Blow-up solutions of degenerate parabolic problems

P. SAWANGTONG[1] and W. JUMPEN[2,3*]

[1]Dept of Mathematics, Faculty of Applied Science
King Mongkut's University of Technology North Bangkok, Thailand
[2]Dept of Mathematics, Faculty of Science, Mahidol University,Thailand
[3]Center of Excellence in Mathematics, PERDO, CHE, Thailand
pa_sawangtong@yahoo.com and scwjp@mahidol.ac.th
*Corresponding author

Abstract: - In this article, we study the degenerate parabolic problem, $x^q u_t - (x^\beta u_x)_x = x^q f(u),$ satisfying the Dirichlet boundary condition and a nonnegative initial condition where q and β are given constants and f is a suitable function. We show that under certain conditions the degenerate parabolic problem has a blow-up solution and the blow-up set of such a blow-up solution is the whole domain of $x.$ Furthermore, we give the sufficient condition to blow-up in finite time. Finally, we generalize the degenerate parabolic problem into the general form, $k(x)u_t - (p(x)u_x)_x = k(x)f(u).$ Under appropriate assumptions on functions k, p and $f,$ we still obtain the same results as the previous problem.

Key-Words: - Degenerate parabolic problems, Finite time blow-up, Blow-up set, Blow-up solution.

1 Introduction

Let T be any positive real number and q and β be positive constants with $q > 0,$ $\beta \in [0,1)$ and $q + \beta \neq 0.$ Let $I = (0,1),$ $Q_T = I \times (0,T),$ $\bar{I}$ and $\bar{Q}_T$ be their closure of I and $Q_T,$ respectively. In this article, we study the degenerate parabolic initial-boundary value problem,

$$\left.\begin{aligned} x^q u_t - (x^\beta u_x)_x &= x^q f(u(x,t)),\ (x,t) \in Q_T, \\ u(0,t) &= 0 = u(1,t),\ t \in (0,T), \\ u(x,0) &= u_0(x),\ x \in \bar{I}, \end{aligned}\right\} \tag{1}$$

where f and u_0 are suitable functions. In 1985, C. E. Mueller and F. B. Weissler [12] studied the behavior of solutions to the semilinear heat equation

$$\left.\begin{aligned} u_t &= \Delta u - \lambda u + f(u),\ (x,t) \in \Omega \times (0,\infty), \\ u(x,t) &= 0,\ (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x),\ x \in \Omega, \end{aligned}\right\} \tag{2}$$

where Ω is $\mathbb{R}^n$ or a smooth bounded domain in $\mathbb{R}^n,$ $\partial\Omega$ denotes the smooth boundary of $\Omega,$ $\Delta = \sum_{i=1}^{n} \partial_i^2,$ $\lambda \geq 0,$ f and u_0 are given function. They proved that, under appropriated hypothesis, solutions of problem (2) blow up in finite time and in fact blow up only at a single point. Further, in 2009, J. P. Pinasco [13] established the blow-up positive solutions of parabolic problems with reaction terms of local and nonlocal type involving a variable exponent,

$$\begin{aligned} u_t &= \Delta u + f(u),\ (x,t) \in \Omega \times (0,\infty), \\ u(x,t) &= 0,\ (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x),\ x \in \Omega, \end{aligned}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega,$ and the source term is of the form $f(u) = a(x)u^{p(x)}$ or $f(u) = a(x)\int_\Omega u^{q(x)}(y,t)dy$ with given functions a, p and $q.$ For blow-up problems of the degenerate semilinear parabolic type, in 1999, C. Y. Chan and W. Y. Chan [3] proved the existence of a blow-up solution of the degenerate semilinear parabolic initial-boundary value problem

$$\left.\begin{aligned} x^q u_t - u_{xx} &= f(u),\ (x,t) \in Q_T, \\ u(0,t) &= 0 = u(1,t),\ t \in (0,T), \\ u(x,0) &= u_0(x),\ x \in \bar{I}, \end{aligned}\right\} \tag{3}$$

where f and u_0 are given functions. They proved existence and uniqueness of a blow-up solution of problem (3) by transforming problem (2) into the equivalent integral equation in terms of its Green's function. Furthermore, in 2006, C. Y. Chan and W. Y. Chan [4] showed that under certain condition on functions f and $u_0,$ a solution u of problem (3) blows up at every point in $I.$ After paper [3] published, in 2004, Y.P. Chen and C.H. Xie [8] discussed the degenerate parabolic equation with the nonlocal term:

$$\left.\begin{aligned} u_t-(x^\beta u_x)_x &= \int_0^1 f(u(x,t))dx \text{ for } (x,t)\in Q_T,\\ u(0,t)&=0=u(1,t) \text{ for } t\in(0,T),\\ u(x,0)&=u_0(x) \text{ for } x\in \bar{I}.\end{aligned}\right\} \quad (4)$$

They consider the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they obtained some sufficient conditions for the global existence and blow-up of a positive solution of problem (4). Additionally, in 2004, Y.P. Chen, Q. Liu and C.H. Xie [7] studied the degenerate nonlinear reaction-diffusion equation with nonlocal source:

$$\left.\begin{aligned} x^q u_t-(x^\beta u_x)_x &= \int_0^1 u^p(x,t)dx \text{ for } (x,t)\in Q_T,\\ u(0,t)&=0=u(1,t) \text{ for } t\in(0,T),\\ u(x,0)&=u_0(x) \text{ for } x\in \bar{I}.\end{aligned}\right\} \quad (5)$$

They established the local existence and uniqueness of a classical solution of problem (5). Under appropriate hypotheses, they also get some sufficient conditions for a global existence and blow-up of a positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution of problem (5) is the whole domain. In 2010, P. Sawangtong, B. Novaprateep and W. Jumpen [15] established the existence of a blow-up solution of the following degenerate parabolic problem with the localized nonlinear source:

$$\left.\begin{aligned} u_t-\frac{1}{k(x)}\big(p(x)u_x\big)_x &= f(u(x_0,t)) \text{ for } (x,t)\in Q_T,\\ u(0,t)&=0=u(1,t) \text{ for } t\in(0,T),\\ u(x,0)&=u_0(x) \text{ for } x\in \bar{I},\end{aligned}\right\} \quad (6)$$

where k, p, f and u_0 are suitable functions. In 2010 P. Sawangtong and W. Jumpen [14] studied the degenerate parabolic problem (6). In this article we continuos to study the degenerate parabolic problem (6) and our objective is to show that before blow-up phenomenon will occur, the degenerate parabolic problem (1) has a unique continuous solution u on the finite time interval T_1 with $T_1>0$ for any $x\in\bar{I}$. Moreover, the sufficient condition to guarantee occurrence of finite time blow-up and the blow-up of such a blow-up solution are given. In order to make more complete, we extend our degenerate parabolic problem (1) into the more general form: $k(x)u_t-(p(x)u_x)_x=k(x)f(u(x_0,t))$ where k and p are given functions. Under some conditions, we also obtain the same results as the degenerate parabolic problem (1). In order to obtain our results for degenerate parabolic problem (1), we need assumptions on functions f and u_0 as follows.

(A) $f\in C^2([0,\infty))$ is convex with $f(0)=0$ and $f(s)>0$ for $s>0$.

(B) $u_0\in C^2(\bar{I})$, $u_0(0)=0=u_0(1)$, u_0 is nonnegative on I and u_0 satisfies the inequality,

$$\frac{d}{dx}\left(x^\beta\frac{du_0}{dx}\right)+x^q f(u_0(x))\geq 0 \text{ on } I. \quad (7)$$

We note that by proposition 2.1 of [12], condition (A) implies that f is increasing and locally Lipschitz continuous on $[0,\infty)$.

A solution u of the degenerate parabolic problem (1) is said to blow-up at the point b in finite time $\hat{T}$ if there exists a sequence (x_n,t_n) with $t_n<\hat{T}$ such that $(x_n,t_n)\to(b,\hat{T})$ and $u(x_n,t_n)\to\infty$. Furthermore, the set consisting of all blow-up points of a blow-up solution is called the blow-up set.

This paper is organized as follows. In section 2, we find associating eigenfunctions and eigenvalues to the degenerate parabolic problem (1). We prove the existence and uniqueness of a solution of problem (1) before blow-up occurs by using the Banach fixed point theorem and give the sufficient condition to blow-up in finite time in section 3 and 4, respectively. In section 5, we give the blow-up set of such a blow-up solution of the degenerate parabolic problem (1). The extended problem, $k(x)u_t-(p(x)u_x)_x=k(x)f(u)$, of the degenerate parabolic problem, $x^q u_t-(x^\beta u_x)_x=x^q f(u)$, is studied in the last section.

2 Eigenvalues and Eigenfunctions

By using separation of variables [9] and [11] on the homogenous problem corresponding to problem (1), we obtain the singular eigenvalue problem,

$$\left.\begin{aligned} \frac{d}{dx}\left(x^\beta\frac{d\varphi}{dx}\right)+\lambda x^q\varphi(x)&=0 \text{ for } x\in I,\\ \varphi(0)=0&=\varphi(1).\end{aligned}\right\} \quad (8)$$

Let $\varphi(x)=x^{\frac{1-\beta}{2}}y(x)$. Then

$$\varphi'(x) = x^{\frac{1-\beta}{2}} y'(x) + \left(\frac{1-\beta}{2}\right) x^{\frac{-\beta-1}{2}} y(x) \qquad (9)$$

and

$$\varphi''(x) = x^{\frac{1-\beta}{2}} y''(x) + (1-\beta) x^{\frac{-\beta-1}{2}} y'(x) + \left(\frac{1-\beta}{2}\right)\left(\frac{-\beta-1}{2}\right) x^{\frac{-\beta-3}{2}} y(x). \qquad (10)$$

Substituting equation (9) and (10) in equation (8), we obtain

$$x^{\beta}\left[x^{\frac{1-\beta}{2}} y''(x) + (1-\beta) x^{\frac{-\beta-1}{2}} y'(x) + \left(\frac{1-\beta}{2}\right)\left(\frac{-\beta-1}{2}\right) x^{\frac{-\beta-3}{2}} y(x)\right]$$

$$+\beta x^{\beta-1}\left[x^{\frac{1-\beta}{2}} y'(x) + \left(\frac{1-\beta}{2}\right) x^{\frac{-\beta-1}{2}} y(x)\right] + \lambda x^{q} x^{\frac{1-\beta}{2}} y(x) = 0$$

or

$$x^{\frac{1+\beta}{2}} y''(x) + x^{\frac{\beta-1}{2}} y'(x) + \left[\lambda x^{q} x^{\frac{1-\beta}{2}} - \left(\frac{1-\beta}{2}\right)^{2} x^{\frac{\beta-3}{2}}\right] y(x) = 0. \qquad (11)$$

Dividing both sides of equation (11) by $x^{\frac{1+\beta}{2}}$, we get

$$y''(x) + \frac{1}{x} y'(x) + \left[\lambda x^{q-\beta} - \left(\frac{1-\beta}{2}\right)^{2} \frac{1}{x^{2}}\right] y(x) = 0. \qquad (12)$$

Multiplying both side of equation (12) by x^2, the singular eigenvalue problem (8) becomes

$$\left.\begin{array}{c} x^{2} y''(x) + x y'(x) + \left[\lambda x^{q-\beta+2} - \left(\frac{1-\beta}{2}\right)^{2}\right] y(x) = 0, \\ y(0) \text{ is bounded and } y(1) = 0. \end{array}\right\} \qquad (13)$$

Again, we set $x = z^{\frac{2}{q-\beta+2}}$. Then

$$y'(x) = \left(\frac{q-\beta+2}{2}\right) z^{\frac{q-\beta}{q-\beta+2}} y'(z)$$

and

$$y''(x) = \left(\frac{q-\beta+2}{2}\right)^{2} z^{\frac{2(q-\beta)}{q-\beta+2}} y''(z) + \left(\frac{q-\beta}{2}\right)\left(\frac{q-\beta+2}{2}\right) z^{\frac{q-\beta-2}{q-\beta+2}} y'(z)$$

From equation (13), we have

$$z^{\frac{4}{q-\beta+2}}\left[\left(\frac{q-\beta+2}{2}\right)^{2} z^{\frac{2(q-\beta)}{q-\beta+2}} y''(z) + \left(\frac{q-\beta}{2}\right)\left(\frac{q-\beta+2}{2}\right) z^{\frac{q-\beta-2}{q-\beta+2}} y'(z)\right]$$

$$+\left(\frac{q-\beta+2}{2}\right) z y'(z) + \left[\lambda z^{2} - \left(\frac{1-\beta}{2}\right)^{2}\right] y(z) = 0$$

or

$$\left.\begin{array}{c} z^{2} y''(z) + z y'(z) + \left[\frac{4\lambda z^{2}}{(q-\beta+2)^{2}} - \frac{(1-\beta)^{2}}{(q-\beta+2)^{2}}\right] y(z) = 0, \\ y(0) \text{ is bounded and } y(1) = 0. \end{array}\right\} \qquad (14)$$

Thus, we see that equation (14) is a Bessel equation. Its general solution of a Bessel equation (14) is given by

$$y(z) = A J_{\mu}(\omega z) + B J_{-\mu}(\omega z)$$

where $\mu = \frac{1-\beta}{q-\beta+2}$, $\omega = \frac{2\lambda^{\frac{1}{2}}}{q-\beta+2}$, A and B are arbitrary constants and J_μ denotes the Bessel function of the first kind of order $\mu (> 0)$. Turning to the boundary condition, at $z = 0$ leads to $B = 0$ and then we obtain

$$y(z) = A J_{\mu}(\omega z). \qquad (15)$$

The boundary condition at $x = 1$ gives the following equation,

$$J_{\mu}(\omega) = 0. \qquad (16)$$

Then, by equation (15), the appropriate eigenfunctions φ_n of the singular eigenvalue problem (15) are

$$\varphi_n(x) = A x^{\frac{1-\beta}{2}} J_{\mu}\left(\omega_n x^{\frac{q-\beta+2}{2}}\right) \qquad (17)$$

where ω_n is the n^{th} root of equation (16). In order to obtain the orthonormal property of φ_n with the weight function x^q,

$\int_0^1 x^q \varphi_n(x)\varphi_m(x)dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$, we use the orthogonality of Bessel functions, that is,

$$\int_0^1 xJ_\mu(\omega_n x)J_\mu(\omega_m x)dx = \begin{cases} \frac{1}{2}J_{\mu+1}^2(\omega_n) & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases},$$

to determine the value of constant A. To do so, we consider

$$\int_0^1 x^q \varphi_n^2(x)dx = A^2 \int_0^1 x^{q-\beta+1} J_{\mu+1}^2(\omega_n x^{\frac{q-\beta+2}{2}})dx. \qquad (18)$$

Let $y = x^{\frac{q-\beta+2}{2}}$. Then $dy = \left(\frac{q-\beta+2}{2}\right) x^{\frac{q-\beta}{2}} dx.$ Let us consider the right-hand side of equation (18)

$$A^2 \int_0^1 x^{q-\beta+1} J_{\mu+1}^2(\omega_n x^{\frac{q-\beta+2}{2}})dx = \frac{2A^2}{q-\beta+2}\int_0^1 yJ_\mu^2(\omega_n y)dy$$

$$= \frac{A^2}{q-\beta+2} J_{\mu+1}^2(\omega_n). \quad (19)$$

It follows from (18) and (19) that

$$\int_0^1 x^q \varphi_n^2(x)dx = \frac{A^2}{q-\beta+2} J_{\mu+1}^2(\omega_n).$$

Since the right-hand side of equation (18) must equal to 1, the value of constant A is determined by $A = \frac{(q-\beta+2)^{\frac{1}{2}}}{\left|J_{\mu+1}(\omega_n)\right|}$. Hence, eigenfunctions φ_n of the singular eigenvalue problem (17) are defined by

$$\varphi_n(x) = \frac{(q-\beta+2)^{\frac{1}{2}} x^{\frac{1-\beta}{2}} J_\mu(\omega_n x^{\frac{q-\beta+2}{2}})}{\left|J_{\mu+1}(\omega_n)\right|}. \qquad (20)$$

Further, it follows from [1] that

$\lambda_n = O(n^2)$ as $n \to \infty$.

Lemma 2.1 For any $x \in \bar{I}$, $\left|\varphi_n(x)\right| \le c_0 x^{\frac{1-\beta}{2}} \lambda_n^{\frac{1}{4}}$ for some positive constant c_0.

Proof. By [1], we have

$$\left|J_\mu(\omega_n x^{\frac{q-\beta+2}{2}})\right| \le 1 \text{ for any } \mu > 0. \qquad (21)$$

It follows from [6] that

$$\frac{1}{\left|J_{\mu+1}(\omega_n)\right|} \le \left(\frac{\pi \lambda_n^{\frac{1}{2}}}{q-\beta+2}\right)^{\frac{1}{2}} c_0 \qquad (22)$$

where c_0 is some positive constant. Therefore, by equations (20), (21) and (22), the proof of this lemma is complete.

3 Local existence of blow-up solution

In this section, we will show the existence and uniqueness of a nonnegative continuous solution of a degenerate parabolic problem (1). Green's function $G(x,t,\xi,\tau)$ corresponding to the degenerate parabolic problem (1) is determined by the following system, for any x and ξ in I, and t and τ in $(0,T)$,

$$\left.\begin{aligned} x^q G_t(x,t,\xi,\tau) - (x^\beta G_x(x,t,\xi,\tau))_x &= \delta(x-\xi)\delta(t-\tau), \\ G(x,t,\xi,\tau) &= 0 \text{ for } t < \tau, \\ G(0,t,\xi,\tau) = 0 &= G(1,t,\xi,\tau), \end{aligned}\right\} \qquad (23)$$

where δ is the Dirac delta function. Let

$$G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x) \qquad (24)$$

Substituting equation (24) into equation (23), we obtain

$$x^q \sum_{n=1}^{\infty} a_n'(t)\varphi_n(x) - \sum_{n=1}^{\infty} a_n(t)\frac{d}{dx}\left(x^\beta \frac{d\varphi_n}{dx}\right) = \delta(x-\xi)\delta(t-\tau).$$

Multiplying both sides by φ_n and then integrating both sides with respect to x over its domain, we have

$$\int_0^1 x^q \varphi_n(x) \sum_{n=1}^{\infty} a_n'(t)\varphi_n(x)dx - \int_0^1 \varphi_n(x)\sum_{n=1}^{\infty} a_n(t)\frac{d}{dx}\left(x^\beta \frac{d\varphi_n}{dx}\right)dx$$
$$= \int_0^1 \varphi_n(x)\delta(x-\xi)\delta(t-\tau)dx.$$

By the orthonormal property of eigenfunctions φ_n and the property of Dirac delta function, we get

$$a_n'(t) + \lambda_n a_n(t) = \varphi_n(\xi)\delta(t-\tau)$$

or

$$\frac{d}{dt}\left(e^{\lambda_n t}a_n(t)\right) = \varphi_n(\xi)\delta(t-\tau)e^{\lambda_n t}.$$

Integrating both sides from t to t_1 with $t_1 < \tau$, we obtain

$$\int_{t_1}^{t} \frac{d}{ds}\left(e^{\lambda_n s}a_n(s)\right) = \int_{t_1}^{t} \varphi_n(\xi)\delta(x-\xi)e^{\lambda_n s}ds$$

or

$e^{\lambda_n t}a_n(t) - e^{\lambda_n t_1}a_n(t_1) = \varphi_n(\xi)e^{\lambda_n t}.$

Since $G(x,t,\xi,\tau) = 0$ for $t < \tau$, $a_n(t_1) = 0$ for all n. We then obtain that $a_n(t) = \varphi_n(\xi)e^{-\lambda_n(t-\tau)}$ for any n. Therefore the Green's function corresponding to problem (1) is defined by

$$G(x,t,\xi,\tau) = \sum_{n=1}^{\infty}\varphi_n(x)\varphi_n(\xi)e^{-\lambda_n(t-\tau)} \text{ for } t > \tau, \qquad (25)$$

where φ_n and λ_n are eigenfunctions and eigenvalues of the singular eigenvalue problem (8), respectively.

Lemma 3.1 For $t > \tau$, $G(x,t,\xi,\tau)$ is continuous for $(x,t,\xi,\tau) \in \overline{I} \times (0,T] \times \overline{I} \times [0,T)$.

Proof. By lemma 2.1 and equation (25), we obtain that

$$\left|\sum_{n=1}^{\infty}\varphi_n(x)\varphi_n(\xi)e^{-(t-\tau)}\right| \le c_0^2 x^{\frac{1-\beta}{2}}\xi^{\frac{1-\beta}{2}}\sum_{n=1}^{\infty}\lambda_n^{\frac{1}{2}}e^{-(t-\tau)}$$
$$\le c_0^2\sum_{n=1}^{\infty}\lambda_n^{\frac{1}{2}}e^{-(t-\tau)},$$

which converges uniformly. We then get the result.

Next lemma shows the positivity of Green's function G.

Lemma 3.2 For any $(x,t,\xi,\tau) \in I \times (0,T] \times I \times [0,T)$, $G > 0$ with $t > \tau$.

Proof. The proof of this lemma is similar to that of lemma 4.c of [5].

To derive the equivalent integral equation of a degenerate parabolic problem (1), let us consider the adjoint operator L^* defined by

$$L^* = -x^q\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(x^\beta\frac{\partial}{\partial x}\right)$$

corresponding to the operator

$$L = x^q\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(x^\beta\frac{\partial}{\partial x}\right)$$

of a degenerate parabolic problem (1). By Green's second identity, we obtain the equivalent integral equation to problem (1) given by

$$u(x,t) = \int_0^1 \xi^q G(x,t,\xi,0)u_0(\xi)d\xi + \int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)f(u(\xi,\tau))d\xi d\tau. \qquad (26)$$

Theorem 3.3 There exists a finite time $T_1 > 0$ such that a degenerate parabolic problem has a unique continuous solution u on the finite time interval $[0,T_1]$ for any $x \in \overline{I}$.

Proof. Let M be a positive constant with $M > \max_{x\in I} u_0(x) + 1$. Locally Lipschitz continuity of f implies that for any $|u| \le M$ and $|v| \le M$ there is a positive constant L depending on M such that $|f(u) - f(v)| \le L|u - v|$. Further, since $\int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)d\xi d\tau \to 0$ as $t \to 0$, there exists a finite time T_1 such that

$$f(M)\int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)d\xi d\tau < 1 \text{ for } t \in [0,T_1]$$

and

$$L\int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)d\xi d\tau < 1 \text{ for } t \in [0,T_1].$$

We next define the space E by

$$E = \left\{u \in C(\overline{Q}_{T_1}) \text{ such that } \sup_{(x,t)\in\overline{Q}_{T_1}}|u(x,t)| \le M\right\}.$$

Then, the space E is a Banach space equipped with the norm $|u|_E = \sup_{(x,t)\in\overline{Q}_{T_1}}|u(x,t)|$. Let Φ be a mapping defined by

$$\Phi u(x,t) = \int_0^1 \xi^q G(x,t,\xi,0)u_0(\xi)d\xi + \int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)f(u(\xi,\tau))d\xi d\tau.$$

In order to apply the Banach fixed point theorem we would like show that Φ maps E into itself and Φ is a contraction mapping. Let u and v be any element in E. We then have that

$$|\Phi u(x,t)| \le \left|\int_0^1 \xi^q G(x,t,\xi,0)u_0(\xi)d\xi\right| + \left|\int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)f(u(\xi,\tau))d\xi d\tau\right|. \qquad (27)$$

Let us consider the following auxiliary problem,

$$\left.\begin{aligned} x^q u_t - (x^\beta u_x)_x &= 0,\ (x,t) \in Q_{T_1}, \\ u(0,t) = 0 &= u(1,t),\ t \in (0,T), \\ u(x,0) &= u_0(x),\ x \in \overline{I}. \end{aligned}\right\} \qquad (28)$$

From (26), a solution u of problem (28) is given by, for any $(x,t) \in \overline{Q}_{T_1}$, $u(x,t) = \int_0^1 \xi^q G(x,t,\xi,0)u_0(\xi)d\xi.$

On the other hand, Maximum principle for parabolic type [10] implies that $u(x,t) \le \max_{x\in I} u_0(x)$ on $\overline{Q}_{T_1}$. We then obtain that, for any $(x,t) \in \overline{Q}_{T_1}$,

$\int_0^1 \xi^q G(x,t,\xi,0)d\xi \le 1.$ From inequality (27), we get that, for any $(x,t)\in \overline{Q}_{T_1}$,

$$|\Phi u(x,t)| \le \max_{x\in \overline{I}} u_0(x) + f(M)\int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)d\xi d\tau < M.$$

This shows that $\Phi u \in E$ for any $u\in E.$ Next, locally Lipschitz continuity of f yields

$$|\Phi u(x,t)-\Phi v(x,t)|$$
$$\le \int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)\left|f(u(\xi,\tau))-f(v(\xi,\tau))\right|d\xi d\tau$$
$$\le L\int_0^t\int_0^1 \xi^q G(x,t,\xi,\tau)d\xi d\tau |u-v|_E .$$

By definition of T_1, we obtain that Φ is a contraction mapping. Hence, by the Banach fixed point theorem, the equivalent integral equation (26) has a unique continuous solution u on $\overline{Q}_{T_1}$. The proof of this theorem therefore is complete.

Theorem 3.4 Let $T_{\max}$ be the supremum of all T_1 such that the solution u of the degenerate parabolic problem (1) is bounded. If $T_{\max}$ is finite, then $\sup_{(x,t)\in \overline{Q}_{T_{\max}}} |u(x,t)|$ is unbounded as t converges to $T_{\max}$.

Proof. Suppose that $\sup_{(x,t)\in \overline{Q}_{T_{\max}}} |u(x,t)|$ is finite. Let N be any positive constant with $N > \sup_{(x,t)\in \overline{Q}_{T_{\max}}} |u(x,t)|+1.$ By theorem 3.1, there exists a finite time $T_2 > T_{\max}$ such that the degenerate parabolic problem has a unique continuous solution on $\overline{Q}_{T_2}$. We then obtain a contradiction to definition of $T_{\max}$. Hence, we get the result.

Lemma 3.5 If $v\in C(\overline{Q}_T)\cap C^{2,1}(Q_T)$ satisfies

$$x^q v_t - (x^\beta v_x)_x \ge B(x,t)v(x,t),\ (x,t)\in Q_T,$$
$$v(0,t)\ge 0 \text{ and } v(1,t)\ge 0,\ t\in(0,T),$$
$$v(x,0)\ge 0,\ x\in \overline{I},$$

where $B(x,t)$ is bounded and nonnegative on $\overline{Q}_T$, then $v(x,t)\ge 0$ on $\overline{Q}_T$.

Proof. Let $\beta'\in(\beta,1)$ be a positive constant and

$$w(x,t) = v(x,t)+\eta(1+x^{\beta'-\beta})e^{ct}$$

where η is any positive constant and c is a positive constant to be determined. We see that, on the parabolic boundary, $w>0.$ Let us consider that for any $(x,t)\in Q_T$,

$$x^q w_t - (x^\beta w_x)_x - B(x,t)w(x,t)$$
$$\ge \eta e^{ct} c x^q (1+x^{\beta'-\beta}) - \eta e^{ct}(\beta'-\beta)(\beta'-1)\frac{1}{x^{2-\beta'}}$$
$$-\eta e^{ct}\left(\max_{(x,t)\in \overline{Q}_T} B(x,t)\right)(1+x^{\beta'-\beta})$$
$$\ge \eta e^{ct} c x^q (1+x^{\beta'-\beta}) + \eta e^{ct}(\beta'-\beta)(1-\beta')\frac{1}{x^{2-\beta'}}$$
$$-2\eta e^{ct}\left(\max_{(x,t)\in \overline{Q}_T} B(x,t)\right). \tag{29}$$

If $\max_{(x,t)\in \overline{Q}_T} B(x,t) \le \frac{1}{2}(\beta'-\beta)(1-\beta'),$ then, by equation (29),

$$x^q w_t - (x^\beta w_x)_x - B(x,t)w(x,t)$$
$$\ge \eta e^{ct} c x^q (1+x^{\beta'-\beta}) + \eta e^{ct}(\beta'-\beta)(1-\beta')\left[\frac{1}{x^{2-\beta'}}-1\right]$$
$$\ge 0.$$

On the other hand, we assume that

$$\max_{(x,t)\in \overline{Q}_T} B(x,t) > \frac{1}{2}(\beta'-\beta)(1-\beta').$$

Let $x_0(>0)$ be the root of equation,

$$\max_{(x,t)\in \overline{Q}_T} B(x,t) = \frac{1}{2x^{2-\beta'}}(\beta'-\beta)(1-\beta'),$$

and let $c = \frac{2}{x_0^q(1+x_0)^{\beta'-\beta}}\left(\max_{(x,t)\in \overline{Q}_T} B(x,t)\right).$ It follows from equation (29) that

if $x > x_0$, then the definition of c yields

$$x^q w_t - (x^\beta w_x)_x - B(x,t)w(x,t)$$
$$\ge \eta e^{ct}\left[c x^q(1+x^{\beta'-\beta}) - 2\max_{(x,t)} B(x,t)\right]$$
$$\ge \eta e^{ct}\left[c x_0^q(1+x_0^{\beta'-\beta}) - 2\max_{(x,t)} B(x,t)\right]$$
$$\ge 0,$$

and if $x\le x_0$, then

$$x^q w_t - (x^\beta w_x)_x - B(x,t)w(x,t)$$
$$\ge \eta e^{ct} c x^q (1+x^{\beta'-\beta}) + \eta e^{ct}(\beta'-\beta)(1-\beta')\frac{1}{x_0^{2-\beta'}}$$
$$-2\eta e^{ct}\left(\max_{(x,t)\in \overline{Q}_T} B(x,t)\right)$$
$$\ge 0.$$

Therefore

$$x^q w_t - (x^\beta w_x)_x - B(x,t)w(x,t)\ge 0 \text{ on } Q_T .$$

We would like to show that $w>0$ on Q_T. Suppose that there exists a point (x_1,t_1) with $w(x_1,t_1)\le 0.$ We define the set $A=\{t \text{ such that } w(x,t)\le 0 \text{ for some } x\in I\}$ is non-empty. Let $\tilde{t}$ denotes its infimum. Since $w(x,0)>0,$

we have $0<\tilde{t}<T$. Then there exists some $x_2 \in I$ such that $w(x_2,\tilde{t})=0,$ $w_t(x_2,\tilde{t})\le 0$ and $w_x(x_2,\tilde{t})=0.$ Since w attains its local minimum at $(x_2,\tilde{t}),$ we have $w_{xx}(x_2,\tilde{t})\ge 0.$ Thus

$$0 \ge x_2^q w_t(x_2,\tilde{t}) \ge x_2^q w_t(x_2,\tilde{t}) - (x_2^\beta w_x(x_2,\tilde{t}))_x - B(x_2,\tilde{t})w(x_2,\tilde{t}) > 0.$$

This contradiction shows that $w>0$ on $\overline{Q}_T$. As $\eta \to 0,$ we will get the result.

We will give additional properties of the solution u of the degenerate parabolic problem (1) in the next lemma.

Lemma 3.6 $u \ge u_0$ and $u_t \ge 0$ on $\overline{Q}_T$.

Proof. Let $w(x,t)=u(x,t)-u_0(x)$ on $\overline{Q}_T$. For any $(x,t)\in Q_T$, equation (7) implies that

$$\begin{aligned} x^q w_t - (x^\beta w_x)_x &= x^q f(u) + \frac{d}{dx}\left(x^\beta \frac{du_0}{dx}\right) \\ &\ge x^q \left(f(u) - f(u_0)\right) \\ &= x^q f'(\zeta_1) w(x,t) \end{aligned}$$

where ζ_1 is a positive constant between u and u_0. Further, on the parabolic boundary, $w\ge 0.$ Then lemma 3.5 yields that $u \ge u_0$ on $\overline{Q}_T$. Let h be any positive constant with $h\in(0,T)$ and $z(x,t)=u(x,t+h)-u(x,t)$ on $\overline{Q}_{T-h}$. For any $(x,t)\in Q_{T-h}$, we obtain

$$\begin{aligned} x^q z_t - (x^\beta z_x)_x &= x^q f(u(x,t+h)) - x^q f(u(x,t)) \\ &= x^q f'(\zeta_2) z(x,t) \end{aligned}$$

where ζ_2 is a positive constant between $u(x,t+h)$ and $u(x,t)$. Moreover, since $u\ge u_0$ on Q_T, we have that $z(x,0)=u(x,h)-u_0(x)\ge 0$ for $x\in \overline{I}$ and $z(0,t)=0=z(1,t)$ for $t\in(0,T-h)$. Lemma 3.5 implies that $u_t \ge 0$ on $\overline{Q}_T$. Therefore the proof of this lemma is complete.

4 A sufficient condition to blow-up in finite time

In this section, we will give the sufficient condition to ensure occurrence of blow-up in finite time. Let $\varphi_1(>0)$ be the first eigenfunction of the singular eigenvalue problem (8) and $\lambda_1(>0)$ its corresponding eigenvalue. Further, we assume that

$$\int_0^1 x^q \varphi_1(x)dx = 1.$$

We define the function H by

$$H(t)=\int_0^1 x^q u(x,t)\varphi_1(x)dx. \tag{30}$$

We note that since $\lim_{s\to\infty}\frac{f(s)}{s}\to\infty,$ there exists a positive constant z_0 such that

$$f(s)-\lambda_1 s>0 \text{ for any } s\ge z_0. \tag{31}$$

Theorem 4.1 Let

$$\int_{z_0}^{\infty}\frac{ds}{f(s)-\lambda_1 s}<\infty.$$

Then, for any initial function u_0 such that

$$H(0)=\int_0^1 x^q u_0(x)\varphi_1(x)dx \ge z_0,$$

the solution u of a degenerate parabolic problem (1) blows up in finite time.

Proof. Suppose that u exists for all time $t\ge 0$ for any $x\in\overline{I}$. By multiplying equation (1) both side by φ_1 and integrating with respect to x over its domain, we have

$$\frac{dH(t)}{dt}+\lambda_1 H(t)=\int_0^1 x^q f(u(x,t))\varphi_1(x)dx. \tag{32}$$

By convexity of f, we can apply the Jensen's inequality to equation (32) and then we obtain

$$\frac{dH(t)}{dt}\ge f(H(t))-\lambda_1 H(t).$$

From equation (30), we differentiate the function H with respect to x and then we have

$$\frac{dH(t)}{dt}=\int_0^1 x^q u_t(x,t)\varphi_1(x)dx.$$

Thus lemma 3.6 yields that $\frac{dH(t)}{dt}>0.$ We further get that $H(t)\ge z_0$ for all $t.$ By equation (31), we have

$$\frac{dH(t)}{dt}\ge f(H(t))-\lambda_1 H(t)\ge 0 \text{ for } t>0 \text{ and } H(0)\ge z_0.$$

So we separate variables to find

$$t\le \int_{H(0)}^{H(t)}\frac{ds}{f(s)-\lambda_1 s}\le \int_0^{\infty}\frac{ds}{f(s)-\lambda_1 s}<\infty.$$

Hence t is finite and a contradiction is achieved. The solution can not exist for all positive time.

5 Blow-up set

In this section, the blow-up set for u of problem (1) is shown.

Theorem 5.1 The blow-up set of a solution u of the degenerate parabolic problem (1) is $\overline{I}$.

Proof. From equation (26), there are two positive constants c_1 and c_2 such that

$$\sup_{(x,t)\in \overline{Q}_{T_{\max}}} u(x,t) \le c_1 + c_2 \int_0^t \int_0^1 f(u(\xi,\tau))d\xi d\tau.$$

Theorem 3.2 implies that as $t \to T_{\max}$,

$$\int_0^{T_{\max}} \int_0^1 f(u(\xi,\tau))d\xi d\tau \to \infty.$$

On the other hand, there are two positive constant c_3 and c_4 such that

$$u(x,t) \ge c_3 + c_4 \int_0^t \int_0^1 f(u(\xi,\tau))d\xi d\tau \text{ for any } (x,t) \in Q_{T_{\max}}. \tag{33}$$

As $t \to T_{\max}$, we obtain that, by (33), $u(x,t) \to \infty$ for all $x \in I$. Furthermore, for $x \in \{0,1\}$, we can find a sequence $\{(x_n,t_n)\}$ such that $\lim_{n\to\infty} u(x_n,t_n) \to \infty.$ Hence, the blow-up set of a solution of a degenerate parabolic problem (1) is $\overline{I}$.

6 Generalized problem

In this section, we extend our degenerate parabolic initial-boundary value problem (1) in more general form by replacing function coefficients of u_t and u_x, x^q and x^β, by functions $k(x)$ and $p(x)$, respectively. We now consider the following degenerate parabolic initial-boundary value problem,

$$\left.\begin{aligned} k(x)u_t - (p(x)u_x)_x &= k(x)f(u),\ (x,t)\in Q_T,\\ u(0,t) = 0 &= u(1,t),\ t\in(0,T),\\ u(x,0) &= u_0(x),\ x\in \overline{I}, \end{aligned}\right\} \tag{34}$$

where k and p are determined. In order to obtain the same results as a degenerate parabolic problem (1), we have to assume the following.

(C) $k \in C(\overline{I})$, $k(0)=0$ and k is positive on $(0,1]$.

(D) $p \in C^1(\overline{I})$, $p(0)=0$, p is positive on $(0,1]$ and p' is positive on $\overline{I}$.

As obtaining equation (8) the corresponding singular eigenvalue problem of (34) is defined by

$$\left.\begin{aligned} \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + \lambda k(x)\phi(x) = 0 \text{ for } x\in I,\\ \phi(0) = 0 = \phi(1). \end{aligned}\right\} \tag{35}$$

We notice that it follows from conditions (C) and (D) that the point $x=0$ is a singular point of problem (35). From equation (35), we can rewrite the corresponding singular eigenvalue problem (35) in the following equivalent form,

$$\left.\begin{aligned} x^2\phi''(x) + x\left[x\frac{p'(x)}{p(x)}\right]\phi'(x) + x^2\left[\lambda\frac{k(x)}{p(x)}\right]\phi(x) = 0 \text{ on } I,\\ \phi(0) = 0 = \phi(1). \end{aligned}\right\} \tag{36}$$

To ensure the existence of eigenfunctions ϕ_n and eigenvalues λ_n, we need an additional condition on functions k and p.

(E) The limit of $\frac{xp'(x)}{p(x)}$ and $\frac{x^2k(x)}{p(x)}$ are finite as x converges to 0 and $\frac{xp'(x)}{p(x)}$ and $\frac{x^2k(x)}{p(x)}$ are analytic at $x=0$.

We note that theorem 5.7.1 [2] implies that eigenfunctions ϕ_n and eigenvalues λ_n of a corresponding singular eigenvalue problem (36) exist. Moreover completeness of eigenfunctions ϕ_n of a singular eigenvalue problem (36) results from the next hypothesis.

(F) $\int_0^1\int_0^1 H(x,\xi)^2 k(x)k(\xi)d\xi dx$ is finite where H is the corresponding Green's function to problem (36).

Well-known properties of eigenfunctions ϕ_n and eigenvalues λ_n are shown in next lemma referred to [xx].

Lemma 6.1

6.1.1. $\int_0^1 k(x)\phi_n(x)\phi_m(x)dx = \begin{cases} 1 & \text{for } m = m,\\ 0 & \text{for } m \ne n. \end{cases}$

6.1.2. All eigenvalues are real and positive.

6.1.3. Eigenfunctions are complete with the weight function k.

6.1.4. $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ and $\lim_{n\to\infty}\lambda_n = \infty.$

6.1.5. $\int_0^1 p(x)\phi_n'(x)\phi_n'(x)dx = \begin{cases} \lambda_n & \text{for } n = m,\\ 0 & \text{for n} \ne \text{m}. \end{cases}$

6.1.6. For any $n \in \mathbb{N}$, $\phi_n \in C^\infty((0,1])$.

The Green's function G corresponding to the degenerate parabolic initial-boundary value problem (34) is determined by the following problem: let x,ξ be in I and t,τ in $(0,T)$,

$$\left.\begin{aligned} k(x)G_t - \left(p(x)G_x\right)_x &= \delta(x-\xi)\delta(t-\tau),\\ G(0,t,\xi,\tau) = 0 &= G(1,t,\xi,\tau),\\ G(x,t,\xi,\tau) &= 0 \text{ for } t > \tau, \end{aligned}\right\} \tag{37}$$

where δ is the Dirac delta function. As obtaining equation (25), the corresponding Green's function of (37) defined by

$$G(x,t,\xi,\tau)=\sum_{n=1}^{\infty}\phi_n(\xi)\phi_n(x)e^{-\lambda_n(t-\tau)} \text{ for } x,\xi\in\bar{I} \text{ and } 0\le\tau<t\le T.$$

The following lemma is due to properties of G corresponding to problem (34).

Lemma 6.2. Assume that $\lambda_n=O(n^s)$ for some $s>1$ as $n\to\infty$.

6.2.1. G is continuous for $x,\xi\in\bar{I}$ and $0\le\tau<t<T$.

6.2.2. G is positive for $x,\xi\in\bar{I}$ and $0\le\tau<t<T$.

As equation (26), the equivalent integral equation to the extended degenerate parabolic problem (34) is given by

$$u(x,t)=\int_0^1 k(\xi)G(x,t,\xi,\tau)u_0(\xi)d\xi +\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(\xi,\tau))d\xi d\tau.$$

Next theorem shows the existence of an unique solution of the extended degenerate parabolic problem (34) before blow-up occurs

Theorem 6.3 There exists a finite time $T_2>0$ such that the extended degenerate parabolic problem (34) has a unique continuous solution u on the finite time interval $[0,T_2]$ for any $x\in\bar{I}$.

Proof. The proof of this theorem is similar to that of theorem 3.3.

Let $\tilde{T}_{\max}$ be the supremum of all T_2 such that the solution u of the extended degenerate parabolic problem (34) is bounded. The following theorem says that the solution of the extended problem (34) blows up in finite time if $\tilde{T}_{\max}$ is finite.

Theorem 6.4 if $\tilde{T}_{\max}$ is finite, then $\sup\limits_{(x,t)\in\bar{Q}_{\tilde{T}_{\max}}}|u(x,t)|$ is unbounded as t converges to $\tilde{T}_{\max}$.

Proof. The proof of this theorem is similar to that of theorem 3.4.

Furthermore we give the additional properties of a solution u of the extended degenerate parabolic problems (34). that is, positivity and increasing in t of u. In order to obtain results, we need the following lemma.

Lemma 6.5 Let v be a classical solution of the following problem:

$$v_t-\frac{1}{k(x)}\left(p(x)v_x\right)_x\ge B(x,t)v(x,t) \text{ for } (x,t)\in Q_T,$$
$$v(0,t)=0=v(1,t) \text{ for } t\in(0,T),$$
$$v(x,0)=u_0(x)\ge 0 \text{ for } x\in\bar{I},$$

where $B(x,t)$ is a nonnegative and bounded function on $\bar{Q}_T$. Then $v(x,t)\ge 0$ for any $(x,t)\in\bar{Q}_T$.

Proof. Let η be any positive constant. Let

$$w(x,t)=v(x,t)+\eta(1+x^2)e^{ct}$$

where c is some positive constant with

$$c\ge\max_{x\in\bar{I}}p'(x)+\max_{x\in\bar{I}}p(x)+\max_{x\in\bar{I}}k(x)\max_{(x,t)\in\bar{I}\times[0,T]}B(x,t).$$

Let us consider

$$\begin{aligned}&k(x)w_t-(p(x)w_x)_x-k(x)B(x,t)w(x,t)\\&=k(x)v_t-(p(x)v_x)_x-k(x)B(x,t)v(x,t)+c\eta(1+x^2)e^{ct}\\&\quad-2\eta e^{ct}\left[xp'(x)+p(x)\right]-k(x)B(x,t)\eta(1+x^2)e^{ct}\\&\ge\eta e^{ct}\left\{c(1+x^2)-2\left[xp'(x)+p(x)\right]-k(x)B(x,t)(1+x^2)\right\}\\&\ge 2\eta e^{ct}\left\{c-\left[\max_{x\in\bar{I}}p'(x)+\max_{x\in\bar{I}}p(x)\right]\right.\\&\quad\left.-\max_{x\in\bar{I}}k(x)\max_{(x,t)\in\bar{I}\times[0,T]}B(x,t)\right\}\end{aligned}$$

By the definition of c, we have that for any $(x,t)\in\bar{Q}_T$, $k(x)w_t-(p(x)w_x)_x-k(x)B(x,t)w(x,t)\ge 0$.

We see that $w(x,t)\ge 0$ for $(x,t)\in\{0,1\}\times(0,T)\cup\bar{I}\times\{0\}$. We next would like to show that $w(x,t)>0$ for any $(x,t)\in\bar{Q}_T$. Suppose that there exists a point (x_1,t_1) with $w(x_1,t_1)\le 0$. We define the set A by

$$A=\{t \text{ such that } w(x,t)\le 0 \text{ for some } x\in I\}.$$

Thus, the set A is non-empty. Let $\tilde{t}=\inf A$. Since $w(x,0)=u_0(x)+\eta(1+x^2)>0$ for $x\in I$, we obtain that $\tilde{t}>0$. Furthermore, since A is closed, by the definition of $\tilde{t}$, there exists a point x_2 in I such that

$$w(x_2,\tilde{t})=0,\ w_t(x_2,\tilde{t})\le 0 \text{ and } w_x(x_2,\tilde{t})=0.$$

Moreover, we also get that $w_{xx}(x_2,\tilde{t})\ge 0$ because w attains its local minimum at the point x_2. Then we have that

$$\begin{aligned}0&\ge k(x_2)w_t(x_2,\tilde{t})\\&\ge k(x_2)w_t(x_2,\tilde{t})-p(x_2)w_{xx}(x_2,\tilde{t})-p'(x_2)w_x(x_2,\tilde{t})\\&\quad-k(x_2)B(x_2,\tilde{t})w(x_2,\tilde{t})\\&>0.\end{aligned}$$

Therefore, we get a contradiction. This shows that $w(x,t)>0$ for any $(x,t)\in\bar{Q}_T$. Since η is arbitrary, we let $\eta\to 0^+$ and then we obtain the desired result.

Lemma 6.6 Let u be a continuous solution of the extended degenerate parabolic problem (34). Then $u(x,t) \ge u_0(x)$ and $u_t(x,t) \ge 0$ for any $(x,t) \in \overline{Q}_{T_1}$.

To ensure that a solution u of the extended degenerate parabolic problem (34) blows up in finite time, we give the condition to guarantee the occurrence for blow-up in finite time. Let $\phi_1(>0)$ be the first eigenfunction of the singular eigenvalue problem (35) and $\lambda_1(>0)$ its corresponding eigenvalue. Moreover we suppose

$$\int_0^1 k(x)\phi_1(x)dx = 1.$$

We construct the function H by

$$H(t) = \int_0^1 k(x)u(x,t)\phi_1(x)dx.$$

Notice that since $\lim_{s\to\infty} \frac{f(s)}{s} \to \infty,$ there exists a positive constant z_0 such that

$f(s) - \lambda_1 s > 0$ for any $s \ge z_0$.

Theorem 6.5 Assume that

$$\int_{z_0}^{\infty} \frac{ds}{f(s)-\lambda_1 s} < \infty.$$

Then, for any initial function u_0 with

$$H(0) = \int_0^1 k(x)u_0(x)\phi_1(x)dx \ge z_0,$$

the solution u of the extended degenerate parabolic problem (34) blows up in finite time.

Proof. By modifying the proof of theorem 4.1, this theorem is proven.

The last theorem concern the blow-up set of the extended degenerate parabolic problem (34).

Theorem 6.6 The blow-up set of a solution u of the extended degenerate parabolic problem (34) is $\overline{I}$.

Proof. The proof of this theorem is similar to that of theorem 5.1.

7 Conclusion

In this work, we obtain four main results for the degenerate parabolic problem (1) which are the theorem 3.3, 3.4, 4.1 and 5.1. The first main result, the theorem 3.3, says that there is a finite time T with $T > 0$ such that the degenerate parabolic problem (1) has a unique solution u on the time interval $[0,T]$ for any x in $\overline{I}$. Theorem 3.3 can be proven by the Banach fixed point theorem. Let $T_{\max}$ be the supremum of all T such that the solution u of the degenerate parabolic problem (1) is bounded. Theorem 3.4 shows that the solution u of our degenerate parabolic problem (1) blows up if $T_{\max}$ is finite. In fact, $T_{\max}$ may be not finite. This is the reason why theorem 4.1 is constructed. Theorem 4.1 is the sufficient condition to blow-up in finite time. The last main result of problem (1), theorem 5.1, indicates that the blow-up set of the degenerate parabolic problem (1) is $\overline{I}$. We finally extend our degenerate parabolic problem (1) into the general form: $k(x)u_t - (p(x)u_x)_x = k(x)f(u(x,t))$ where k and p are given functions. Under some conditions, we also obtain the same results as the previous problem, that is, theorem 6.3-6.6.

Acknowledgment

Authors would like to thank the Department of Mathematics, Faculty of Science, Mahidol University for financial support during the preparation of this paper.

References:

[1] M. Abramowitz, and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Grapminimumhs and Mathematical Tables*, Washington: National Burean of Standards, Applied Mathematics Series 55, 1985.

[2] W. E. Boyce and R. C. Diprima, *Elementary Differential Equations and boundary Value Problems.*, 8th edition, John Wiley & Sons, Inc., 2005.

[3] C. Y. Chan, and W. Y. Chan, "Existence of classical solutions for degenerate semilinear parabolic problems," *Appl. Math. and Comp.*, Vol. 101, 1999, pp. 125-149.

[4] C. Y. Chan, and W. Y. Chan, "Complete blow-up of solutions for a degenerate semilinear parabolic first initial-boundary value problems," *Appl. Math. and Comp.*, Vol. 177, 2006, pp. 777-784.

[5] C. Y. Chan, and B. W. Wong, "Existence of classical solutions for semilinear parabolic problems," *Quart. Appl. Math.*, Vol. 53, 1995, pp. 201-213.

[6] C. Y. Chan, and B. M. Wong, "Periodic solutions of singular linear and semilinear parabolic problems," *Quart. Appl. Math.*, Vol. 47, 1989, pp. 142-174.

[7] Y. Chen, Q. Liu, and C. Xie, “Blow-up for degenerate parabolic equations with nonlocal source,” *Proceedings of the American Mathematical Society*, Vol. 132, No. 1, May 2003, pp. 135-145.

[8] Y. P. Chen, C. H. Xie, “Blow-up for degenerate, singular , semilinear parabolic equations with nonlocal source,” *Acta Mathematica Sinica*, Vol. 47, No 1, 2004, pp. 41-50.

[9] K. T. Chiang, G. C. Kuo, K. J. Wang, Y. F. Hsiao and K. Y. Kung , “Transient temperature analysis of a cylindrical heat equation,” *WSEAS Transactions on Mathematics*, Vol. 8, 2009, pp. 309-319.

[10] A. Friedman, *Partial Differential Equation of Parabolic Type*, N.J.: Prentice-Hall, Englewood, 1964.

[11] K. Y. Kung and S. C. Lo, “Transient analysis of two-Dimensional cylindrical fin with various surface heat effects,” *WSEAS Transactions on Mathematics*, Vol. 3, 2004, pp. 120-125.

[12] C. Mueller, and F. Weissler, “Single point blow-up for a general semilinear heat equation,” *Indiana Univ. Math. J.*, Vol. 34, No. 4, 1985, pp. 881-913.

[13] J. P. Pinasco, “Blow-up for parabolic and hyperbolic problems with variable exponents,” *Nonlinear Analysis*, vol. 71, pp. 1094-1099, 2009.

[14] P. Sawangtong and W. Jumpem, “Existence of a blow-up solution for a degenerate parabolic initial-boundary value problem,” *Proceeding of the 4th international conference on applied mathematics, simulation, modeling (ASM’10)*, 2010 , pp. 13-18.

[15] P. Sawangtong, B. Novaprateep and W. Jumpen, “Complete blow-up for a degenerate semilinear parabolic problem with a localized nonlinear term,” *Proceeding of the international conference on theoretical and applied mechanics 2010 (MECHANICS’10) and the international conference on fluid mechanics and heat and mass transfer 2010 (FLUID-HEAT’10)*, 2010, pp. 94-99.

[16] H. F. Weinberger, *A first Course in Partial Differential Equation*. John Wiley & Sons, Inc., 1965.