

The Effect of Time Scales on SIS Epidemic Model

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Abstract: - The distribution of diseases is one of the most interesting real-world phenomena which can be systematically studied through a mathematical model. A well-known simple epidemic model with surprising dynamics is the SIS model. Usually, the time domains that are widely used in mathematical models are limited to real numbers for the case of continuous time or to integers for the case of discrete time. However, a disease pandemic such as an influenza pandemic regularly disappears from a population and then recurs after a period of time. Additionally, collecting actual data continuously is time-consuming, relatively expensive, and really impractical. It seems that using a continuous-time model to describe observed data may not always be possible due to time domain conflict. The purpose of this paper is, therefore, to study the qualitative behavior of SIS models on continuous, discrete, and mixed continuous-discrete time scales. We investigate their dynamic behavior and examine how this behavior changes in the different time scale domains. We show that the dynamic behavior can change in a systematic manner from simple stable steady-state solutions for the continuous time domain to complicated chaotic solutions for the discrete-time domain.

Key-Words: - Bifurcation, Chaos, Limit cycles, Period doubling, SIS epidemic model, Time scales calculus.

1 Introduction

Mathematical modelling has a long and rich history, spanning many fields, not only in the physical sciences but also in the biological sciences. This modelling typically treats time as a discrete variable or a continuous variable. For discrete time, a model is formulated by difference equation(s), for example, [1]-[6]. For continuous time, a model is represented by differential equation(s), for example, [7]. Unlike physical data, biological data are difficult to collect continuously. A variety of time measurements (seconds, hours, days, weeks, months, or years) are regularly used [8]. Sometimes

observed data reveal periodic patterns in time series [9], [10]. Obviously, the traditional categories of time variable as being wholly continuous or wholly discrete should be reconsidered and the effects of mixed continuous-discrete time scales on the behavior of disease models should be examined.

Fortunately, Stefan Hilger introduced the theory of time scales in 1988 in order to unify continuous (\mathbb{R}) and discrete (\mathbb{Z}) analysis [11]. Since then, the time scales calculus has been gradually developed and continuously extended. The time scales calculus shares the general ideas of traditional calculus; however, it broadens these ideas for use with

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functions whose time variable can be an arbitrary nonempty closed subset of the real numbers. Time scales theory has also been used in the study of first order dynamic equations [12], first order dynamical systems [13]-[15], numerical results [16], and a variety of applications, including a plant population model [17], economic model [18], predator-prey model [19], and West Nile virus model [20].

An outline of this paper is as follows. Since time scale calculus is relatively new and there are many notations, definitions, and concepts, we introduce some of these important and useful ideas in section 2. In section 3, an SIS epidemic model is selected as a representative of a mathematical model. This simple model reveals surprising qualitative behaviour even for time domains that are either entirely real numbers or entirely integers. We then modify the SIS model for time scales that are a mixture of continuous and discrete regions. In section 4, we give a theoretical analysis of the dynamic behaviour of the SIS model on each of the different time scales. In section 5, numerical solutions of the SIS model on the different time scales are presented and the results compared with results from the theoretical analysis. Finally, we discuss the results and draw conclusions.

2 Preliminaries

In order to explain the later analytical methods more clearly, we begin by introducing the main concepts, notation and definitions of calculus on time scales.

2.1 Time Scales

A time scale is an arbitrary nonempty closed subset of the real numbers [17], [21]. A time scale is usually denoted by the symbol \mathbb{T} . Forward and backward jump operators are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$ and \emptyset denotes the empty set. A point $t \in \mathbb{T}$ is called *left-dense* if $t > \inf \mathbb{T}$ and $\rho(t) = t$, *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *right-scattered* if $\sigma(t) > t$. The set \mathbb{T}^κ is defined to be \mathbb{T} if \mathbb{T} does not have a left-scattered maximum m ; otherwise it is \mathbb{T} without this left-scattered maximum. All definitions are summarized in Table 1.

TABLE 1
CLASSIFICATION OF POINTS

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

The *graininess function* $\mu: \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. The interval notation in time scale \mathbb{T} is defined by $[a, b] = \{t \in \mathbb{T} \mid a \leq t \leq b\}$.

2.2 Delta Derivatives

The (*delta*) derivative of $f: \mathbb{T} \rightarrow \mathbb{R}$ at point $t \in \mathbb{T}^\kappa$ is defined as follows. Assume $f: \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then $f^\Delta(t)$ is defined to be the number (provided it exists) with the property that for all $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$.

Another useful formula for the relationship concerning the (*delta*) derivative is given by

$$f^\Delta(t) = \begin{cases} \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} & \text{if } \mu(t) = 0 \\ \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} & \text{if } \mu(t) > 0. \end{cases} \quad (1)$$

To avoid separate discussion of the two cases $\mu(t) = 0$ and $\mu(t) > 0$, there is another useful formula which holds when f is delta differentiable at $t \in \mathbb{T}^\kappa$, namely

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t). \quad (2)$$

2.3 Integration

Definition 1 A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is called regulated provided its right-side limits exist (finite) at all right-dense points in \mathbb{T} and its left-sided limits exist (finite) at all left-dense points in \mathbb{T} .

Definition 2 A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist (finite) at left-dense points in \mathbb{T} . The set of rd-continuous functions $f: \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f: \mathbb{T} \rightarrow \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C_{rd}^1 = C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R}).$$

Theorem 1 (Existence of Pre-Antiderivatives). Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that

$$F^\Delta(t) = f(t) \text{ holds for all } t \in D.$$

Definition 3 Assume $f: \mathbb{T} \rightarrow \mathbb{R}$ is a regulated function. Any function F as in Theorem 1 is called a pre-antiderivative of f . The indefinite integral of a regulated function is defined by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f .

Definition 4 A function $F: \mathbb{T} \rightarrow \mathbb{R}$ is called an antiderivative of a rd-continuous function f if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. If $t_0 \in \mathbb{T}$ then $F(t) = \int_{t_0}^t f(s)\Delta s$ for $t \in \mathbb{T}$. For all $m, n \in \mathbb{T}$ and $m < n$, the Cauchy integral and the infinite integral are defined by

$$\int_m^n f(t)\Delta t = F(n) - F(m)$$

and

$$\int_m^\infty f(t)\Delta t = \lim_{n \rightarrow \infty} \int_m^n f(t)\Delta t, \text{ respectively.}$$

2.4 First-Order Dynamic Equations

The nontrivial function, $z(t)$, is called the solution of the dynamic system

$$z^\Delta(t) = f(t, z(t)), \quad z \in \mathbb{R}^n, \quad t \in \mathbb{T} \tag{3}$$

when $z(t) \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^n)$ and satisfies (3). If $z(t)$ also satisfies the initial condition

$$z(t_0) = z_0, \tag{4}$$

then $z(t)$ is called the solution of initial value problem (3) and (4).

2.5 Example

The theory of time scale is useful not only for understanding the relationship between difference and differential equations [21], [22] but also for understanding time scales that are a combination between continuous and discrete time.

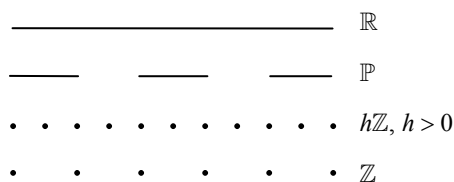


Fig.1 Examples of time scales.

In the case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) = 0,$$

$$f^\Delta(t) = f'(t), \quad \int_m^n f(t)\Delta t = \int_m^n f(t)dt,$$

where $f'(t) = df(t)/dt$ is the right-derivative of $f(t)$.

Thus, the time scales operators reduce to the corresponding continuous operators.

In the case $\mathbb{T} = \mathbb{P}_{l,h} := \bigcup_{k=0}^\infty [k(l+h), k(l+h)+l]$ where $l, h > 0$ and $k \in \mathbb{N}_0$, we have

$$\sigma(t) = \begin{cases} t & \text{if } t \in \bigcup_{k=0}^\infty [k(l+h), k(l+h)+l) \\ t+h & \text{if } t \in \bigcup_{k=0}^\infty \{k(l+h)+l\}, \end{cases}$$

$$\rho(t) = \begin{cases} t & \text{if } t \in \bigcup_{k=0}^\infty (k(l+h), k(l+h)+l] \\ t-h & \text{if } t \in \bigcup_{k=1}^\infty \{k(l+h)\}, \end{cases}$$

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^\infty [k(l+h), k(l+h)+l) \\ h & \text{if } t \in \bigcup_{k=0}^\infty \{k(l+h)+l\}, \end{cases}$$

$$f^\Delta(t) = \begin{cases} f'(t) & \text{if } t \in \bigcup_{k=0}^\infty [k(l+h), k(l+h)+l) \\ \frac{f(\sigma(t)) - f(t)}{h} & \text{if } t \in \bigcup_{k=0}^\infty \{k(l+h)+l\}, \end{cases}$$

$$\int_m^n f(t)\Delta t = \begin{cases} \int_m^n f(t)dt & \text{if } m, n \in [k(l+h), k(l+h)+l), \\ & k \in \mathbb{N}_0 \\ hf(m) & \text{if } m \in \{k(l+h)+l\} \text{ and } \\ & n = \sigma(m), k \in \mathbb{N}_0. \end{cases}$$

In the case $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}, h > 0\}$, i.e., where the points are equally spaced points in time, we have

$$\sigma(t) = t+h, \quad \rho(t) = t-h, \quad \mu(t) = h,$$

$$f^\Delta(t) = \frac{f(t+h) - f(t)}{h}, \quad \int_m^n f(t)\Delta t = \sum_{t=m}^{n-1} hf(t).$$

Therefore, every point is isolated. This time scales is considered as discrete and the delta derivative operator corresponds to the forward finite difference operator (Δ) and the delta integration corresponds to summation. Assume $t_0 \in \mathbb{T}$ and it is convenient to let $t_0 > 0$. The time scale interval $[0, t_0)_{\mathbb{T}}$ is defined by $[0, t_0)_{\mathbb{T}} = [0, t_0)_{\mathbb{T}} \cap \mathbb{T}$.

3 SIS Epidemic Models on Time Scales

For the SIS epidemic model on time scales \mathbb{T} , $S(t)$ represents the number of susceptible individuals at time t and $I(t)$ is the number of infectious individuals at time t . We assume that the total population size is a constant N . Let γ be the recovery rate of an infectious individual who then returns to the susceptible population. Then $\gamma I(t)$ represents the total number of infectious individuals who recover per unit time at the time t . Let α be the disease virulence per unit time, i.e., the rate of infection of a susceptible person due to contact with an infectious person. Then $(\alpha/N)S(t)I(t)$ represents the infection rate at which the susceptible population contracts the disease at time t . Thus the SIS epidemic system for time scales can be written in the following form:

$$S^\Delta(t) = I(t) \left(-\frac{\alpha}{N} S(t) + \gamma \right), \quad S(t) \geq 0 \tag{5}$$

$$I^\Delta(t) = I(t) \left(\frac{\alpha}{N} S(t) - \gamma \right), \quad I(t) \geq 0 \tag{6}$$

with positive initial conditions $S(0)$ and $I(0)$ satisfying $S(0) + I(0) = N$. The total population size remains constant and thus

$$S(t) + I(t) = S(\sigma(t)) + I(\sigma(t)) = N \text{ for } t \geq 0.$$

Another assumption is that the population is homogeneously mixed at all times. The parameters α, γ, N are all positive constants.

For the continuous time scale, the system becomes

$$\frac{dS}{dt} = I(t) \left(-\frac{\alpha}{N} S(t) + \gamma \right), \quad S(t) \geq 0 \tag{7}$$

$$\frac{dI}{dt} = I(t) \left(\frac{\alpha}{N} S(t) - \gamma \right), \quad I(t) \geq 0 \tag{8}$$

The system can be changed to be a single equation by substituting $I(t) = N - S(t)$ into (7). Therefore,

$$\frac{dS}{dt} = \frac{\alpha}{N} S^2(t) - (\alpha + \gamma)S(t) + \gamma N \tag{9}$$

An exact solution of (9) can be obtained by integrating dt/ds using the method of partial fractions. The result is:

$$S(t) = \frac{\left[\frac{C_0 \gamma}{\alpha} e^{(\alpha-\gamma)t} - 1 \right] N}{\left[C_0 e^{(\alpha-\gamma)t} - 1 \right]},$$

where $C_0 = (S_0 - N)/(S_0 - \frac{\gamma N}{\alpha})$ and $S_0 = S(0)$.

Obviously, the asymptotic behavior of $S(t)$ for large t is

$$S(t) = \begin{cases} \frac{\gamma N}{\alpha} & \text{for } \alpha > \gamma \\ N & \text{for } \alpha < \gamma \end{cases}$$

Therefore, the solution of (9) is non-oscillatory and reaches an equilibrium point.

For any other time scales, the SIS epidemic model is

$$\begin{aligned} S^\Delta(t) &= \frac{S(\sigma(t)) - S(t)}{\mu(t)} = \frac{S(t+h) - S(t)}{h} \\ &= I(t) \left(-\frac{\alpha}{N} S(t) + \gamma \right), \quad S(t) \geq 0. \end{aligned} \tag{10}$$

$$\begin{aligned} I^\Delta(t) &= \frac{I(\sigma(t)) - I(t)}{\mu(t)} = \frac{I(t+h) - I(t)}{h} \\ &= I(t) \left(\frac{\alpha}{N} S(t) - \gamma \right), \quad I(t) \geq 0. \end{aligned} \tag{11}$$

The system can be changed to a single equation as before

$$\begin{aligned} S^\Delta(t) &= \frac{\alpha}{N} S^2(t) - (\alpha + \gamma)S(t) + \gamma N \\ &:= F(S(t)) \end{aligned} \tag{12}$$

The equation can also be written as a difference equation:

$$\begin{aligned} S(\sigma(t)) &= \frac{\alpha \mu}{N} S^2(t) + (1 - \alpha \mu - \gamma \mu)S(t) + \gamma N \mu \\ &:= f(S(t)) \end{aligned} \tag{13}$$

The continuous time scale is useful for mathematical models but the seasonal time scale also occurs in the real world.

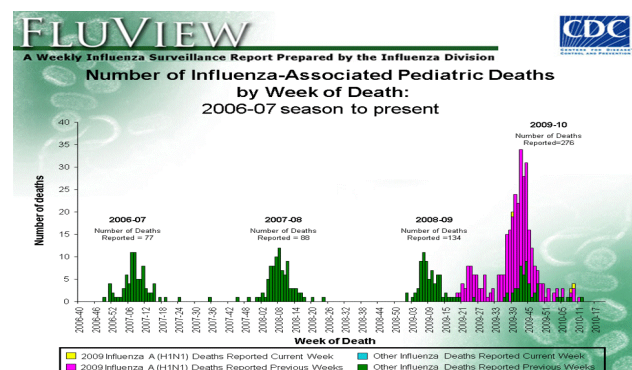


Fig. 2 The number of influenza-associated Pediatric Deaths. The data shows periodic outbreaks of disease. [9]

In the case $\mathbb{T} = \mathbb{P}_{t,h} := \bigcup_{k=0}^{\infty} [k(l+h), k(l+h) + l]$ where

$$l, h > 0, \text{ a solution of (12) is } S(t) = \frac{\left[\frac{C_0 \gamma}{\alpha} e^{(\alpha-\gamma)t} - 1 \right] N}{\left[C_0 e^{(\alpha-\gamma)t} - 1 \right]},$$

$$\text{where } C_0 = \frac{(S_0 - N)}{\left(S_0 - \frac{\gamma N}{\alpha} \right)} \text{ and } S_0 = S(k(l+h))$$

$$\text{for } t \in \bigcup_{k=0}^{\infty} [k(l+h), k(l+h)+l] \text{ and}$$

$$S((k+1)(l+1)) = \frac{\alpha}{N} S^2(k(l+1)+l) + (1-\alpha-\gamma)S(k(l+1)+l) + \gamma N$$

$$\text{for } t \in \bigcup_{k=0}^{\infty} \{k(l+h)+l\}$$

In the case $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}, h > 0\}$, the graininess function is defined by $\mu(t) = (t+h) - t = h$. The analytical solution of (13) for all values of parameters is still unknown, although numerical solutions can be obtained for any given parameter values. Therefore, qualitative analysis is a useful tool.

4 Qualitative analysis of SIS epidemic models

4.1 Equilibrium Points

For a natural disease process, each parameter is assumed to be positive and each variable is non-negative. Therefore, the region of interest is

$$\Gamma = \{(S, I) \in \mathbb{R}^2 \mid S \geq 0, I \geq 0, S + I = N\}$$

The equilibrium point or the steady state (time-independent) solution is obtained by setting $S(\sigma(t)) = S(t) = f(S(t)) = S^*$ in (13).

$$\frac{\alpha}{N} S^2 - (\alpha + \gamma)S + \gamma N = 0. \tag{14}$$

$$\text{Therefore, } S_{1,2}^* = \frac{((\alpha + \gamma) \pm |\alpha - \gamma|)N}{2\alpha}.$$

There are two equilibrium points for both $\alpha < \gamma$ and $\alpha > \gamma$, namely, the disease-free equilibrium point $(S_1^*, I_1^*) = (N, 0)$ and the endemic equilibrium point $(S_2^*, I_2^*) = (\gamma N / \alpha, N - \gamma N / \alpha)$. However, this second equilibrium point only satisfies the conditions $0 < S(t) < N, 0 < I(t) < N$ when $\alpha > \gamma$. Consequently, we first consider $\alpha > \gamma$.

4.2 Stability

We consider a first-order dynamic equation in the following form:

$$x^\Delta(t) = F(t, x(t))$$

where $x(t)$ is the value of x at time t .

The conditions for asymptotic stability of equilibrium points, x^* , are obtained by linearization of the equations [23]. For the discrete time scale, the condition is that $|dF/dx| < 1$, where $x = x^*$, and for the continuous time scale, the condition is that the real part of $\frac{dF}{dx} < 0$, where $x = x^*$.

To determine the asymptotic stability of the discrete case (13) we look at $S(t)$ close to S^* where S^* is the equilibrium point and define

$$S(t) = S^* + \tilde{S}(t) \tag{15}$$

where $\tilde{S}(t)$ is a small quantity termed a perturbation of the equilibrium point S^* . Then,

$$\begin{aligned} \tilde{S}(\sigma(t)) &= S(\sigma(t)) - S^* \\ &= f(S(t)) - S^* \\ &= f(S^* + \tilde{S}(t)) - S^* \end{aligned} \tag{16}$$

and a Taylor series expansion of $f(S(t))$ about the point S^* gives:

$$f(S^* + \tilde{S}(t)) = f(S^*) + \left(\frac{df}{dS} \Big|_{S^*} \right) \tilde{S}(t) + O(\tilde{S}^2(t)).$$

$O(\tilde{S}^2(t))$ is very small and can be neglected. The linear approximation for (16) is:

$$\tilde{S}(\sigma(t)) = \left(\frac{df}{dS} \Big|_{S^*} \right) \tilde{S}(t) = a\tilde{S}(t).$$

Thus, if $|a| < 1$, then the equilibrium point is asymptotically stable.

For the period time scale,

$$\mathbb{T} = \mathbb{P}_{l,h} := \bigcup_{k=0}^{\infty} [k(l+h), k(l+h)+l]$$

where $l, h > 0$ and $k \in \mathbb{N}_0$,

Consider the continuous interval

$$t \in [k(l+h), k(l+h)+l]$$

From $\frac{dS}{dt} = \frac{\alpha}{N} S^2 - (\alpha + \gamma)S + \gamma N$ the equilibrium points are $\bar{S}_{1,2} = \frac{\gamma N}{\alpha}, N$.

Stability of the steady-state solution on the continuous interval is the same as the stability of the steady-state solution of the differential equation. By using linearization from (12) and (15), we have

$$\frac{d(S^* + \tilde{S}(t))}{dt} = F(S^* + \tilde{S}(t))$$

$$\frac{d\tilde{S}(t)}{dt} = F(S^*) + \left(\frac{dF}{dS} \Big|_{S^*} \right) \tilde{S}(t) + \left(\frac{d^2 F}{dS^2} \Big|_{S^*} \right) \tilde{S}^2(t) + \dots$$

$$\frac{d\tilde{S}(t)}{dt} = (\gamma - \alpha)\tilde{S}(t) = \lambda\tilde{S}(t)$$

Therefore, $\tilde{S}(t) = Ce^{\lambda t}$; $0 \leq t < l$ $\tilde{S}(0) = \tilde{S}_0$,

$$\tilde{S}(t) = \tilde{S}_0 e^{\lambda t}.$$

At $t = l$; $\tilde{S}(l) = \tilde{S}_0 e^{\lambda l}$

At $t = k(l+h) + l$

$$\begin{aligned} S((k+1)(l+h)) &= \frac{\alpha}{N} S^2(k(l+h) + l) \\ &\quad - (\alpha + \gamma - 1)S(k(l+h) + l) + \gamma N \\ &:= G(S(k(l+h) + l)) \end{aligned}$$

To find equilibrium point we solve

$$\begin{aligned} S(k(l+h) + l) &= \frac{\alpha}{N} S^2(k(l+h) + l) \\ &\quad - (\alpha + \gamma - 1)S(k(l+h) + l) + \gamma N \end{aligned}$$

The equilibrium points are $S_{1,2}^* = \frac{\gamma N}{\alpha}, N$.

At $k = 0$;

$$S^* + \tilde{S}(l+1) = G(S^* + \tilde{S}(l))$$

$$= G(S^*) + \left(\frac{dG}{dS} \Big|_{S^*} \right) \tilde{S}(l) + \left(\frac{d^2G}{dS^2} \Big|_{S^*} \right) \tilde{S}^2(l) + \dots$$

$$\begin{aligned} \tilde{S}(l+1) &= (\gamma - \alpha + 1)\tilde{S}(l) \\ &= (\gamma - \alpha + 1)\tilde{S}_0 e^{\lambda l} \end{aligned}$$

$$\tilde{S}(t) = Ce^{\lambda t}$$

$$\begin{aligned} \tilde{S}(l+1) &= Ce^{\lambda(l+1)} \\ &= (\gamma - \alpha + 1)\tilde{S}_0 e^{\lambda l} \end{aligned}$$

$$C = (\gamma - \alpha + 1)\tilde{S}_0 e^{-\lambda}$$

$$\tilde{S}(t) = (\gamma - \alpha + 1)\tilde{S}_0 e^{\lambda(t-1)} \text{ where } l+1 \leq t \leq 2l+1.$$

For the general solution

$$S(t) = \left(\frac{\gamma - \alpha + 1}{e^\lambda} \right)^k S_0 e^{\lambda t} \text{ where } t \in [k(l+h), k(l+h) + l]$$

At $t = k(l+h)$,

$$S(k(l+h)) = \left(\frac{\gamma - \alpha + 1}{e^\lambda} \right)^k S_0 e^{\lambda k(l+h)}$$

At $t = (k+1)(l+h)$,

$$S((k+1)(l+h)) = \left(\frac{\gamma - \alpha + 1}{e^\lambda} \right)^{k+1} S_0 e^{\lambda(k+1)(l+h)}$$

Therefore,

$$S((k+1)(l+h)) = \left(\frac{\gamma - \alpha + 1}{e^\lambda} \right) e^{\lambda(l+h)} S(k(l+h))$$

The stable equilibrium occurs when

$$\left| \left(\frac{\gamma - \alpha + 1}{e^\lambda} \right) e^{\lambda(l+h)} \right| < 1$$

For example, for $h=1$ we can rearrange this inequality and obtain

$l < -\frac{\ln|\gamma - \alpha + 1|}{\lambda}$ where λ is eigenvalue of the differential equation on the continuous region of length l

Here $\lambda = \text{Re}(\gamma - \alpha)$

Theorem 2. If the inequalities $-2 < (\alpha - \gamma)\mu < 0$ hold, then a disease-free equilibrium point $S_1^* = N$ is locally asymptotically stable. Otherwise, $S_1^* = N$ is unstable.

Proof. For (13), the asymptotic stability is given by

$$|a|_{S_1^* = N} = \left| \frac{2\alpha\mu S}{N} + (1 - \alpha\mu - \gamma\mu) \right|_{S_1^* = N} = |1 + \mu(\alpha - \gamma)| < 1.$$

Theorem 3. If the inequality $0 < (\alpha - \gamma)\mu < 2$ holds, then an endemic equilibrium point $S_2^* = \gamma N / \alpha$ is locally asymptotically stable. Otherwise, $S_2^* = \gamma N / \alpha$ is unstable.

Proof. For (13), the stability is determined by

$$|a|_{S_2^*} = \left| \frac{2\mu\alpha}{N} \frac{\gamma N}{\alpha} + (1 - \alpha\mu - \gamma\mu) \right| = |1 + \mu(\gamma - \alpha)| < 1.$$

4.3 Initial Conditions

Lemma 1. For $(\alpha + \gamma)\mu - 1 < 0$, $\gamma\mu < 1$ and $\alpha > \gamma$, the solutions to the single-population SIS model are positive for all initial conditions $[0, N]$.

Lemma 2. The solutions to SIS epidemic model are positive for all initial conditions if and only if $0 \leq 1 - \gamma\mu < \alpha\mu < (1 + \sqrt{\gamma\mu})^2$ and $\alpha > \gamma$.

Proof. It is similar to the proof in [1].

Lemma 3. For $(\alpha + \gamma)\mu - 1 > 0$, $\alpha > \gamma$, and $\gamma\mu > 1$, the solutions to the single population SIS model are positive for initial conditions $\left(\frac{N(\gamma\mu - 1)}{\alpha\mu}, N \right)$.

Proposition 1. The solutions of the SIS epidemic model of (5) and (6) remain nonnegative and are bounded under conditions stated in Theorems 2, 3 and Lemmas 1-3.

4.4 The Period-Doubling Route to Chaos

To find the period two cycle, we need to find the solutions of $f(f(S(t))) = S(t)$. In addition to the equilibrium point of (13) given by

$$S(t) = S(\sigma(t)) = f(S(t)),$$

there are two more equilibrium points of (13) given by

$$S(t) = f(S(\sigma(t))) = f(f(S(t))) = f^2(S(t)),$$

which form a period 2 cycle.

The two points on the period 2 cycle are

$$\bar{S}_{1,2} = \left(\alpha\mu + \gamma\mu - 2 \pm \sqrt{(\alpha - \gamma)^2 \mu^2 - 4} \right) N / (2\alpha\mu).$$

The period two cycle exists when the square root is real, i.e., when $(\alpha - \gamma)\mu > 2$.

The stability is determined by $|a|_{S=\bar{S}_1}|a|_{S=\bar{S}_2}| < 1$.

From this condition it can be shown that the period 2 cycle $\bar{S}_{1,2}$ is locally asymptotically stable if $2 < (\alpha - \gamma)\mu < \sqrt{6}$, Otherwise the cycle $\bar{S}_{1,2}$ is unstable.

To find the period 2^n cycle, let $f^{2^n}(S(t)) = S(t)$ and solve for equilibrium points (\hat{S}). The stability is considered by $|a|_{S=\hat{S}_1}|a|_{S=\hat{S}_2} \dots |a|_{S=\hat{S}_n}| < 1$. It is extremely complicated, if not impossible, to find these higher-order limit cycles by analytical methods, and therefore numerical methods are useful.

5 Numerical Result

This section shows the numerical solutions for many different type of time scales. We begin by looking at how the behavior of the solutions changes for a combination of continuous and discrete time scales.

5.1 Combination of continuous and discrete time scales.

For the same values of parameters $\alpha = 3.6$, $\gamma = 0.9$ and $N = 100$, there are various behaviors of the solution depending on the values of the parameters l and h . Fig. 3 shows that the positive equilibrium is asymptotically stable on the continuous time scale $\mathbb{P}_{1,0}$ i.e., for discrete jump $h = 0$. Fig. 4 shows that the positive equilibrium on time scale $\mathbb{P}_{1,1}$ is also asymptotically stable, i.e., for continuous region and jump both of length 1. The gaps in the solution are due to the discrete time jumps μ of length $h = 1$. In the real world application, the mosquito population increases drastically with the onset of heavy rainfall. Therefore, the dengue fever has a high in transmission in rainy season [24]. Fig. 5 shows that the positive equilibrium on time scale $\mathbb{P}_{0,00001,1}$ is unpredictable. Fig. 3-Fig. 5 show that as the length of the continuous interval increases the positive stable equilibrium point is reached more quickly. The first jumps in the values of S in Fig. 4 and Fig. 5 are the same and are from $S(t) = 23.7578$ to $S(t+1) = 27.1672$. They are the same because the

size of the jump in S is fixed by the S value before the jump is 23.7578 and the length of the time jump $h = 1$. However, later jumps in S in the two figures are different because the size of S at the end of a continuous region depends on the length of the continuous region.

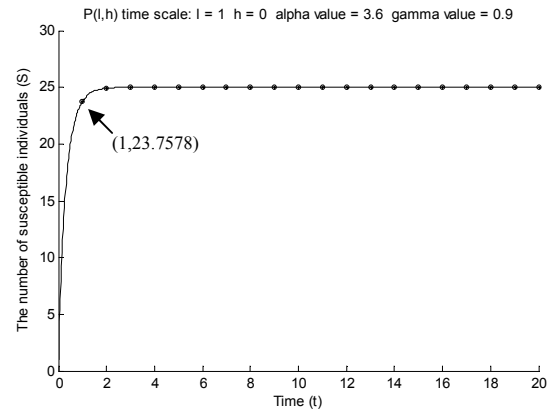


Fig. 3 The time series solution of (12) on time scale $\mathbb{P}_{1,0}$ therefore $\mu = 0$ with $\alpha = 3.6$, $\gamma = 0.9$ and $N = 100$. The result appears as a non-oscillatory solution. For $h = 0$, the time scale is a continuous time scale.

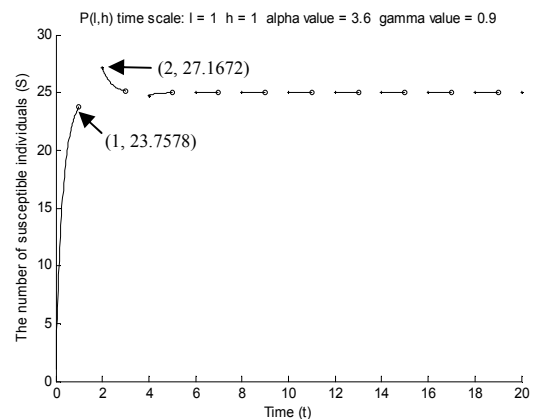


Fig. 4 The time series solution of (12) on time scale $\mathbb{P}_{1,1}$ therefore $\mu = 1$ with $\alpha = 3.6$, $\gamma = 0.9$ and $N = 100$. The result appears as a non-oscillatory solution which is similar to the result in Fig. 3 for a continuous time scale. The result disappears in some intervals because of the discrete time jump.

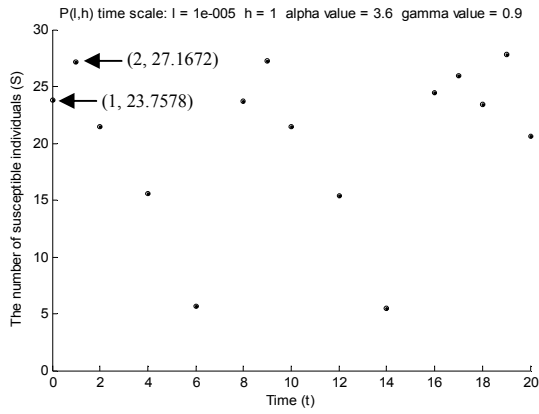


Fig. 5 The time series solution of (12) on time scale $\mathbb{P}_{0.00001,1}$ therefore $\mu=1$ with $\alpha=3.6$, $\gamma=0.9$ and $N=100$. The result appears unpredictable solution. The length of continuous time interval is very small and time scale approximates a discrete time scale.

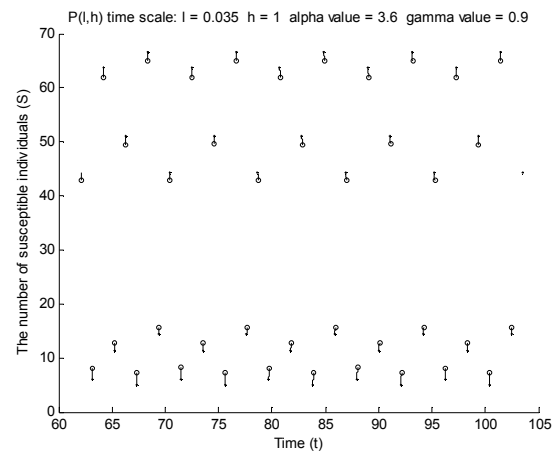


Fig. 8 The time series solution of (12) on time scale $\mathbb{P}_{0.035,1}$ with $\alpha=3.6$, $\gamma=0.9$ and $N=100$. The result appears as a period eight cycle with a very small continuous interval.

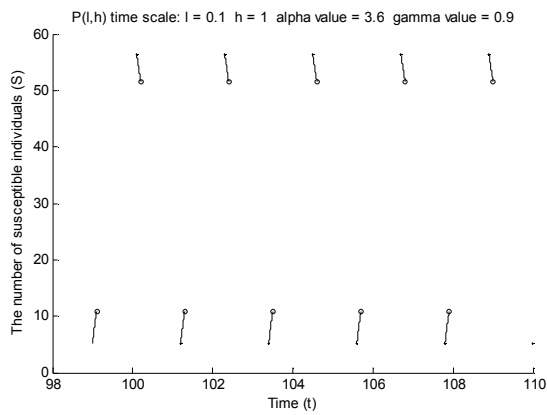


Fig. 6 The time series solution of (12) on time scale $\mathbb{P}_{0.1,1}$ with $\alpha=3.6$, $\gamma=0.9$ and $N=100$. The result appears as a period two cycle with some continuous interval.

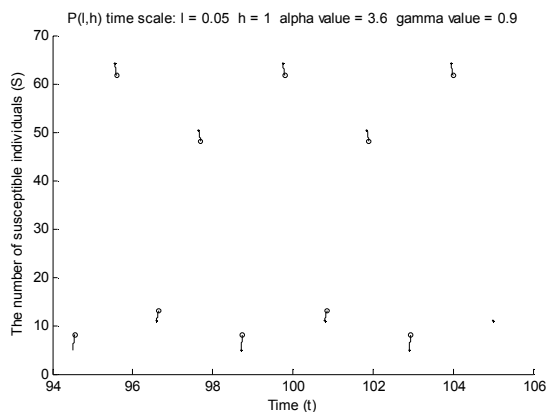


Fig. 7 The time series solution of (12) on time scale $\mathbb{P}_{0.05,1}$ with $\alpha=3.6$, $\gamma=0.9$ and $N=100$. The result appears as a period four cycle with some small continuous interval.

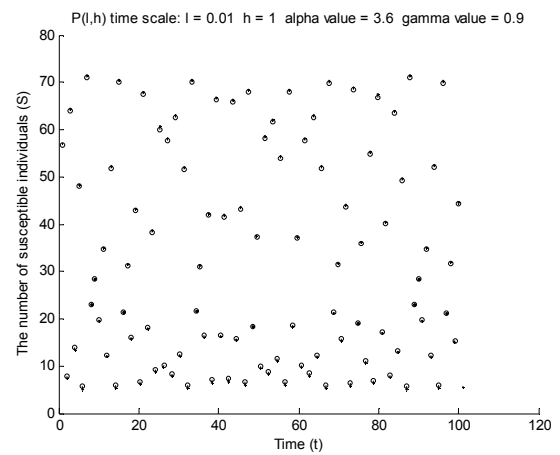


Fig. 9 The time series solution of (12) on time scale $\mathbb{P}_{0.01,1}$ with $\mu=1$, $\alpha=3.6$, $\gamma=0.9$ and $N=100$. The result appears unpredictable solution. The length of continuous time interval is small. The behavior on this time scale is similar to the behavior on a discrete time scale.

Fig. 10 shows how the behavior of the solution changes as the length of the continuous time region is reduced keeping the discrete time jump unchanged. The parameter values are $\alpha=3.6$, $\gamma=0.9$ and $N=100$. The result is the bifurcation path to chaos shown in Fig. 10. Two nontrivial equilibria occur until $l=0.1965$.

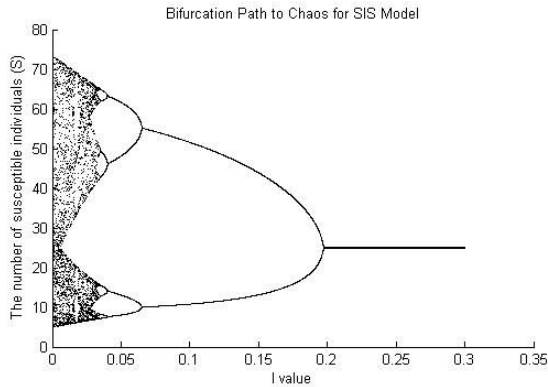


Fig. 10 The bifurcation diagram of l . The parameter values in (12) are: $\alpha = 3.6, \gamma = 0.9$.

We will now look at the discrete time scale case in more detail.

5.2 Discrete Time Scales

In the discrete time scale, the period four cycle can be obtained by numerical computation of the equilibrium points of $f^4(S(t)) = S(t)$. For parameter values $\alpha = 3.6, \gamma = 0.9$, and $N = 100$, the system of dynamic equations has a stable period two cycle when $\mu \in (0.740741, 0.907218)$, and a stable period four cycle when $\mu \in (0.907218, 0.942256)$.

From [3], [25], the SIS epidemic model can be transformed to the discrete logistic model

$$x(\sigma(t)) = f(x(t)) = rx(t)(1 - x(t)),$$

by the substitutions $x(t) = \frac{\alpha\mu I(t)}{N(1 - \gamma\mu + \alpha\mu)}$ and

$$r = 1 - \gamma\mu + \alpha\mu.$$

r is a bifurcation parameter in the logistic model while μ is a bifurcation parameter in the SIS epidemic model. However, as stated above r and μ are related by $r = 1 - \gamma\mu + \alpha\mu$.

$0 < (\alpha - \gamma)\mu < 2$ gives the inequality $1 < 1 + (\alpha - \gamma)\mu < 3$, which corresponds to the condition $1 < r < 3$, which is the condition for asymptotic stability of a non-zero equilibrium point for the logistic model.

From [26], the ratio $(\mu_n - \mu_{n-1}) / (\mu_{n+1} - \mu_n)$ is equivalent to $(r_n - r_{n-1}) / (r_{n+1} - r_n)$ which is called the Myrberg or Feigenbaum number δ .

From analysis (see, e.g., [6]) this ratio approaches a constant,

$$\delta = \lim_{n \rightarrow \infty} (\mu_n - \mu_{n-1}) / (\mu_{n+1} - \mu_n) \approx 4.669202$$

Since

$$\frac{(r_n - r_{n-1})}{(r_{n+1} - r_n)} = \frac{((1 + (\alpha - \gamma)\mu_n) - (1 + (\alpha - \gamma)\mu_{n-1}))}{((1 + (\alpha - \gamma)\mu_{n+1}) - (1 + (\alpha - \gamma)\mu_n))} = \frac{(\mu_n - \mu_{n-1})}{(\mu_{n+1} - \mu_n)}$$

Some numerical estimates of the Feigenbaum number are given in Table II. These estimates are, however, subject to appreciable numerical errors as the limit for δ approaches 0/0.

TABLE 2
The Feigenbaum Constant

n	μ_n	$\mu_n - \mu_{n-1}$	$(\mu_n - \mu_{n-1}) / (\mu_{n+1} - \mu_n)$
1	0.740741		
2	0.907218	0.166477	4.751327
3	0.942256	0.035038	4.656831
4	0.94978	0.007524	4.66685
5	0.951392	0.001612	4.665595
6	0.951738	0.000346	4.688442
7	0.951811	7.37E-05	

Fig. 11 shows how the solution behavior of (13) changes for $\alpha = 3.6, \gamma = 0.9$. For $\mu \in (0, 0.740741)$ the discrete equation has a stable equilibrium point, which corresponds with the stable equilibrium point of the continuous SIS model. For $\mu = 1$, the solution is chaotic.

As shown in [6] (see also [27]), if there exists a period 3 cycle, then there exists chaotic behavior. For $\alpha = 3.6, \gamma = 0.9$, the bifurcation diagrams show a period three cycle for $\mu \in (1.0476, 1.0524)$ and also show chaos.

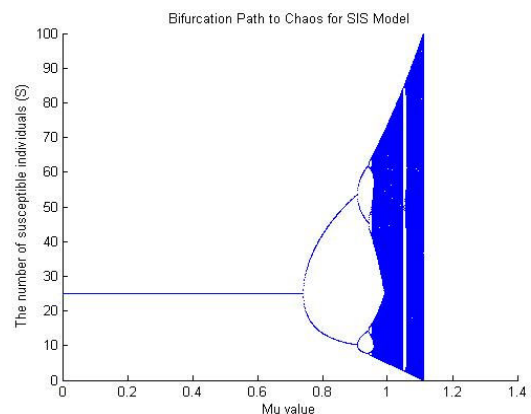


Fig. 11 The bifurcation diagram of μ . The parameter values in (13) are: $\alpha = 3.6, \gamma = 0.9$.

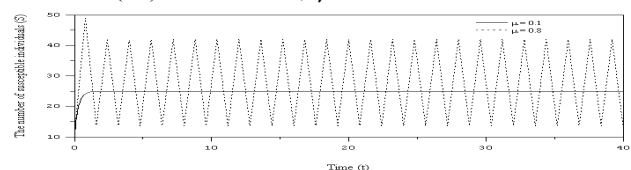


Fig. 12 The time series solution of (13) with $\alpha = 3.6, \gamma = 0.9$ and $N = 100$. The non-oscillatory solution occurs when $\mu = 0.1$ while oscillating period-2 solution occurs when $\mu = 0.8$.

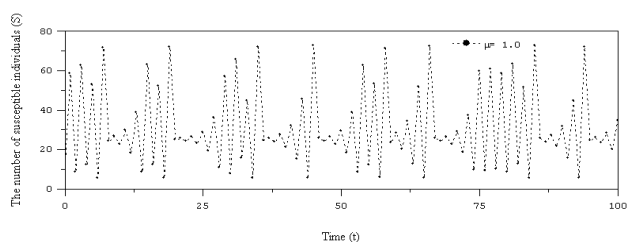


Fig. 13 The time series solution of (13) with $\alpha = 3.6$, $\gamma = 0.9$, and $N = 100$. The chaos occurs when $\mu = 1$.

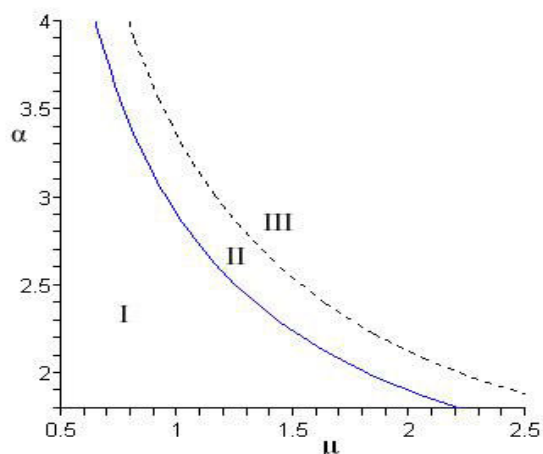


Fig. 14 The x-axis is μ and y-axis is α . Area I is region of stable equilibrium point, area II is region of stable period two cycle, and area III is region of stable higher period cycles and chaos. Parameter value $\gamma = 0.9$.

6 Conclusion

In this paper, the time space of the model is important because the behavior is changed in each model. For the continuous interval, if it is big enough, then the behavior of the model is similar to continuous model. If the continuous interval is small, then the behavior of the model is close to discrete model. Since the collecting of data could not be continuous, therefore the discrete model is important. The results of the analysis in this paper show that for an SIS model, the predictions of the model depend critically on the time scales used.

References:

[1] L. J. S. Allen, Some discrete-time SI, SIS, and SIR epidemic models, *Mathematical Biosciences*, Vol. 124, 1994, pp. 83-105.
 [2] L. J. S. Allen and A. B. Burgin, Comparison of deterministic and stochastic SIS and SIR models in discrete time, *Mathematical Biosciences*, Vol. 163, 2000, pp. 1-33.
 [3] S. Chokprasit, K. Kittivachpokawat, T. Nantawannakorn, E. J. Moore, and P. Khumsup, Teaching Modeling of Disease Transmission using Mathematical Software, in

Proc. First Thailand International Conference on 21st Century Information Technology in Mathematics Education, 2006, pp. 214-224.

- [4] W. M. Getz and J. O. Lloyd-Smith, Basic Methods for Modeling the Invasion and Spread of Infectious Diseases, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, Vol. 71, 2006, pp. 1-23.
 [5] S. R.-J. Jang, Backward bifurcation in a discrete SIS model with vaccination, *Journal of Biological Systems*, Vol.16, 2008, pp. 479-494.
 [6] M. R. S. Kulenovic and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*. Boca Raton, FL, Chapman and Hall/CRC, 2002.
 [7] S. P. Ellner and J. Guckenheimer, *Dynamic Models in Biology*. Princeton, NJ: Princeton University Press, 2006.
 [8] H. Mejhed, S. Boussaa, and N. Mejhed, Data warehouse and the deployment of data mining process to make decision for Leishmaniasis and development of mathematical models for Phlebotomine sandflies seasonality in Marrakech city, *WSEAS TRANSACTIONS on BIOLOGY and BIOMEDICINE*, Vol. 6, 2009, pp. 12-25.
 [9] Centers for Disease Control and Prevention (2010, May 28) [Online] Available <http://www.cdc.gov/flu/weekly/>
 [10] P. Pongsumpun and IM. Tang, Transmission Model for *Plasmodium Vivax* Malaria, *Proceedings of the 3rd WSEAS/IASME International Conference on Dynamical Systems and Control*, Arcachon, France, 2007, pp. 276-281.
 [11] R. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, Dynamic equations on time scales: a survey, *Journal of Computational and Applied Mathematics*, Vol. 141, 2002, pp. 1-26.
 [12] E. Akin, L. Erbe, A. Peterson, and B. Kaymakçalan, Oscillation results for a dynamic equation on a time scale, *Journal of Difference Equations and Applications*, Vol. 7, 2001, pp. 793-810.
 [13] D. R. Anderson, Oscillation and nonoscillation criteria for two-dimensional time-scale systems of first-order nonlinear dynamic equations, *Electronic Journal of Differential Equations*, Vol. 2009, No.24, 2009, pp. 1-13.
 [14] J. Hoffacker and C. C. Tisdell, Stability and instability for dynamic equations on time scales, *Computers and Mathematics with Applications*, Vol. 49, 2005, pp. 1327-1334.
 [15] Y. Xu and Z. Xu, Oscillation criteria for two-dimensional dynamic systems on time scales,

- Journal of Computational and Applied Mathematics*, Vol. 225, 2009, pp. 9-19.
- [16] W. Sae-jie and K. Bunwong, Numerical verification of certain oscillation results on time scales, *Neural, Parallel & Scientific Computations*, Vol. 17, 2009, pp. 317-338.
- [17] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*. Rensselaer, NY: Birkhäuser, 2001.
- [18] F. M. Atici, D. C. Biles, and A. Lebedinsky, An application of time scales to economics, *Mathematical and Computer Modelling*, Vol. 43, 2006. pp. 718-726.
- [19] Z. Siming, L. Yanhui and R. Zhixin, Region qualitative analysis of predator-prey system on time scales, *Ann. of Diff. Eqs.* Vol. 24, 2008, pp. 121-126.
- [20] D. Thomas, M. Weederemann, L. Billings, J. Hoffacker, and R. A. Washington-Allen, When to spray: a time-scale calculus approach to controlling the impact of West Nile virus, *Ecology and Society*, Vol. 14, 2009.
- [21] E. Duke, K. Hall, and R. Oberste-Vorth, Changing time scales I: the continuous case as a limit, *Proceeding of the Sixth WSEAS International Conference on Applied Mathematics*, 2004.
- [22] W. Sae-jie, K. Bunwong, E. J. Moore, Qualitative behavior of SIS epidemic model on time scales, *Proceedings of the 4th International Conference on Applied Mathematics, Simulation, Modelling (ASM '10)*, 2010. pp. 159-164.
- [23] L. Edelstein-Keshet, *Mathematical Models in Biology*. New York: The Random House/Birkhäuser mathematics series, 1988.
- [24] P. Pongsumpun, Dengue Disease Model with the Effect of Extrinsic Incubation Period, *Proceedings of the 2006 WSEAS Internatinal Conference on Mathematical Biology and Ecology*, Florida, USA, 2006, pp. 43-48.
- [25] R. M. May, Simple mathematical models with very complicated dynamics, *Nature*, Vol. 261, 1976, pp. 459-467.
- [26] S. N. Elaydi, *Discrete Chaos with Applications in Science and Engineering*. Boca Raton, FL, Chapman & Hall/CRC, 2008.
- [27] Tien-yien Li and J. A. Yorke. "Period three implies chaos", *The American Mathematical Monthly*, Vol. 82, 1975, pp. 965-992.