Descriptor Techniques for Modeling of Swirling Fluid Structures and Stability Analysis

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Abstract: In this paper we develop descriptor techniques for modeling swirling hydrodynamic structures. Using the descriptor notation we obtain the generalized eigenvalue formulation for differential-algebraic equations describing the spatial stability of the fluid flow. We describe the general framework for spatial stability investigation of vortex structures in both viscous and inviscid cases. The dispersion relation is analytically investigated and the polynomial eigenvalue problem describing the viscous spatial stability is reduced to a generalized eigenvalue problem in operator formulation using the companion vector technique. A different approach is assessed for spatial inviscid study when the stability model is obtained by means of a class of shifted orthogonal basis and a spectral differentiation matrix is derived to approximate the discrete spatial derivatives. Both schemes applied to a swirling fluid profile provide good results.

Key-Words: Hydrodynamic stability, Swirling flow, Descriptor operators, Spectral collocation.

1 Introduction

The role of the hydrodynamic stability theory in fluid mechanics reaches a special attention, especially when reaserchers deal with problem of minimum consumption of energy. This theory deserves special mention in many engineering fields, such as the aerodynamics of profiles in supersonic regime, the construction of automation elements by fluid jets and the technique of emulsions.

The main interest in recent decades is to use the theory of hydrodynamic stability in predicting transitions between laminar and turbulent configurations for a given flow field. R.E. Langer [1] proposed a theoretical model for transition based on supercritical branching of the solutions of the Navier-Stokes equations. This model was substantiated mathematically by E. Hopf [2] for systems of nonlinear equations close to Navier-Stokes equations. C.C. Lin, a famous specialist in hydrodynamic stability theory, published his first paper on stability of fluid systems in which the mathematical formulation of the problems was essentially diferent from the conservative treatment [3]. The intermittent character of the transition of motions in pipes was identified for the first time by J.C. Rotta [4]. J.T. Stuart in [5] developed an energetic method frequently used in the investigation of transition, method that was

undertaken by D.D. Joseph whose intensive activity has lead to the theory of the global stability of fluid flows [6]. The Nobel laureate Chandrasekhar [7] presents in his study considerations of typical problems in hydrodynamic and hydromagnetic stability as a branch of experimental physics. Among the treated subjects are thermal instability of a layer of fluid heated from below, the Benard problem, stability of Couette flow, and the Kelvin-Helmholtz instability.

Many publications in the field of hydrodynamics are focused on vortex motion as one of the basic states of a flowing continuum and effects that vortex can produce. Such problems may be of interest in the field of aerodynamics, where vortices trail on the tip of each wing of the airplane and stability analyses are needed. Mayer [8] and Khorrami [9] have mapped out the stability of Q-vortices, identifying both inviscid and viscous modes of instability. The mathematical description of the dynamics of swirling flows is hindered by the requirement to consider three-dimensional and effects, singularity nonlinear and various instabilities as in [10, 11, 12].

The numerical simulation is the main instrument to investigate this type of three dimensional unsteady flows. However, the simulation requirements are very expensive even with very powerful computer resources. In these conditions, stability analyses of vortex motions that can help to better understand the dynamical behavior of the flow by offering a significant insight for the physical mechanics of the observed dynamics become very important in flow control problems.

The objective of this paper is to present new instruments that can provide relevant conclusions on the stability of swirling flows, assessing both an analytical methodology and numerical methods. The study involves new mathematical models and simulation algorithms that translate equations into computer code instructions immediately following problem formulations. Classical vortex problems were chosen to validate the code with the existing results in the literature. The paper is outlined as follows: Section 1 gives a brief motivation for the study of hydrodynamic stability using computer aided techniques. The dispersion equation governing the linear stability analysis for swirling flows against normal mode perturbations is derived in Section 2. The analytical investigation of the dispersion relationship is included in Section 3. In Section 4 a nodal collocation method is proposed for viscous stability investigations and in Section 5 a modal collocation method is developed, based on shifted orthogonal expansions, assessing different boundary conditions. In Section 6 the hydrodynamic models are applied upon the velocity profile of a Q-vortex. Section 7 concludes the paper.

2 Problem Formulation

Hydrodynamic stability theory is concerned with the response of a laminar flow to a disturbance of small or moderate amplitude. If the flow returns to its original laminar state one defines the flow as stable, whereas if the disturbance grows and causes the laminar flow to change into a different state, one defines the flow as unstable. Instabilities often result in turbulent fluid motion, but they may also take the flow into a different laminar, usually more complicated state. Stability theory deals with the mathematical analysis of the evolution of disturbances superposed on a laminar base flow. In many cases one assumes the disturbances to be small so that further simplifications can be justified. In particular, a linear equation governing the evolution of disturbances is desirable. As the disturbance velocities grow above a few percent of the base flow, nonlinear effects become important and the linear equations no longer accurately predict the disturbance evolution. Although the linear equations have a limited region of validity they are important in detecting physical growth mechanisms and identifying dominant disturbance types.

The equations governing the general evolution of fluid flow describing the conservation of mass and momentum are known as the Navier-Stokes equations [13]. They describe the conservation of mass and momentum. For an incompressible fluid, using Cylindrical coordinates (z,r,θ) , the equations read

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_{\theta}}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0, \qquad (1)$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{\text{Re}} \Delta u_z, (2)$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = = -\frac{\partial p}{\partial r} + \frac{1}{\text{Re}} \left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right), \quad (3)$$

$$\frac{\partial u_{\theta}}{\partial t} + u_{r} \frac{\partial u_{\theta}}{\partial r} + \frac{u_{\theta}}{r} \frac{\partial u_{\theta}}{\partial \theta} + u_{z} \frac{\partial u_{\theta}}{\partial z} + \frac{u_{r} u_{\theta}}{r} = = -\frac{\partial p}{\partial \theta} + \frac{1}{\text{Re}} \left(\Delta u_{\theta} - \frac{u_{\theta}}{r^{2}} + \frac{2}{r^{2}} \frac{\partial u_{r}}{\partial \theta} \right), \quad (4)$$

where (u_z, u_r, u_θ) are the velocity components, $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian, p is the pressure and the radial and axial coordinates

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for these equation were considered normalized by a reference dimension.

The evolution equations for the disturbance can be derived by considering the basic state $\{\overline{U} = (u_z, u_r, u_\theta), p\}$ and a perturbed state $\{\overline{V} = (v_z, v_r, v_\theta), \pi\}$, with the disturbance being of order $0 \prec \delta \prec \prec 1$

$$\overline{U} = \left[U(r), 0, W(r), P(r) \right] + \delta \overline{V}$$
(5)

Consistent with the parallel mean flow assumption is that the functional form for the mean part of the velocity components only involves the cross-stream coordinate and also zero mean radial velocity. The linearized equations are obtained after substituting the expressions for the components of the velocity and pressure field into the Navier Stokes equations and only considering contributions of first order in delta. For high Reynolds numbers a restrictive hypothesis to neglect viscosity can be imposed in some problems. The linearized equations in descriptor formulation are

 $L \cdot S = 0, \quad S = (v_r \quad v_\theta \quad v_z \quad \pi)^T$ (6)

and the elements of matrix L being

$$\begin{split} L_{11} &= \partial_t + \frac{W}{r} \partial_\theta + U \partial_z - \frac{1}{\operatorname{Re}} \Delta - \frac{1}{r^2 \operatorname{Re}} \\ L_{12} &= -\frac{2W}{r} + \frac{2}{r^2 \operatorname{Re}} \partial_\theta , \\ L_{13} &= 0 , \\ L_{14} &= \partial_r , \\ L_{21} &= W' + \frac{W}{r} - \frac{2}{r^2 \operatorname{Re}} \partial_\theta , \\ L_{22} &= \partial_t + \frac{W}{r} \partial_\theta + U \partial_z - \frac{1}{\operatorname{Re}} \Delta + \frac{1}{r^2 \operatorname{Re}} , \\ L_{23} &= 0 , \\ L_{24} &= \frac{1}{r} \partial_\theta , \\ L_{31} &= U' , \\ L_{32} &= 0 , \\ L_{33} &= \partial_t + \frac{W}{r} \partial_\theta + U \partial_z - \frac{1}{\operatorname{Re}} \Delta , \\ L_{34} &= \partial_z , \\ L_{41} &= \partial_r + \frac{1}{r} , \\ L_{42} &= \frac{1}{r} \partial_\theta , \\ L_{43} &= \partial_z , \\ L_{44} &= 0 , \end{split}$$

where $\partial_{\{t,z,r,\theta\}}$ denote the partial derivative operators and primes denote derivative with respect to radial coordinate. In linear stability analysis the disturbance components of velocity are shaped into normal mode form, given here

$$\{v_z, v_r, v_\theta, \pi\} = \{F(r), iG(r), H(r), P(r)\}E(t, z, \theta)$$
 (7)
where $E(t, z, \theta) \equiv e^{i(kz+m\theta-oot)}$, F, G, H, P represent
the complex amplitudes of the perturbations, k is
the complex axial wavenumber, m is the tangential
integer wavenumber and ω represents the complex

frequency. The hydrodynamic equation of dispersion is obtained, where we have explicitly decomposed into operators that multiply ω and the different powers of k

 $(\omega M_{\omega} + M + kM_k + k^2 M_{k^2}) \cdot (F \quad G \quad H \quad P)^T = 0.(8)$ The matrices are given explicitly by

$$M_{\omega} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_{k} = \begin{pmatrix} -U & 0 & 0 & 0 \\ 0 & Ui & 0 & 0 \\ 0 & 0 & Ui & i \\ 0 & 0 & i & 0 \end{pmatrix},$$
$$M_{k^{2}} = \begin{pmatrix} i/\operatorname{Re} & 0 & 0 & 0 \\ 0 & i/\operatorname{Re} & 0 & 0 \\ 0 & 0 & i/\operatorname{Re} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the elements of matrix M are

$$\begin{split} M_{11} &= -\frac{mW}{r} - \frac{i}{\text{Re}} d_{rr} - \frac{i}{r \text{Re}} d_r + \frac{i(m^2 + 1)}{r^2 \text{Re}}, \\ M_{12} &= -\frac{2W}{r} + \frac{2im}{r^2 \text{Re}}, M_{13} = 0, M_{14} = d_r, \\ M_{21} &= iW' + \frac{iW}{r} + \frac{2m}{r^2 \text{Re}}, \\ M_{22} &= \frac{imW}{r} - \frac{1}{\text{Re}} d_{rr} - \frac{1}{r \text{Re}} d_r + \frac{m^2}{r^2 \text{Re}}, M_{23} = 0, \\ M_{24} &= \frac{im}{r}, M_{31} = iW', M_{32} = 0, \\ M_{33} &= \frac{imW}{r} - \frac{1}{\text{Re}} d_{rr} - \frac{1}{r \text{Re}} d_r + \frac{m^2}{r^2 \text{Re}}, \\ M_{34} &= 0, M_{41} = id_r + \frac{i}{r}, M_{42} = \frac{im}{r}, \\ M_{43} &= M_{44} = 0, \end{split}$$

where prime denotes differentiation with respect to the radius and d_r and d_{rr} mean the differentiation operators of first and second order, respectively.

3 The Analytical Investigation of the Dispersion Relationship

The nature of the instability of the basic flow has been widely investigated either analytically, numerically or experimentally.

Depending on whether the frequency is real and the wavenumber is complex or viceversa, the stability investigations are classified as temporal or spatial stability, respectively. In this way, a temporal stability analysis of normal modes imply that the ω roots $\omega = \omega_r + i \cdot \omega_i$, $\omega_r = \operatorname{Re}(\omega)$, $\omega_i = \operatorname{Im}(\omega)$, of the dispersion relation $D(\omega) = 0$ are obtained as functions of the real values of k. In this conditions, a characterization of the stability of the basic flow is: the basic flow is unstable if, for some real k, the growth rate, $\omega_i = \text{Im}(\omega)$ is positive. If the growth rate is negative for all real k then the basic flow is stable.

Conversely, solving the dispersion relation for the complex wavenumber, $k = k_r + i \cdot k_i$, $k_r = \text{Re}(k)$, $k_i = \text{Im}(k)$, when ω is given real leads to the spatial branches $k(\omega, \Upsilon)$ where by Υ we denoted the set of all other physical parameters involved. The growth of the wave solution in spatial case depends on the imaginary part of the axial wavenumber, as described in the next formula

$$e^{-k_{i}z} \begin{cases} F_{r}\cos(k_{r}z+\Theta) - F_{i}\sin(k_{r}z+\Theta) + \\ i[F_{r}\sin(k_{r}z+\Theta) + F_{i}\cos(k_{r}z+\Theta)] \end{cases}, \\ \Theta \equiv m\theta - \omega t,$$
(9)

When temporal stability analysis is assessed for a given axial wavenumber, the dispersion relation is translated into an operator eigenvalue problem of form

where

$$\bar{Sv} = \omega \bar{Pv} \tag{10}$$

$$S = \begin{pmatrix} k & d_r & \frac{m}{r} & 0 \\ 0 & \frac{m}{r}W - kU & \frac{2W}{r} & d_r \\ 0 & d_rW + \frac{W}{r} & \frac{m}{r}W + kU & \frac{m}{r} \\ \frac{mW}{r} & U & 0 & k \end{pmatrix},$$
$$= \begin{pmatrix} F \\ G \\ H \\ P \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In almost all studies one of this type of instability or both are investigated. However, in certain cases, these classifications can become arbitrary without a careful examination on the propagative character of the instability waves. This examination is related with the measure of the group velocity of the wavepackets [14], i.e. a further characterization of the impulse response of the system is necessary at the local level of description. Therefore the concepts of local convective and absolute instability provide a rigorous justification of selecting spatial or temporal stability [14]. The existence of spatially localized linear disturbances covering the entire flow with time and infinitely grow at all points of the flow defines an absolutely unstable state. Conversely, localized disturbances reaching a maximum value growing downstream and leaving a stabilizing flow behind them characterize the term of convective instability. An occurrence of a saddle point in the wavenumber values space may be related with the process of transition between the convective to absolute instability. In this case, temporal stability calculations are required [14]. Chomaz [15] emphasized that a transition between the convective and absolute instability of the trivial steady state take place at the point where a front between the rest state and the nontrivial steady state is stationary in a frame moving with the groups velocity.

Linear criteria of absolute instability can be established for the case of branching dispersion relationship within the case of supercritical bifurcations, yet for the case of subcritical bifurcations these criteria cease to hold [16]. The disturbances can only be amplified in a convectively unstable system, whereas absolutely unstable systems can generate them.

4 Nodal Collocation Approach For Spatial Stability Including Viscosity

When viscosity is included as parameter of spatial stability analysis, a given ω leads to a polynomial eigenvalue problem of form

$$\begin{pmatrix} M_0 + kM_k + k^2 M_{k^2} \end{pmatrix} \begin{pmatrix} F & G & H & P \end{pmatrix}^T = 0,$$

$$M_0 \equiv \omega M_\omega + M.$$
 (11)

In general, the direct solution of the polynomial eigenvalue problems can be heavy. For this case, we can transform the polynomial eigenvalue problem into a generalized eigenvalue problem, using the companion vector method, assessed also in [11]. We augment the system with the variable

$$\widehat{\Psi} = \begin{pmatrix} kF & kG & kH \end{pmatrix}^T, \quad \widehat{S} = \begin{pmatrix} F & G & H & P \end{pmatrix}^T (12)$$

The eigenvalue problem describing the spatial hydrodynamic stability for a viscous fluid system reads now

$$\begin{pmatrix} M_0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{S} \\ \hat{\Psi} \end{pmatrix} + k \begin{pmatrix} M_k & M_{k^2} \\ -I & 0 \end{pmatrix} \begin{pmatrix} \hat{S} \\ \hat{\Psi} \end{pmatrix} = 0 \quad (13)$$

where the first row is the polynomial eigenvalue problem (11) and the second row enforces the definition of $\widehat{\Psi}$.

The collocation method is associated with a grid of clustered nodes x_j and weights w_j (j = 0,...,N). The collocation nodes must cluster near the boundaries to diminish the negative effects of the Runge phenomenon [17]. Another aspect is that the convergence of the interpolation function on the clustered grid towards unknown solution is

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extremely fast. We recall that the nodes x_0 and x_N coincide with the endpoints of the interval [a,b], and that the quadrature formula is exact for all polynomials of degree $\leq 2N-1$, i. e.,

$$\sum_{j=0}^{N} v(x_j) w_j = \int_a^b v(x) w(x) dx, \qquad (14)$$

for all v from the space of test functions.

Let $\{\Phi_{\ell}\}_{\ell=0,N}$ a finite basis of polynomials relative to the given set of nodes, not necessary being orthogonal. If we choose a basis of nonorthogonal polynomials we refer to it as a nodal basis, Lagrange polynomials for example. In nodal approach, each function of the nodal basis is responsible for reproducing the value of the polynomial at one particular node in the interval. When doing simulations and solving PDEs, a major problem is one of representing a deriving functions on a computer, which deals only with finite integers. In order to compute the radial and pressure derivatives that appear in our mathematical model, the derivatives are approximated by differentiating a global interpolative function built trough the collocation points. We choose $\{\Phi_i\}_{i=0}$ given by

Lagrange's formula

$$\Phi_i(r) = \frac{\omega_N(r)}{\omega'_N(r_i)(r-r_i)}, \text{ where } \omega_{N(r)} = \prod_{m=1}^N (r-r_m).$$

We used for this approach the interpolative spectral differentiation matrix $\Delta_{(N+1)\times(N+1)}$, having the entries

$$\begin{split} &\Delta_{00} = \frac{2N^2 + 1}{6}, \Delta_{NN} = -\frac{2N^2 + 1}{6}, \\ &\Delta_{jj} = \frac{-\xi_j}{2(1 - \xi_j^2)}, \ j = 1, \dots, N - 1, \\ &\Delta_{ij} = \frac{\lambda_i}{\lambda_j} \frac{(-1)^{i+j}}{(\xi_i - \xi_j)}, i \neq j, i, j = 1, \dots, N - 1, \\ &\lambda_i = \begin{cases} 2 & if \ i = 0, N \\ 1 & otherwise \end{cases}. \end{split}$$

derived in [17].

We made use of the conformal transformation

$$r(\xi) = \frac{\left\lfloor 1 + b \exp(-a) \right\rfloor r_{\max}}{\left\lceil 1 + b \exp\left(-a\frac{1-\xi}{2}\right) \right\rceil} \left(\frac{1-\xi}{2}\right)$$
(15)

that maps the standard interval $\xi \in [-1, 1]$ onto the physical range of our problem $r \in [0, r_{\text{max}}]$. Because large matrices are involved, we numerically solved the eigenvalue problem using the Arnoldi type algorithm [17], which provides entire eigenvalue and eigenvector spectrum (Figure 1).



Fig.1 The "Y" shape of the eigenvalue spectra, in temporal stability analysis of a Q-vortex:

a) Stable fluid system in non-axisymmetrical case m = 2, Re = 8000, k = 3.5.

b) Unstable fluid system in axisymmetrical case m = 0, Re = 8000, k = 3.5.

5 Modal Collocation With Orthogonal Basis For Inviscid Stability Analysis

The collocation method became a widely used technique in many applications of systems control. The efficiency of the collocation based algorithms was exposed in [20], for solving the Hartree-Fock equations of the self-consistent field in large atomic and molecular systems.

The collocation method that we present in this section has the peculiar feature that can approximate the perturbation field for all types of boundary conditions, especially when the boundary limits are described by sophisticated expressions. We consider the mathematical model of an inviscid columnar vortex derived in [18] whose velocity profile is written as $\underline{V}(r) = [U(r), 0, W(r)]$.

$$G' + \frac{G}{r} + \frac{mH}{r} + kF = 0, \qquad (16)$$

$$\left(\omega - \frac{mW}{r} - kU\right)G - \frac{2WH}{r} + P' = 0, \qquad (17)$$

$$\left(-\omega + \frac{mW}{r} + kU\right)H + \left(W' + \frac{W}{r}\right)G + \frac{mP}{r} = 0,(18)$$

$$\left(-\omega + \frac{mW}{r} + kU\right)F + U'G + kP = 0.$$
⁽¹⁹⁾

We assume for this model that the radial amplitude of the velocity perturbation at the wall is negligible, i.e. $G(r_{wall}) = 0$, for a truncated radius distance r_{wall} selected large enough such that the numerical results do not depend on that truncation of infinity. We have at r = 0

$$(|m|>1), F = G = H = P = 0,$$
 (20)

$$(m=0), \quad G=H=0, F, P \ finite,$$
 (21)

$$(m = \pm 1), \quad H \pm G = 0, F = P = 0.$$
 (22)

and at $r = r_{wall}$

$$(|m|>1), \quad F = G = H = P = 0,$$
 (23)

$$(m = 0), \quad \frac{2W_{wall} H}{r_{wall}} - P' = 0, G = 0,$$

$$HkU_{wall} - \omega H = 0, FkU_{wall} - \omega F + kP = 0, (24)$$

$$(m = \pm 1), \quad \frac{2W_{wall} H}{r_{wall}} - P' = 0, G = 0,$$

$$r_{wall} H (kU_{wall} - \omega) \pm HW_{wall} \pm P = 0 = 0,$$

 $r_{wall}F(kU_{wall}-\omega)\pm FW_{wall}+kr_{wall}P=0$, (25) where U_{wall} and W_{wall} are the axial and the tangential velocity respectively, calculated at domain limit r_{wall} .

A different approach is obtained by taking as basis functions simple linear combinations of orthogonal polynomials. These are called bases of *modal* type, i. e., such that each basis function provides one particular pattern of oscillation of lower and higher frequency. We approximate the perturbation amplitudes as a truncated series of shifted Chebyshev polynomials

$$(F,G,H,P) = \sum_{k=1}^{N} (f_k,g_k,h_k,p_k) \cdot T_k^*,$$
 (26)

where T_k^* are shifted Chebyshev polynomials on the physical domain $[0, r_{wall}]$.

The Chebyshev polynomial $T_n(\xi)$ of the first kind is a polynomial in ξ of degree *n*, defined by the relation

$$T_n(\xi) = \cos n\theta, \quad \xi = \cos \theta.$$
 (27)

If the range of the variable ξ is the interval [-1,1], the the range of the corresponding variable θ can be taken as $[0, \pi]$. These ranges are traversed in opposite directions since x = -1 corresponds to $\theta = \pi$ and x = 1 corresponds to $\theta = 0$. Since the range $[0, r_{wall}]$ is more convenient to use than the range [-1,1] to discretize our hydrodynamic stability problem, we map the independent variable r in $[0, r_{wall}]$ to the variable ξ in [-1,1] by the linear transformation

$$\xi = \frac{2r}{r_{wall}} - 1 \quad \Leftrightarrow \quad r = \frac{r_{wall}}{2}\xi + \frac{r_{wall}}{2}. \tag{28}$$

The shifted Chebyshev polynomial of the first kind $T_n^*(r)$ of degree n-1 in r on $[0, r_{max}]$ are given by

$$T_n^*(r) = T_n(\xi) = T_n\left(\frac{2r}{r_{wall}} - 1\right).$$
(29)

The shifted Chebyshev polynomials defined as described above meet the relations

$$T_n^*(0) = (-1)^{n+1}, \quad T_n^*(r_{wall}) = 1,$$
 (30)

relations that we will frequently use in our future calculations and let

$$\left(f,g\right)_{w} = \int_{0}^{r_{wall}} wfg \, dr \tag{31}$$

be the inner product in the Hilbert space $L_w^2(0, r_{wall})$,

$$w(r) = \left(\sqrt{1 - \left(\frac{2r}{r_{wall}} - 1\right)^2}\right)^{-1}$$
. Then we have the next

properties

$$(T_n^*, T_m^*)_w = 0$$
, $n \neq m$, $n, m = 1..N$, (32)

$$(T_n^*, T_n^*)_w = r_{wall} \frac{\pi}{2} , n = 1,$$
 (33)

$$(T_n^*, T_n^*)_w = r_{wall} \frac{\pi}{4}$$
, $n = 2..N$. (34)

The clustered Chebyshev Gauss grid $\Xi = (\xi_j)_{1 \le j \le N}$ in [-1,1] is defined by relation

$$\xi_{j+1} = \cos \frac{\pi (j+N-1)}{N-1},$$

$$\xi_{j+1} \in [-1,1], \ j = 0 \dots N-1.$$
(35)

This formula has the advantage that in floatingpoint arithmetic it yields nodes that are perfectly symmetric about the origin, being clustered near the boundaries and diminishing the negative effects of the Runge phenomena [17, 19]. This collocation nodes are the roots of Chebyshev polynomials and distribute the error evenly and exhibit rapid convergence rates with increasing numbers of terms.

In order to approximate the derivatives of the unknown functions, we express the derivative of the shifted Chebyshev polynomial T_n^* as a difference between the previous and the following term

$$T_{n}^{*'}(r) = \frac{r_{wall}}{4} \frac{(n-1)}{r(r_{wall}-r)} \left[T_{n-1}^{*}(r) - T_{n+1}^{*}(r) \right], \quad n \ge 2.$$
(36)

Let us consider

$$F(r) = f_1 T_1^*(r) + \sum_{k=2}^N f_k T_k^*(r).$$
(37)

By differentiating (37) results

$$F'(r) = f_1 T_1^{*'}(r) + \sum_{k=2}^{N} f_k T_k^{*'}(r).$$
 (38)

But $T_1^{*'}(r) = 0$ and involving relation (36) leads to

$$F'(r) = \sum_{k=2}^{N} f_k \frac{r_{wall}}{4} \frac{(k-1)}{r(r_{wall}-r)} \Big[T_{k-1}^*(r) - T_{k+1}^*(r) \Big].$$
(39)

The interpolative differentiation matrix D that approximates the discrete derivatives has the elements

 $D_{m,n} = E_n(r_m), m = 2..N - 1, n = 2..N - 1, \quad (40)$ where for k = 2..N - 1

$$E_{k}(r) = \frac{(k-1)}{r(r_{wall} - r)} \left[T_{k-1}^{*}(r) - T_{k+1}^{*}(r) \right] .$$
(41)

The eigenvalue problem governing the inviscid stability analysis appears now as a system of 4N equations, with the boundary conditions included as equations of the system. A special situation occur for the cases $m = \pm 1$, when only seven relations define the boundary conditions. To regain the eightth equation we choose the third relation from the mathematical model and we compute it in the extreme node $r = r_{wall}$.

We have chosen this relation for several reasons. We observed that the equations that not contain the axial perturbation F are the second and the third. The second equation contains the derivative of the pressure perturbation that cannot be computed in extreme nodes because the interpolative derivative matrix may produce singularities as a result of expression of $E_k(r)$. The remain possibility is actually the third equation symmetrized.

The hydrodynamic model reads, for j = 2..N - 1

$$G' + \frac{1}{r_j} \sum_{k=1}^{N} g_k T_k^* (r_j) + \frac{m}{r_j} \sum_{k=1}^{N} h_k T_k^* (r_j) + k \sum_{k=1}^{N} f_k T_k^* (r_j) = 0, \qquad (42)$$

$$\begin{bmatrix} \omega - \frac{mW}{r_j} - kU \end{bmatrix}_{k=1}^{N} g_k T_k^* (r_j) - \frac{2W}{r_j} \sum_{k=1}^{N} h_k T_k^* (r_j) + P' = 0,$$
(43)

$$\left[-\omega + \frac{mW}{r_j} + kU \right] \sum_{k=1}^{N} h_k T_k^* (r_j) + \left[W' + \frac{W}{r_j}\right] \sum_{k=1}^{N} g_k T_k^* (r_j) + \frac{m}{r_j} \sum_{k=1}^{N} p_k T_k^* (r_j) = 0, (44)$$

$$\left[-\omega + \frac{mW}{r_j} + kU \right] \sum_{k=1}^{N} f_k T_k^* (r_j) + U' \sum_{k=1}^{N} g_k T_k^* (r_j) + k \sum_{k=1}^{N} p_k T_k^* (r_j) = 0,$$
(45)

$$kr_{wall}U_{wall}\sum_{k=1}^{N}h_{k} + (mW_{wall} - r_{wall}\omega)\sum_{k=1}^{N}h_{k} + (W_{wall} + r_{wall}W'_{wall})\sum_{k=1}^{N}g_{k} + m\sum_{k=1}^{N}p_{k} = 0, \quad (46)$$

$$\sum_{1}^{N} (-1)^{k+1} g_k \pm \sum_{1}^{N} (-1)^{k+1} h_k = 0, \qquad (47)$$

$$\sum_{1}^{N} (-1)^{k+1} f_k = \sum_{1}^{N} (-1)^{k+1} p_k = 0, \qquad (48)$$

$$\frac{2W_{wall}}{r_{wall}}\sum_{1}^{N}h_{k}-p_{2}\frac{2}{r_{wall}}-\sum_{k \text{ odd}}^{N}p_{k}\frac{2(k-1)}{r_{wall}}\left[\sum_{\substack{r=k-1\\k \text{ even}}}^{2}2\right]-\\-\sum_{k \text{ even}}^{N}p_{k}\frac{2(k-1)}{r_{wall}}\left[\sum_{\substack{r=k-1\\k \text{ odd}}}^{2}2+1\right]=0,$$
(49)

$$\sum_{1}^{N} g_{k} = 0, \qquad (50)$$

$$kU_{wall}r_{wall}\sum_{1}^{N}h_{k} + \left(\pm W_{wall} - \omega r_{wall}\right)\sum_{1}^{N}h_{k} \pm \pm \sum_{1}^{N}p_{k} = 0, \qquad (51)$$

$$k \left(U_{wall} r_{wall} \sum_{1}^{N} f_k + r_{wall} \sum_{1}^{N} p_k \right) + \left(\pm W_{wall} - \omega r_{wall} \right) \sum_{1}^{N} f_k = 0.$$
(52)

Let us denote by $[r] = diag(r_i)$, $\left[\frac{1}{r}\right] = diag(1/r_i)$,

$$\begin{split} & \left[\eta\right] = (\eta_{ij})_{\substack{2 \le i \le N-1, \\ 1 \le j \le N}}, \quad \eta_{ij} = T_j^*(r_i), \quad \left[U\right] = diag(U(r_i)) \\ & \left[W\right] = diag(W(r_i)), \quad 2 \le i \le N-1. \end{split}$$
 Written in matrix

formulation, the hydrodynamic model reads

$$\left(kM_{k}+\omega M_{\omega}+mM_{m}+M_{0}\right)s=0$$

 $\overline{s} = (f_1, ..., f_N, g_1, ..., g_N, h_1, ..., h_N, p_1, ..., p_N)^T$, (53) where M_k , M_{ω} , M_m and M_0 are square matrices of dimension 4N and the elements being matrix blocks

where D represents the interpolative derivative matrix.

6 Model Validation On a Q-Vortex Profile

Swirling flows models have been assessed in literature with applications to various optimization fluid motion control problems. The and hydrodynamics of rotating machines where confined vortices are developed due to the turbine rotation have been investigated in various surveys [21-24]. An experimental investigation of the suction side boundary layer of a large scale turbine cascade has been performed in [22] to study the effect of Reynolds number on the boundary layer transition process at large and moderate Reynolds numbers. The boundary element approach is assessed in [23] for the problem of the compressible fluid flow around obstacles. The system is analyzed with respect to different operating conditions, for understanding its behavior. In [24] oscillations and rotations of a liquid droplet are simulated numerically using the level set method, and the combined effects of oscillation amplitude and rotation rate on the drop-shape oscillation is studied.

In this section we assume the velocity profile of Q-Vortex, written in form

$$U(r) = a + e^{-r^2}, \quad W(r) = \frac{q}{r} \left(1 - e^{-r^2}\right), \quad (54)$$

where q represents the swirl number and a provides a measure of free-stream axial velocity. We perform a spatial stability analysis using the collocation method described above. The spectra of the eigenvalue problem governing the spatial stability is depicted in Figure 2.



Fig.2 Spectra of the hydrodynamic eigenvalue problem computed at $\omega = 0.01$, m = -3, a = 0, q = 0.1, for N = 100 collocation nodes.

It is noticeable that the eigenvalue with the largest imaginary part defines the most unstable mode. In Table 1 we have compared the results obtained by this method with those of Olendraru et al. [25], in the non axisymmetrical case |m| > 1.

Table 1. Comparative results of the most amplified k-spatial wave at a = 0, q = 0.1, $\omega = 0.01$ for the case of the counter-rotating mode m = -3: eigenvalue with largest imaginary part $k_{cr} = (k_r, k_i)$ and critical distance of the most amplified perturbation r_c .

Shooting method [25]	
$k_{cr} = (0.506, -0.139)$	$r_c = 1.0005$
Collocation method	
$k_{cr} = (0.50819, -0.14192)$	$r_c = 0.971$
Error 0.79%	2.94%



Fig.3 Plot of the most unstable eigenfunctions for case $\omega = 0.01$, m = -3, a = 0, q = 0.1, N = 100, considering the critical eigenvalue with the largest imaginary part $k_{cr} = 0.50819 - 0.14192i$, without

stabilization (a) and with Lanczos stabilization (b).

Radial distribution of the velocity perturbation mode is depicted in Figure 3. Figure 3a shows the profiles without a stabilization and the Gibbs phenomenon occurs. In Figure 3b a smoothing procedure was applied by multiplication with a Lanczos σ factor [19]

$$(F,G,H,P) = \sum_{k=1}^{N} \sigma_k \cdot (f_k, g_k, h_k, p_k) \cdot T_k^*,$$

$$\sigma_k = \frac{N}{2\pi k} \sin \frac{2\pi k}{N}, \ 1 \le k \le N.$$
(55)

Performing a closer analysis, we observed the behavior of the growth rate $-k_i$ and the axial wavenumber k_r as functions of real frequency. We denote by the critical frequency ω_{cr} , the temporal frequency corresponding to maximum $-k_i$ for a given omega. Figure 4 presents the results obtained by collocation method for axisymmetrical mode m = 0.



Fig.4 Results for axisymmetrical mode m = 0: a) Plot of spatial growth rate as a function of real frequency.

b) Plot of the axial wavenumber as a function of real

frequency.

7 Conclusions

In this paper we developed hydrodynamic models using spectral differential operators to investigate the spatial stability of swirling fluid systems, using two different methods.

When viscosity is considered as a valid parameter of the fluid, the hydrodynamic model is implemented using a nodal Lagrangean basis and the eigenvalue problem describing the viscous spatial stability is solved using the companion vector method. The second model for inviscid study is assessed for the construction of a certain class of shifted orthogonal expansion functions. The choice of the grid and of the trial basis eliminates the singularities and the spectral differentiation matrix was derived to approximate the discrete derivatives. The models were applied to a Q-vortex structure, the scheme based on shifted Chebyshev polynomials providing good results.

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