}{2})^\sigma)$$

Case 2. $\gamma < \tau$.

By Theorem 2, if $\gamma < \tau$, then $u(x) \to \infty$ as $|x| \to \infty$. Then, it follows from (11) that $\lim_{|x|\to\infty} \frac{f(u(x))}{u(x)^q} = k_2$.

So for some $\varepsilon > 0$ small enough, there exists a large $R_3 > R_1$ such that if $|x| > R_3$, we have

$$(k_2 - \varepsilon)u^q \le f(u) \le (k_2 + \varepsilon)u^q$$

and

$$(k_2 - \varepsilon)k_*^{\,q-1} - (k_2 + \varepsilon) > 0.$$

Then we deduce, for $x \in \Omega_0$

$$\begin{aligned}
&\Delta(u_2(x) - k_* u_1(x)) \\
&= -a(x)(u_2(x) - k_* u_1(x)) + b(x)(f(u_2) - k_* f(u_1)) \\
&\ge -a(x)(u_2(x) - k_* u_1(x)) + b(x)((k_2 - \varepsilon)u_2{}_2^q \\
&\quad - k_*(k_2 + \varepsilon)u_1^q) \\
&\ge -a(x)(u_2(x) - k_* u_1(x)) + b(x)((k_2 - \varepsilon)u_2{}_1^q \\
&\quad - k_*(k_2 + \varepsilon)u_1^q) \\
&\ge -2\alpha_2 |x|^\gamma (u_2(x) - k_* u_1(x)) + b(x)k_* u_1^q \\
&\quad ((k_2 - \varepsilon)k_*^{q-1} - (k_2 + \varepsilon)) \\
&\ge -M|x_*|^\gamma (u_2(x) - k_* u_1(x)) + \frac{1}{2}k_* |x|^\tau \\
&\quad A_1^q \beta_1 |x|^{\frac{(\gamma-\tau)^q}{q-1}} ((k_2 - \varepsilon)k_*^{q-1} - (k_2 + \varepsilon)) \\
&\ge -M|x_*|^\gamma (u_2(x) - k_* u_1(x)) + m_2 k_* |x_*|^\sigma
\end{aligned}$$

where

$$M = 2\alpha_2 \max((\frac{1}{2})^\gamma, (\frac{3}{2})^\gamma), \sigma = \tau + \frac{(\gamma - \tau)q}{q-1}$$

$$m_2 = \frac{1}{2}A_1^q \beta_1((k_2 - \varepsilon)k_*^{q-1} - (k_2 + \varepsilon))\min((\frac{1}{2})^\sigma, (\frac{3}{2})^\sigma)$$

Case3. $\gamma = \tau$.

It follows from (11) that there exists a large $R_4 > R_1$ such that $f(u) = Cu^q$. Then we deduce, for $x \in \Omega_0$

$$\begin{aligned}
&\Delta(u_2(x) - k_* u_1(x)) \\
&= -a(x)(u_2(x) - k_* u_1(x)) + b(x)(f(u_2) - k_* f(u_1))
\end{aligned}$$

$\geq -a(x)(u_2(x) - k_* u_1(x)) + b(x)(Cu_{2\,2}^{\,q} - Cu_1^q)$

$\geq -a(x)(u_2(x) - k_* u_1(x)) + b(x)C(k_*^q u_{2\,1}^{\,q} - k_* u_1^q)$

$\geq -2\alpha_2 |x|^\gamma (u_2(x) - k_* u_1(x)) + b(x)C(k_*^q u_1^q - k_* u_1^q)$

$\geq -M|x_*|^\gamma (u_2(x) - k_* u_1(x))$

$+ \frac{1}{2}|x|^\tau A_1^q \beta_1 |x|^{\frac{(\lambda-\tau)q}{q-1}} C(k_*^q - k_*)$

$\geq -M|x_*|^\gamma (u_2(x) - k_* u_1(x)) + m_2 k_* |x|^\sigma$

where

$$M = 2\alpha_2 \max((\frac{1}{2})^\gamma, (\frac{3}{2})^\gamma), \sigma = \tau + \frac{(\gamma-\tau)q}{q-1}$$

$$m_3 = \frac{1}{2}\beta_1 A_1^q C(k_*^{q-1} - k_*) \min((\frac{1}{2})^\sigma, (\frac{3}{2})^\sigma)$$

Overall, for

$R > \max\{R_1, R_2, R_3, R_4\}, m = \min\{m_1, m_2, m_3\} > 0$,

we have

$\Delta(u_2(x) - k_* u_1(x)) \geq -M|x_*|^\gamma (u_2(x) - k_* u_1(x))$

$+ mk_* |x_*|^\sigma$

With these preparations, we now define

$$\omega(x) = (2N)^{-1} mk_* |x_0|^\delta (r^2 - |x - x_0|^2)$$

Clearly $\omega(x) > 0$ in $B_r(x_0)$

and

$$\Delta\omega = -mk_* |x_0|^{\delta^2}$$

It follows that, for $x \in \Omega_0$

$$\Delta(u_2 - k_* u_1 \geq -M|x_0|^\gamma (u_2 - k_* u_1) \geq -M|x_0|^\gamma (u_2 - k_* u_1 + \omega) \qquad (28)$$

If we denote by $\lambda_1(\Omega)$ the first eigenvalue of $-\Delta$ over Ω under homogeneous Dirichlet boundary conditions, we have

$$\lambda_1(\Omega_0) \leq (B_r(x_0)) = r^2 \lambda_1(B_1(x_0))$$

Therefore, by the definition of r_0, we obtain

$$\lambda_1(\Omega_0) \geq r_0^{-2} |X_0|^\gamma \lambda_1,$$

where $\lambda_1 = \lambda_1(B_1(x_0))$ is independent of x_0. we now choose $r_0 \in (0,1)$ small enough so that

$$r_0^{-2}\lambda_1 > M$$

And hence

$$\lambda_1(\Omega_0) \geq M|x_0|^\gamma$$

Then by the maximum principle (see [5]), due to (28),

$u_2(x_0) - k_* u_1(x_0) + \omega(x_0) \leq \max_{\partial\Omega_0}(u_2 - k_* u_1 + \omega)$

We observe that the maximum of $u_2 - k_* u_1 + \omega$ over $\partial\Omega_0$ has to be achieved by some $y_0 \in \partial B_r(x_0)$ since any $y_0 \in \partial\Omega_0 \setminus \partial B_r(x_0)$ satisfies, by the definition of Ω_0, $u_2(y) = k_* u_1(y)$ and hence

$u_2(y) - k_* u_1(y) + \omega(y) = \omega(y) \leq \omega(x_0)$

$\geq u_2(x_0) - k_* u_1(x_0) + \omega(x_0)$

Thus we can find $y_0 \in \partial\Omega_0$ satisfying $|y_0 - x_0| = r$ (hence $\omega(y_0) = 0$) such that

$u_2(y_0) - k_* u_1(y_0) = u_2(y_0) - k_* u_1(y_0) + \omega(y_0)$

$\geq u_2(x_0) - k_* u_1(x_0) + \omega(x_0)$

$> \omega(x_0) = (2N)^{-1} mk_* |x_0|^\delta r^2$

$= (2N)^{-1} mk_* r_0^2 |x_0|^{(\gamma-\tau)/(q-1)}$

$\geq dk_* |y_0|^{(\gamma-\tau)/(q-1)}$

where

$d = (2N)^{-1} mr_0^2$

$\bullet \min\{(1/2)^{-(\gamma-\tau)/(q-1)}, (2/3)^{-(\gamma-\tau)/(q-1)}) > 0\}$

and we have used (27). Making use of (26), we finally deduce

$$u_2(y_0) - k_* u_1(y_0) > dk_* |y_0|^{(\gamma-\tau)/(q-1)} \geq c_1 \mu_2^{-1} k_* u_1(y_0)$$

Therefore we can take $c_0 = d\mu_2^{-1}$ and the proof is complete.

Proof of Theorem 3. By Lemma 7 above and Theorem 2, under conditions (4) and (8), problem (1) possesses a minimal positive solution u_1 and a maximal positive solution u_2 and any positive solution of (1) satisfies (9) and (10).

Let $k_1 = \overline{\lim}_{|x|\to\infty} \dfrac{u_2}{u_1}$.

By (4) and (5) we know that $k_1 \geq 1$ is finite. If $k_1 = 1$, then for any $\varepsilon > 0$ there exists $R_\tau > 0$ such that for all x satisfying $|x > R_\tau|$

$$u_2(x) \leq (1+\varepsilon)u_1(x)$$

Since $(1+\varepsilon)u_1$ is a supersolution of (7), we apply Lemma 5 over $\Omega = B_r(0)$, $R > R_\tau$, and deduce

$$u_2(x) \leq (1+\varepsilon)u_1(x) \qquad x \in R^N$$

Letting $\varepsilon \to 0$ we obtain $u_1 \equiv u_2$. This complete the proof.

Next we will prove the result is true when $k_1 > 1$. Therefore there exists a constant $k \in (1, k_1)$ and a sequence $\{x_n\}$ such that

$$|x_n| \to \infty, \qquad u_2(x)/u_1(x) > k, \qquad n = 1,2\cdots$$

We are now ready to apply Lemma 8. Let R, r_0 and $c_0 = c_0(R,k)$ be determined by Lemma 8. We recall that R satisfies $R^{-1-(\gamma/2)} < 1/2$. We first find an integer $j > 1$ such that

$$(1+c_0)^j k > \sup_{|x|>R} \frac{u_2}{u_1}$$

Since $|x_n| \to \infty$, we can then find n_0 large enough such that

$$|x_{n_0}|(1/2)^j k > R$$

Taking $x_0 = x_{n_0}$ and $k_* = k$ in Lemma 8, we can find $y_0 = y_1$ such that

$$|y_1 - x_0| = r_0 |x_0|^{-\gamma/2}, \quad u_2(y_1) > (1+c_0) u_1(y_1)$$

Clearly

$$|y_1| \geq |x_0| - r_0 |x_0|^{-\gamma/2} \geq |x_{n_0}|(1 - R^{-1-(\gamma/2)})$$
$$> |x_{n_0}|(1/2) > R$$

We now take $x_0 = y_1$ and $k_* = (1+c_0)k$ in Lemma 8, and we can find y_2 such that

$$|y_2 - y_1| = r_0 |y_1|^{-\gamma/2}, \quad u_2(y_2) > (1+c_0)^2 k u_1(y_2)$$

Let us note that

$$|y_2| \geq |y_1|(1/2) \geq |x_{n_0}|(1/2)^2 > R$$

We can repeat the above process until we obtain y_j which satisfies

$$u_2(y_j) > (1+c_0)^j k u_1(y_j), \quad |y_j| \geq |x_{n_0}|(1/2)^j > R$$

Therefore

$$\frac{u_2(y_j)}{u_1(y_j)} \geq (1+c_0)^j k > \sup_{|x|>R} \frac{u_2}{u_1}$$

This contradiction completes our proof.

Remark. If $a(x) = a(|x|)$ and $b(x) = b(|x|)$, then the unique positive solution of (7) must be radially symmetric solution, we can use the methods in [7,8,22] to obtain the analytic solution.

4 Proof of the Main Theorem

In this section, we will span the positive solution of problem (1) to whole space, and use the results in section 3 to prove the uniqueness Theorem 4.

To start, we should prove the following lemma.

Lemma 9. Assume u_1 to be an arbitrary positive solution of problem (1), letting

$$u = \begin{cases} u_1, x \in T \\ u_2, x \in R^N \setminus T \end{cases}$$

where $u_2 = u_1(x_1, x_2, \cdots, x_N), \quad x \in R^N \setminus T$,then u is the positive solution of the following problem.

$$-\Delta u = a(x)u - b(x)\int(u), x \in R^N.$$

Proof. For any $R > 0$, we denote

$$\Gamma = B_R \cap \partial T, \ \Omega_1 = B_R \cap T, \ \Omega_2 = B_R \cap \Omega_1$$

By a simple computation, we can obtain that u_2 is a positive solution of

$$-\Delta u = a(x)u - b(x)f(u), x \in \Omega_2, \frac{\partial u}{\partial n} = 0.$$

Next we will show that

$$u = \begin{cases} u_1 \mid \Omega_1 & x \in \Omega_1 \\ u_2 \mid \Omega_2 & x \in \Omega_2 \end{cases}$$

is a positive solution of the following equation

$$-\Delta u = a(x)u - b(x)f(u), \quad x \in B_{R.} \tag{29}$$

For $\forall \varphi \in C_c^\infty(B_R)$, since

$$\begin{aligned}
&(-\Delta u, \varphi)_{L^2(B_R)} \\
&= \int_{B^R} -\Delta u.\varphi dx \\
&= \int_{B^R} DuD\varphi dx - \int_{\partial B_R} \frac{\partial u}{\partial v} \varphi dx \\
&= \int_{\Omega_1} DuD\varphi + \int_{\Omega_2} DuD\varphi dx \\
&= \int_{\Omega_1} -\Delta u_1 \varphi dx + \int_{\partial\Omega_1} \frac{\partial u_1}{\partial v} \varphi dx + \int_{\Omega_2} -\Delta u_2 \varphi dx \\
&+ \int_{\partial\Omega_2} \frac{\partial u_2}{\partial v} \varphi dx \\
&= \int_{\Omega_1} [a(x)u_1 - b(x)f(u_1)]\varphi dx \\
&+ \int_{\Omega_2} [a(x)u_2 - b(x)f(u_2)]\varphi dx \\
&= \int_{B_R} [a(x)u - b(x)f(u)]\varphi dx \\
&= (a(x)u - b(x)f(u), \varphi)_{L^2(B_R)}
\end{aligned}$$

Hence u is a positive solution of problem (29). It follows from the arbitrary of R that

$$u = \begin{cases} u_1 & x \in T \\ u_2 & x \in R^N \setminus T \end{cases}$$

is a positive solution of

$$-\Delta u = a(x)u - b(x)f(u), x \in R^N.$$

The proof is complete.

Now we are ready to complete the proof of Theorem 4.

Proof of Theorem 4. Let $u_1(x)$ and $v_1(x)$ be two arbitrary positive solutions of (1). By Lemma 9, letting

$$u = \begin{cases} u_1, x \in T \\ u_2, x \in R^N \setminus T \end{cases}$$

and

$$v = \begin{cases} v_1, x \in T \\ v_2, x \in R^N \setminus T \end{cases}$$

We know that $u(x)$ and $v(x)$ corresponding to u_1 and v_1, respectively, is the positive solution of

$$-\Delta u = a(x)u - b(x)u^q \ \ x \in R^N.$$

By Theorem 3 above, the problem in whole space has only one positive solution. It follows that

$$u(x) = v(x), x \in R^N$$

Thus $\quad u_1 = v_1, x \in T$

This completes our proof.

5 Conclusion

In this paper, under less restricted conditions on coefficients $a(x)$ and $b(x)$, we use the same method in [15, 16] to handle with more complicated degenerate logistic cases. If the volume of the set D={ $x : x \in R^N$, $b(x) = 0$ } is small enough we obtain existence and uniqueness theorem for a class of semilinear equations with Neumann boundary value in unbounded domain in R^N. It improves the previous result.

Acknowledgement: Supported by Chinese NSF (10971046), Chinese NSF of Hebei Province (A20060 00 605) and SRC for ROSC, SEM.

References:

[1] G. A. Afrouzi, K. J. Brown, On Adiusive Logistic Equation, J. Math. Anal. Appl., Vol. 225, 1998, pp. 326- 339.

[2] S. B. Angenent. Uniqueness of the solutions of a semilinear boundary value problem, Math. Ann. 272 (1985), 129-138.

[3] H. Beresticky, L. Caffarelli, L. Nirenberg, Monotonenicity for elliptic equations in unbounded domain, Commun. Pure Appl. Math. 50 (1997), 1089-1112.

[4] H. Beresticky, L. Caffarelli, L. Nirenberg, Further qualitative properties for elliptic equations in unbounded domains. Annli. Scuola Norm. Sup. Pisa 25 (1997), 69-94.

[5] H. Beresticky, L. Nirenberg, S. R. S. Varadhan, The Principal Eigenvalue and Maximum Pri- nciple for Second Order Elliptic Operators in General Domains, Comm. Pure Appl. Math., 199 4, pp. 47-92.

[6] B. Bianchini,M. Rigoli, Non Existence and Uniq ueness of Positive Solutions of Yamabe Type Equations on Nonpositively Curved Mainfolds, Tran. Amer.Math.soc., Vol.349,1997, pp. 4735-4774.

[7] G. Bognar, E. ROzgonyi, The Local Analytic Solution to Some Nonlinear Diusion-reaction Problem,WSEAS Transactions on Mathematics, Vol.7, No. 6, 2008, pp.382-395.

[8] G. Bognar, E. ROzgonyi, The Power Series Solutions of Some Nonlinear Initial Value Problems, WSEAS Transactions on Mathematics, Vol.6, No.5, 2006, pp.627-635.

[9] F-C Cirstea, V. Radulescu, Existence and Uniqu- eness of Blow-up Solutions for a Class of Logis- tic Equations, Commun. Contemp. Math., Vol.4, No. 3, 2002, pp.559-586.

[10] J. Diaz, Nonlinear Partial Dierential Equation and Free Boundary Problems, Vol.1, Elliptic Equations, Pitman Research Notesin Math., Vol.106, Boston,1985.

[11] W. Dong, Positive Solutions for Logistic Type Quasilinear Elliptic Equations on R^N J. Math., Anal. Appl., Vol.290, 2004, pp.469-480.

[12] W. Dong, Symmetry for Boundary Blow-up Solutions of Elliptic Equations in a Half-space, Non.Anal., Vol.58, 2004, pp.159-173.

[13] W. Dong, Y. Du, Unbounded Principal Eigenfun- ctios and the Logistic Equation on R^N, Bull. Australia Math.Soc., Vol.67, 2003, pp.413-427.

[14] W. Dong, Pang, Uniqueness of Boundary Blow-up Solutions on Exterior Domainof R^N R ,J. Math. Anal. Appl., Vol.330,2007, pp. 654-664.

[15] W. Dong and N. Wang , Existence and uniqueness of positive solutions for Neumenn problem in a half space, WSEAS Transactions on Mathematics, issue 11, Vol. 7, 2008, 637-646.

[16] C. Dang and W. Dong, Uniqueness of Positive Solutions for Semilinear Neumann Problems in a Half Space, 13th WSEAS International Conference on APPLIED MATHEMATICS, DEC 15-17, 2008, 252-254.

[17] Y. Du, Z. M. Guo, Symmetry for Elliptic Equa- tions in a Half Space With Outstong Maximum Principle, Proc.Roy.Soc.Edinburgh, Vol.134, 2004, pp.259-270.

[18] Y. Du, Huang, Blow-up Solutions for a Class of Semilinear Elliptic and Parabolic Equations, Siam.J.Math.Anal., Vol.1, No.31,1999, pp.1-18.

[19] Y. Du, Lishan Liu, Remarks on the Uniqueness Problem of the Logistic Equation on the Entire Space, Bull. Australia Math. Soc., Vol.73, 2006, pp.129-137.

[20] Y. Du, L. Ma, Logistic Type Equations on R^N By a Squeezing Method in Volving Boundary Blow-up Solutions, J.London Math. Soc.,(2) Vol.64, 2001, pp.107-124.

[21] D. Gilbarg, N. S. Trudinger, Elliptic Partial Dierential Equations of Second Order, Second Edition,Springer,Berlin, 1983.

[22] P. Lima, L. Morgado, A Symptotic and Numeric- al Approximation of a Boundary Value Problem for a Quasi-linear Dierential Equation, Wseas Transactionson Mathematics, Vol.5, No.6, 2007, pp.639-647.

[23] M. Otani, T. Teshima, On the Rsteigen Value of Some Quasi-linear Elliptic Equations, Proc.Ja- pan Acad. Ser., A64, 1988, 8-10 MR89h: 35257.

[24] A. Ratto, M. Rigoli, L. Veron, Scalar Curvature and Conformal Deformations of Noncompact Riemannian Manifolds, Math.Z., Vol. 225, 1997, pp.395-426.

[25] C. Sweezy, Weighted Norm Inequalities for Caloric Functions on Classical Domains, WSEA- S,Trans-actionon Mathematics, Vol.3, No.3, 2004 , pp. 578- 583.

[26] C. Sweezy, Gradient Norm Inequalities for Weak Solutions to Parabolic Equations on Bounded Domains with and without Weights, WSEAS Transactions on System,Vol.4, No.12, 2005, pp. 2196-2203.

[27] D.Sweezy, Rearrangement Invariant Sets Relat- ed to Subspace of BMO, WSEAS Transactions on Math , Vol.6, No.2, 2007.

[28] Wheeden, R. L., Wilson, J. M., Weighted Norm Estimates for Gradients of Half Space Extesions, Indiana U.Math. J., Vol.44, 1995, pp.917 -969.