Blow-up solutions of degenerate parabolic problems

P. SAWANGTONG1 and W. JUMPEN2,3*

1Dept of Mathematics, Faculty of Applied Science
King Mongkut’s University of Technology North Bangkok, Thailand
2Dept of Mathematics, Faculty of Science, Mahidol University, Thailand
3Center of Excellence in Mathematics, PERDO, CHE, Thailand
pa_sawangtong@yahoo.com and scwjp@mahidol.ac.th
*Corresponding author

Abstract: - In this article, we study the degenerate parabolic problem, 
\( \frac{\partial u}{\partial t} - (x^q u_x)_x = x^\beta f(u) \), satisfying the Dirichlet boundary condition and a nonnegative initial condition where \( q \) and \( \beta \) are given constants and \( f \) is a suitable function. We show that under certain conditions the degenerate parabolic problem has a blow-up solution and the blow-up set of such a blow-up solution is the whole domain of \( x \). Furthermore, we give the sufficient condition to blow-up in finite time. Finally, we generalize the degenerate parabolic problem into the general form, \( \frac{\partial u}{\partial t} - (p(x) u_k)_x = k(x) f(u) \). Under appropriate assumptions on functions \( k, p \) and \( f \), we still obtain the same results as the previous problem.

Key-Words: - Degenerate parabolic problems, Finite time blow-up, Blow-up set, Blow-up solution.

1 Introduction
Let \( T \) be any positive real number and \( q \) and \( \beta \) be positive constants with \( q > 0, \beta \in [0,1) \) and \( q + \beta \neq 0 \). Let \( I = (0,1), \ Q_r = I \times (0,T), \ T \) and \( \overline{Q_r} \) be their closure of \( I \) and \( Q_r \), respectively. In this article, we study the degenerate parabolic initial-boundary value problem,

\[
\begin{align*}
\frac{\partial u}{\partial t} - (x^q u_x)_x &= x^\beta f(u), \quad (x,t) \in \overline{Q_r}, \\
u(0,t) &= 0 = u(1,t), \quad t \in (0,T), \\
u(x,0) &= u_0(x), \quad x \in T,
\end{align*}
\]

where \( f \) and \( u_0 \) are suitable functions. In 1985, C. E. Mueller and F. B. Weissler [12] studied the behavior of solutions to the semilinear heat equation

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + f(u), \quad (x,t) \in \Omega \times (0,\infty), \\
u(x,t) &= 0, \quad (x,t) \in \partial \Omega \times (0,\infty), \\
u(x,0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with a smooth boundary \( \partial \Omega \), and the source term is of the form \( f(u) = a(x) u^{p(x)} \) or \( f(u) = a(x) \int_{\Omega} u^{p(x)}(y,t) \, dy \) with given functions \( a, p \) and \( q \). For blow-up problems of the degenerate semilinear parabolic type, in 1999, C. Y. Chan and W. Y. Chan [3] proved the existence of a blow-up solution of the degenerate semilinear parabolic initial-boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - (p(x) u_k)_x &= f(u), \quad (x,t) \in \overline{Q_r}, \\
u(0,t) &= 0 = u(1,t), \quad t \in (0,T), \\
u(x,0) &= u_0(x), \quad x \in T,
\end{align*}
\]

where \( f \) and \( u_0 \) are given functions. They proved the existence and uniqueness of a blow-up solution of problem (3) by transforming problem (2) into the equivalent integral equation in terms of its Green’s function. Furthermore, in 2006, C. Y. Chan and W. Y. Chan [4] showed that under certain condition on functions \( f \) and \( u_0 \), a solution \( u \) of problem (3) blows up at every point in \( I \). After paper [3] published, in 2004, Y.P. Chen and C.H. Xie [8]...
discussed the degenerate parabolic equation with the nonlocal term:

\[ u_t - (x^2 u_x)_x = \int_0^1 f(u(x,t))dx \quad \text{for } (x,t) \in Q_T, \]
\[ u(0,t) = 0 = u(1,t) \quad \text{for } t \in (0,T), \]
\[ u(x,0) = u_0(x) \quad \text{for } x \in I. \quad (4) \]

They considered the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they obtained some sufficient conditions for the global existence and blow-up of a positive solution of problem (4). Additionally, in 2004, Y.P. Chen, Q. Liu and C.H. Xie [7] studied the degenerate nonlinear reaction-diffusion equation with nonlocal source:

\[ x^s u_t - (x^q u_x)_x = \int_0^1 u^q(x,t)dx \quad \text{for } x \in I. \]
\[ u(0,t) = 0 = u(1,t) \quad \text{for } t \in (0,T), \]
\[ u(x,0) = u_0(x) \quad \text{for } x \in I. \quad (5) \]

They established the local existence and uniqueness of a classical solution of problem (5). Under appropriate hypotheses, they also get some sufficient conditions for a global existence and blow-up of a positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution of problem (5) is the whole domain. In 2010, P. Sawangtong, B. Novaprateep and W. Jumpen [15] established the existence of a blow-up solution of the following degenerate parabolic problem with the localized nonlinear source:

\[ u_t - \frac{1}{k(x)} (p(x)u_x)_x = f(u(x,t)) \quad \text{for } x \in I. \]
\[ u(0,t) = 0 = u(1,t) \quad \text{for } t \in (0,T), \]
\[ u(x,0) = u_0(x) \quad \text{for } x \in I. \quad (6) \]

where \( k, p, f \) and \( u_0 \) are suitable functions. In 2010 P. Sawangtong and W. Jumpen [14] studied the degenerate parabolic problem (6). In this article we continuos to study the degenerate parabolic problem (6) and our objective is to show that before blow-up phenomenon will occur, the degenerate parabolic problem (1) has a unique continuous solution \( u \) on the finite time interval \( T_i \) with \( T_i > 0 \) for any \( x \in I \). Moreover, the sufficient condition to guarantee occurrence of finite time blow-up and the blow-up of such a blow-up solution are given. In order to make more complete, we extend our degenerate parabolic problem (1) into the more general form:

\[ k(x)u_t - (p(x)u_x)_x = k(x)f(u(x,t)) \quad \text{where } k \text{ and } p \text{ are given functions. Under some conditions, we also obtain the same results as the degenerate parabolic problem (1).} \]

In order to obtain our results for degenerate parabolic problem (1), we need assumptions on functions \( f \) and \( u_0 \) as follows.

(A) \( f \in C^1([0,\infty)) \) is convex with \( f(0) = 0 \) and \( f(s) > 0 \) for \( s > 0 \).

(B) \( u_0 \in C^2(\bar{I}), u_0(0) = u_0(1), u_0 \) is nonnegative on \( I \) and \( u_0 \) satisfies the inequality,

\[ \frac{d}{dx} \left( x^\beta \frac{du_0}{dx} \right) + x^\beta f(u_0(x)) \geq 0 \text{ on } I. \quad (7) \]

We note that by proposition 2.1 of [12], condition (A) implies that \( f \) is increasing and locally Lipschitz continuous on \([0,\infty)\).

A solution \( u \) of the degenerate parabolic problem (1) is said to blow up at the point \( b \) in finite time \( \hat{T} \) if there exists a sequence \((x_n, t_n)\) with \( t_n < \hat{T} \) such that \((x_n, t_n) \to (b, \hat{T}) \) and \( u(x_n, t_n) \to \infty \). Furthermore, the set consisting of all blow-up points of a blow-up solution is called the blow-up set.

This paper is organized as follows. In section 2, we find associating eigenfuctions and eigenvalues to the degenerate parabolic problem (1). We prove the existence and uniqueness of a solution of problem (1) before blow-up occurs by using the Banach fixed point theorem and give the sufficient condition to blow-up in finite time in section 3 and 4, respectively. In section 5, we give the blow-up set of such a blow-up solution of the degenerate parabolic problem (1). The extended problem, \( k(x)u_t - (p(x)u_x)_x = k(x)f(u), \) of the degenerate parabolic problem, \( x^s u_t - (x^q u_x)_x = x^s f(u), \) is studied in the last section.

### 2 Eigenvalues and Eigenfunctions

By using separation of variables [9] and [11] on the homogenous problem corresponding to problem (1), we obtain the singular eigenvalue problem,

\[ \frac{d}{dx} \left( x^\beta \frac{d\phi}{dx} \right) + \lambda x^\beta \phi(x) = 0 \quad \text{for } x \in I, \]
\[ \phi(0) = 0 = \phi(1). \]

Let \( \phi(x) = x^{1-\beta} y(x) \). Then
\[
\varphi(x) = x^\frac{1-\beta}{2} y'(x) + \left(\frac{1-\beta}{2}\right) x^\frac{-\beta-1}{2} y(x) \quad (9)
\]

and
\[
\varphi^*(x) = x^\frac{1-\beta}{2} y^*(x) + (1-\beta)x^\frac{-\beta-1}{2} y'(x) + \left(\frac{1-\beta}{2}\right) x^\frac{-\beta-1}{2} y(x). \quad (10)
\]

Substituting equation (9) and (10) in equation (8), we obtain
\[
x^\mu \left[ x^\frac{1-\beta}{2} y^*(x) + (1-\beta)x^\frac{-\beta-1}{2} y'(x) + \left(\frac{1-\beta}{2}\right) x^\frac{-\beta-1}{2} y(x) \right] + \beta x^\beta \left[ x^\frac{1-\beta}{2} \frac{1-\beta}{2} y(x) \right] + \lambda x^\beta y^2(x) = 0
\]
or
\[
x^\frac{1-\beta}{2} y^*(x) + x^\frac{-\beta-1}{2} y'(x) + \left[ \lambda x^\beta - \left(\frac{1-\beta}{2}\right)^2 \right] y(x) = 0. \quad (11)
\]

Dividing both sides of equation (11) by \(x^\frac{1-\beta}{2}\), we get
\[
y^*(x) + \frac{1}{x} y'(x) + \left[ \lambda x^\beta - \frac{1-\beta}{2} \frac{1}{x^2} \right] y(x) = 0. \quad (12)
\]

Multiplying both side of equation (12) by \(x^2\), the singular eigenvalue problem (8) becomes
\[
x^2 y^*(x) + x y'(x) + \left[ \lambda x^\beta - \frac{1-\beta}{2} \frac{1}{x^2} \right] y(x) = 0,
\]
\[y(0) \text{ is bounded and } y(1) = 0. \quad (13)
\]

Again, we set \(x = z^\frac{2}{\beta+2}\). Then
\[
y'(z) = \left(\frac{q-\beta+2}{2}\right)^{\frac{2}{\beta+2}} z^{\frac{-\beta-2}{\beta+2}} y'(z)
\]
and
\[
y^*(z) = \left(\frac{q-\beta+2}{2}\right)^{\frac{2}{\beta+2}} z^{\frac{-\beta-2}{\beta+2}} y^*(z) + \left(\frac{q-\beta}{2}\right)^{\frac{2}{\beta+2}} z^{\frac{-\beta-2}{\beta+2}} y(z)
\]
From equation (13), we have
\[
y^*(z) = \left(\frac{q-\beta+2}{2}\right)^{\frac{2}{\beta+2}} z^{\frac{-\beta-2}{\beta+2}} y^*(z) + \left(\frac{q-\beta+2}{2}\right)^{\frac{2}{\beta+2}} z^{\frac{-\beta-2}{\beta+2}} y(z)
\]
\[\int \left[ x^\beta \varphi_n(x) \varphi_m(x) \right] dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}
\]
to determine the value of constant $A$. To do so, we consider
\[
\int_0^1 x^n \varphi_n^2(x) dx = A^2 \int_0^1 x^{n+2} J_{n+1}^2(\omega_n x) \frac{x^{-1}}{q - \beta + 2} dx.
\] (18)

Let $y = x^{-\frac{1}{2}}$. Then $dy = \left(\frac{q - \beta + 2}{2}\right)^{\frac{1}{2}} x^{-\frac{3}{2}} dx$. Let us consider the right-hand side of equation (18)
\[
A^2 \int_0^1 x^{n+2} J_{n+1}^2(\omega_n x) \frac{x^{-1}}{q - \beta + 2} dx = \frac{2A^2}{q - \beta + 2} \int_0^1 y J_n^2(\omega_n y) dy
\]
\[= \frac{A^2}{q - \beta + 2} J_{n+1}^2(\omega_n). \quad (19)
\]

It follows from (18) and (19) that
\[
\int_0^1 x^n \varphi_n^2(x) dx = \frac{A^2}{q - \beta + 2} J_{n+1}^2(\omega_n).
\]

Since the right-hand side of equation (18) must equal to 1, the value of constant $A$ is determined by
\[
A = \left(\frac{q - \beta + 2}{2}\right)^{\frac{1}{2}} J_{n+1}^2(\omega_n). \quad \text{Hence, eigenfunctions } \varphi_n \text{ of the singular eigenvalue problem (17) are defined by}
\]
\[
\varphi_n(x) = \left(\frac{q - \beta + 2}{2}\right)^{\frac{1}{2}} \frac{1}{J_{n+1}(\omega_n)} \cdot \frac{1}{x^{\frac{1}{2}}} J_{n+1}(\omega_n x) \quad \text{for some positive constant } c_0.
\] (20)

Further, it follows from [1] that $\lambda_n = O(n^2)$ as $n \to \infty$.

**Lemma 2.1** For any $x \in \mathbb{R}$, $|\varphi_n(x)| \leq c_0 x^2 \frac{1}{\lambda_n^2}$ for some positive constant $c_0$.

**Proof.** By [1], we have
\[
\left|J_{n+1}(\omega_n x) \frac{x^{-1}}{q - \beta + 2}\right| \leq 1 \text{ for any } \mu > 0. \quad (21)
\]

It follows from [6] that
\[
\frac{1}{J_{n+1}(\omega_n)} \leq \left(\frac{\pi \lambda_n^2}{q - \beta + 2}\right)^{\frac{1}{2}} c_0 \quad \text{or}
\]

where $c_0$ is some positive constant. Therefore, by equations (20), (21) and (22), the proof of this lemma is complete.

### 3 Local existence of blow-up solution

In this section, we will show the existence and uniqueness of a nonnegative continuous solution of a degenerate parabolic problem (1). Green’s function $G(x, t, \xi, \tau)$ corresponding to the degenerate parabolic problem (1) is determined by the following system, for any $x$ and $\xi$ in $I$, and $t$ and $\tau$ in $(0, T)$,
\[
x^q G_{1}(x, t, \xi, \tau) - (x^q G_{1}(x, t, \xi, \tau))_{\xi} = \delta(x - \xi) \delta(t - \tau),
\]
\[G(x, t, \xi, \tau) = 0 \text{ for } t < \tau,
\]
\[G(0, t, \xi, \tau) = 0 = G(1, t, \xi, \tau). \quad (23)
\]

where $\delta$ is the Dirac delta function. Let
\[
G(x, t, \xi, \tau) = \sum_{n=1}^{\infty} a_n(t) \varphi_n(x) \quad (24)
\]

Substituting equation (24) into equation (23), we obtain
\[
x^q \sum_{n=1}^{\infty} a_n'(t) \varphi_n(x) - \sum_{n=1}^{\infty} a_n(t) \frac{d}{dx} x^q \frac{d}{dx} \varphi_n(x) = \delta(x - \xi) \delta(t - \tau).
\]

Integrating both sides with respect to $x$ over its domain, we have
\[
\int_0^1 x^n \varphi_n(x) \sum_{n=1}^{\infty} a_n'(t) \varphi_n(x) dx - \int_0^1 \varphi_n(x) \sum_{n=1}^{\infty} a_n(t) \frac{d}{dx} x^q \varphi_n(x) dx = \int_0^1 \varphi_n(x) \delta(x - \xi) \delta(t - \tau) dx.
\]

By the orthonormal property of eigenfunctions $\varphi_n$ and the property of Dirac delta function, we get
\[
a_n'(t) + \lambda_n a_n(t) = \varphi_n(\xi) \delta(t - \tau)
\]

or
\[
\frac{d}{dt} e^{\lambda_n t} a_n(t) = \varphi_n(\xi) \delta(t - \tau) e^{\lambda_n t}.
\]

Integrating both sides from $t$ to $t_i$ with $t_i < \tau$, we obtain
\[
\int_{t_i}^{t} \frac{d}{ds} (e^{\lambda_n s} a_n(s)) = \int_{t_i}^{t} \varphi_n(\xi) \delta(s - \xi) e^{\lambda_n s} ds
\]

or
\[ e^{\lambda_n t} a_n(t) - e^{\lambda_n t} a_n(t_0) = \phi_n(\xi) e^{\lambda_n t_0}. \]

Since \( G(x,t,\xi,\tau) = 0 \) for \( t < \tau \), \( a_n(t_0) = 0 \) for all \( n \).

We then obtain that \( a_n(t) = \phi_n(\xi) e^{\lambda_n (t - \tau)} \) for any \( n \).

Therefore the Green’s function corresponding to problem (1) is defined by

\[
G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} \phi_n(x) \phi_n(\xi) e^{\lambda_n (t - \tau)} \quad \text{for } t > \tau, \quad (25)
\]

where \( \phi_n \) and \( \lambda_n \) are eigenfunctions and eigenvalues of the singular eigenvalue problem (8), respectively.

**Lemma 3.1** For \( t > \tau \), \( G(x,t,\xi,\tau) \) is continuous for \( (x,t,\xi,\tau) \in \bar{T} \times (0,T) \times \bar{T} \times (0,T) \).

**Proof.** By Lemma 2.1 and equation (25), we obtain that

\[
\left| \sum_{n=1}^{\infty} \phi_n(x) \phi_n(\xi) e^{\lambda_n (t - \tau)} \right| \leq c_3 x \sum_{n=1}^{\infty} \lambda_n^{-\frac{3}{2}} \left( x^3 + \sum_{n=1}^{\infty} \lambda_n^{-\frac{1}{2}} \right) \leq c_3 \sum_{n=1}^{\infty} \lambda_n^{-\frac{3}{2}} e^\lambda \nonumber,
\]

which converges uniformly. We then get the result.

Next lemma shows the positivity of Green’s function \( G \).

**Lemma 3.2** For any \( (x,t,\xi,\tau) \in I \times (0,T) \times I \times (0,T) \),

\[ G > 0 \] with \( t > \tau \).

**Proof.** The proof of this lemma is similar to that of lemma 4.c of [5].

To derive the equivalent integral equation of a degenerate parabolic problem (1), let us consider the adjoint operator \( L' \) defined by

\[
L' = -x^3 \frac{\partial}{\partial t} - x^\beta \frac{\partial}{\partial \xi} \left( x^\beta \frac{\partial}{\partial x} \right)
\]

corresponding to the operator

\[
L = -x^3 \frac{\partial}{\partial t} + x^\beta \frac{\partial}{\partial \xi} \left( x^\beta \frac{\partial}{\partial x} \right)
\]

of a degenerate parabolic problem (1). By Green’s second identity, we obtain the equivalent integral equation to problem (1) given by

\[
u(x,t) = \int_0^1 \xi^\beta G(x,t,\xi,\tau) u_0(\xi) d\xi + \int_0^1 \xi^\beta G(x,t,\xi,\tau) f(u(\xi,\tau)) d\xi d\tau. \quad (26)
\]

**Theorem 3.3** There exists a finite time \( T_1 > 0 \) such that a degenerate parabolic problem has a unique continuous solution \( u \) on the finite time interval \([0,T_1]\) for any \( x \in \bar{T} \).

**Proof.** Let \( M \) be a positive constant with \( M > \max_{x \in \bar{T}} u_0(x) + 1 \). Locally Lipschitz continuity of \( f \) implies that for any \( |\xi| \leq M \) and \( |\tau| \leq M \) there is a positive constant \( L \) depending on \( M \) such that \( |f(u) - f(v)| \leq L |u - v| \). Further, since

\[
\int_0^1 \xi^\beta G(x,t,\xi,\tau) d\xi d\tau \rightarrow 0 \quad \text{as } t \rightarrow 0,
\]

there exists a finite time \( T_1 \) such that

\[
f(M) \int_0^1 \xi^\beta G(x,t,\xi,\tau) d\xi d\tau < 1 \quad \text{for } t \in [0,T_1]
\]

and

\[
L \int_0^1 \xi^\beta G(x,t,\xi,\tau) d\xi d\tau < 1 \quad \text{for } t \in [0,T_1].
\]

We next define the space \( E \) by

\[
E = \left\{ u \in C(\bar{\Omega},) \text{ such that } \sup_{(x,t) \in \bar{\Omega}} |u(x,t)| \leq M \right\}.
\]

Then, the space \( E \) is a Banach space equipped with the norm \( u_E = \sup_{(x,t) \in \bar{\Omega}} |u(x,t)| \). Let \( \Phi \) be a mapping defined by

\[
\Phi u(x,t) = \int_0^1 \xi^\beta G(x,t,\xi,\tau) u_0(\xi) d\xi + \int_0^1 \xi^\beta G(x,t,\xi,\tau) f(u(\xi,\tau)) d\xi d\tau.
\]

In order to apply the Banach fixed point theorem we would like show that \( \Phi \) maps \( E \) into itself and \( \Phi \) is a contraction mapping. Let \( u \) and \( v \) be any element in \( E \). We then have that

\[
|\Phi u(x,t) - \Phi v(x,t)| \leq L \int_0^1 \xi^\beta G(x,t,\xi,\tau) u_0(\xi) d\xi + \int_0^1 \xi^\beta G(x,t,\xi,\tau) f(u(\xi,\tau)) d\xi d\tau.
\]

Let us consider the following auxiliary problem,

\[
x^\beta u_t - (x^\beta u_x)_x = 0, \quad (x,t) \in \bar{\Omega},
\]

\[
\begin{align*}
u(0,t) &= \varphi_1(t), \
\varphi_2(t) &= \varphi_3(t), \
u(t,0) &= \varphi_1(0), \quad t \in (0,T_1),
\end{align*}
\]

\[
u(x,0) = u_0(x), \quad x \in \bar{T}.
\]

From (26), a solution \( u \) of problem (28) is given by,

\[
u(x,t) = \int_0^1 \xi^\beta G(x,t,\xi,\tau) u_0(\xi) d\xi.
\]

On the other hand, Maximum principle for parabolic type [10] implies that \( u(x,t) \leq \max_{x \in \bar{T}} u_0(x) \) on \( \bar{\Omega} \). We then obtain that, for any \( (x,t) \in \bar{\Omega}, \)
\[
\int_0^1 \xi^\beta G(x, t, \xi, 0) \, d\xi \leq 1. \]
From inequality (27), we get that, for any \((x, t) \in \overline{Q}_T\),
\[
|\Phi u(x, t)| \leq \max_{n \in \mathbb{N}_0} u_0(x) + f(M) \int_0^1 \int_0^1 \xi^\beta G(x, t, \xi, \tau) \, d\xi \, d\tau < M.
\]
This shows that \(\Phi u \in E\) for any \(u \in E\). Next, locally Lipschitz continuity of \(f\) yields
\[
|\Phi u(x, t) - \Phi (v(x, t))| \\
\leq \int_0^1 \int_0^1 \xi^\beta G(x, t, \xi, \tau) \left| f(u(\xi, \tau)) - f(v(\xi, \tau)) \right| \, d\xi \, d\tau \\
\leq L \int_0^1 \int_0^1 \xi^\beta G(x, t, \xi, \tau) \, d\xi \, d\tau \, |u - v|_E.
\]
By definition of \(T\), we obtain that \(\Phi\) is a contraction mapping. Hence, by the Banach fixed point theorem, the equivalent integral equation (26) has a unique continuous solution \(u\) on \(\overline{Q}_T\). The proof of this theorem therefore is complete.

**Theorem 3.4** Let \(T_{\max}\) be the supremum of all \(T_i\) such that the solution \(u\) of the degenerate parabolic problem (1) is bounded. If \(T_{\max}\) is finite, then \(\sup_{(x, t) \in \overline{Q}_{T_{\max}}} |u(x, t)|\) is unbounded as \(t\) converges to \(T_{\max}\).

**Proof.** Suppose that \(\sup_{(x, t) \in \overline{Q}_{T_{\max}}} |u(x, t)|\) is finite. Let \(N\) be any positive constant with \(N > \sup_{(x, t) \in \overline{Q}_{T_{\max}}} |u(x, t)| + 1\). By theorem 3.1, there exists a finite time \(T > T_{\max}\) such that the degenerate parabolic problem has a unique continuous solution on \(\overline{Q}_T\). We then obtain a contradiction to definition of \(T_{\max}\). Hence, we get the result.

**Lemma 3.5** If \(v \in C(\overline{Q}_T) \cap C^1(\overline{Q}_T)\) satisfies
\[
x^\beta v - (x^\beta v)_t \geq B(x, t)v(x, t), \quad (x, t) \in Q_T, \\
v(0, t) \geq 0 \quad \text{and} \quad v(1, t) \geq 0, \quad t \in (0, T), \\
v(x, 0) \geq 0, \quad x \in \overline{T},
\]
where \(B(x, t)\) is bounded and nonnegative on \(\overline{Q}_T\), then \(v(x, t) \geq 0\) on \(\overline{Q}_T\).

**Proof.** Let \(\beta' \in (\beta, 1)\) be a positive constant and
\[
w(x, t) = v(x, t) + \eta(1 + x^{\beta' - \beta})e^{\alpha t}
\]
where \(\eta\) is any positive constant and \(c\) is a positive constant to be determined. We see that, on the parabolic boundary, \(w > 0\). Let us consider that for any \((x, t) \in Q_T,\)
\[
x^\beta w - (x^\beta w)_t - B(x, t)w(x, t) \\
\geq \eta \omega(1 + x^{\beta' - \beta}) - \eta \omega(\beta' - \beta)(\beta' - 1) - \frac{1}{x^{2-\beta}} \\
- \eta \omega \left( \max_{(x, t) \in \overline{Q}_T} B(x, t) \right)(1 + x^{\beta' - \beta}) \\
\geq \eta \omega(1 + x^{\beta' - \beta}) + \eta \omega(\beta' - \beta)(1 - \beta') - \frac{1}{x^{2-\beta}} \\
- 2\eta \omega \left( \max_{(x, t) \in \overline{Q}_T} B(x, t) \right).
\]
If \(\frac{\max_{(x, t) \in \overline{Q}_T} B(x, t)}{2} (\beta' - \beta)(1 - \beta')\), then, by equation (29),
\[
x^\beta w - (x^\beta w)_t - B(x, t)w(x, t) \\
\geq \eta \omega(1 + x^{\beta' - \beta}) + \eta \omega(\beta' - \beta)(1 - \beta') \left[ \frac{1}{x^{2-\beta}} - 1 \right] \\
\geq 0.
\]
On the other hand, we assume that
\[
\max_{(x, t) \in \overline{Q}_T} B(x, t) > \frac{1}{2} (\beta' - \beta)(1 - \beta').
\]
Let \(x_0(> 0)\) be the root of equation,
\[
\frac{\max_{(x, t) \in \overline{Q}_T} B(x, t)}{2x^{\beta' - \beta}} (\beta' - \beta)(1 - \beta'),
\]
and let \(c = \frac{2}{x_0^2(1 + x_0)^{\beta' - \beta}} \left( \max_{(x, t) \in \overline{Q}_T} B(x, t) \right)\). It follows from equation (29) that if \(x > x_0\), then the definition of \(c\) yields
\[
x^\beta w - (x^\beta w)_t - B(x, t)w(x, t) \\
\geq \eta \omega(1 + x^{\beta' - \beta}) - 2\max_{(x, t) \overline{Q}_T} B(x, t) \\
\geq \eta \omega(1 + x^{\beta' - \beta}) - 2\max_{(x, t) \overline{Q}_T} B(x, t) \\
\geq 0,
\]
and if \(x \leq x_0\), then
\[
x^\beta w - (x^\beta w)_t - B(x, t)w(x, t) \\
\geq \eta \omega(1 + x^{\beta' - \beta}) + \eta \omega(\beta' - \beta)(1 - \beta') - \frac{1}{x^{2-\beta}} \\
- 2\eta \omega \left( \max_{(x, t) \in \overline{Q}_T} B(x, t) \right) \\
\geq 0.
\]
Therefore
\[
x^\beta w - (x^\beta w)_t - B(x, t)w(x, t) \geq 0 \text{ on } Q_T.
\]
We would like to show that \(w > 0\) on \(Q_T\). Suppose that there exists a point \((x, t)\) with \(w(x, t) \leq 0\). We define the set
\[
A = \{ t \text{ such that } w(x, t) \leq 0 \text{ for some } x \in I \}
\]
is non-empty. Let \(\tilde{t}\) denote its infimum. Since \(w(x, 0) > 0\),
we have \(0 < \tilde{t} < T\). Then there exists some \(x_t \in I\) such that \(w(x_t, \tilde{t}) = 0\), \(w_t(x_t, \tilde{t}) \leq 0\) and \(w(x_t, \tilde{t}) = 0\). Since \(w\) attains its local minimum at \((x_t, \tilde{t})\), we have \(w_{nt}(x_t, \tilde{t}) \geq 0\). Thus
\[
0 \geq x_t^2 w(x_t, \tilde{t}) + (x_t^2 w_t(x_t, \tilde{t})_x - B(x_t, \tilde{t})w(x_t, \tilde{t}) > 0.
\]
This contradiction shows that \(w > 0\) on \(\overline{Q}_{\tilde{t}}\). As \(\eta \to 0\), we will get the result.

We will give additional properties of the solution \(u\) of the degenerate parabolic problem (1) in the next lemma.

**Lemma 3.6** \(u \geq u_0\) and \(u_t \geq 0\) on \(\overline{Q}_{\tilde{t}}\).

**Proof.** Let \(w(x, t) = u(x, t) - u_0(x)\) on \(\overline{Q}_{\tilde{t}}\). For any \((x, t) \in \overline{Q}_{\tilde{t}}\), equation (7) implies that
\[
x_t^2 w - (x_t^2 w_t)_x = x_t^2 f(u) + \frac{d}{dx} x_t^2 u_{nt}
\]
\[
\geq x_t^2 \left(f(u) - f(u_0)\right)
\]
\[
= x_t^2 f'(\zeta_x)w(x, t)
\]
where \(\zeta_x\) is a positive constant between \(u\) and \(u_0\). Further, on the parabolic boundary, \(w \geq 0\). Then lemma 3.5 yields that \(u \geq u_0\) on \(\overline{Q}_{\tilde{t}}\). Let \(h\) be any positive constant with \(h \in (0, T)\) and \(z(x, t) = u(x, t + h) - u(x, t)\) on \(\overline{Q}_{T-h}\). For any \((x, t) \in \overline{Q}_{T-h}\), we obtain
\[
x_t^2 z_t - (x_t^2 z_t)_x = x_t^2 f(u(x, t+h)) - x_t^2 f(u(x, t))
\]
\[
= x_t^2 f'(\zeta_z)z(x, t)
\]
where \(\zeta_z\) is a positive constant between \(u(x, t+h)\) and \(u(x, t)\). Moreover, since \(u \geq u_0\) on \(\overline{Q}_{\tilde{t}}\), we have that \(z(x, 0) = u(x, h) - u_0(x) \geq 0\) for \(x \in \tilde{T}\) and \(z(0, t) = 0 = z(1, t)\) for \(t \in (0, T-h)\). Lemma 3.5 implies that \(u_t \geq 0\) on \(\overline{Q}_{\tilde{t}}\). Therefore the proof of this lemma is complete.

**4 A sufficient condition to blow-up in finite time**

In this section, we will give the sufficient condition to ensure occurrence of blow-up in finite time. Let \(\varphi_1(>0)\) be the first eigenfunction of the singular eigenvalue problem (8) and \(\lambda_1(>0)\) its corresponding eigenvalue. Further, we assume that
\[
\int_0^1 x^2 \varphi_1(x) dx = 1.
\]
We define the function \(H\) by
\[
H(t) = \int_0^1 x^2 u(x, t) \varphi_1(x) dx.
\]
We note that since \(\lim_{s \to \infty} \frac{f(s)}{s} \to \infty\), there exists a positive constant \(z_0\) such that \(f(s) - \lambda s > 0\) for any \(s \geq z_0\).

**Theorem 4.1** Let
\[
\int z_0 f(s) - \lambda s < \infty.
\]
Then, for any initial function \(u_0\) such that
\[
H(0) = \int_0^1 x^2 u_0(x) \varphi_1(x) dx \geq z_0,
\]
the solution \(u\) of a degenerate parabolic problem (1) blows up in finite time.

**Proof.** Suppose that \(u\) exists for all time \(t \geq 0\) for any \(x \in \tilde{T}\). By multiplying equation (1) both side by \(\varphi_1\) and integrating with respect to \(x\) over its domain, we have
\[
\frac{dH(t)}{dt} + \lambda_1 H(t) = \int_0^1 x^2 f(u(x, t)) \varphi_1(x) dx.
\]
By convexity of \(f\), we can apply the Jensen’s inequality to equation (30) and then we obtain
\[
\frac{dH(t)}{dt} \geq f(H(t)) - \lambda_1 H(t).
\]
From equation (30), we differentiate the function \(H\) with respect to \(x\) and then we have
\[
\frac{dH(t)}{dt} \geq \int_0^1 x^2 u(x, t) \varphi_1(x) dx.
\]
Thus lemma 3.6 yields that \(\frac{dH(t)}{dt} > 0\). We further get that \(H(t) \geq z_0\) for all \(t\). By equation (31), we have
\[
\frac{dH(t)}{dt} \geq f(H(t)) - \lambda_1 H(t) \geq 0\text{ for } t > 0 \quad \text{and} \quad H(0) \geq z_0.
\]
So we separate variables to find
\[
t \leq \int_{H_0}^{H(t)} ds - \lambda_1 s \leq \int_0^1 f(s) - \lambda s < \infty.
\]
Hence \(t\) is finite and a contradiction is achieved. The solution can not exist for all positive time.

**5 Blow-up set**

In this section, the blow-up set for \(u\) of problem (1) is shown.

**Theorem 5.1** The blow-up set of a solution \(u\) of the degenerate parabolic problem (1) is \(\tilde{T}\).

**Proof.** From equation (26), there are two positive
constants $c_1$ and $c_2$ such that
\[
\sup_{(x,t)\in \bar{Q}_{T_{\text{max}}}} u(x,t) \leq c_1 + c_2 \int_0^1 f(u(\xi, \tau))d\xi d\tau.
\]
Theorem 3.2 implies that as $t \to T_{\text{max}}$, 
\[
\int_0^1 \int_0^1 f(u(\xi, \tau))d\xi d\tau \to \infty.
\]
On the other hand, there are two positive constant $c_1$ and $c_2$ such that
\[
u(x,t) \geq c_1 + c_2 \int_0^1 f(u(\xi, \tau))d\xi d\tau \text{ for any } (x,t) \in \bar{Q}_{T_{\text{max}}}.
\]
(33)

As $t \to T_{\text{max}}$, we obtain that, by (33), $u(x,t) \to \infty$ for all $x \in I$. Furthermore, for $x \in (0,1]$, we can find a sequence $\{(x_n, t_n)\}$ such that $\lim_{n \to \infty} u(x_n, t_n) \to \infty$.

Hence, the blow-up set of a solution of a degenerate parabolic problem (1) is $\bar{T}$.

6 Generalized problem

In this section, we extend our degenerate parabolic initial-boundary value problem (1) in more general form by replacing function coefficients of $u_t$ and $u_x$, $x^s$ and $x^q$, by functions $k(x)$ and $p(x)$, respectively. We now consider the following degenerate parabolic initial-boundary value problem,
\[
k(x)u_t - (p(x)u_x)_x = k(x)f(u), \quad (x,t) \in \bar{Q}_T,
\]
\[
\begin{array}{c}
u(0,t) = 0 = u(1,t), \quad t \in (0,T), \\
u(x,0) = u_0(x), \quad x \in \bar{T},
\end{array}
\]
(34)

where $k$ and $p$ are determined. In order to obtain the same results as a degenerate parabolic problem (1), we have to assume the following.

(C) $k \in C(\bar{T})$, $k(0) = 0$ and $k$ is positive on $(0,1]$.

(D) $p \in C^1(\bar{T})$, $p(0) = 0$, $p$ is positive on $(0,1]$ and $p'$ is positive on $\bar{T}$.

As obtaining equation (8) the corresponding singular eigenvalue problem of (34) is defined by
\[
\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + \lambda k(x)\phi(x) = 0 \text{ for } x \in I,
\]
\[
\phi(0) = 0 = \phi(1).
\]
(35)

We notice that it follows from conditions (C) and (D) that the point $x = 0$ is a singular point of problem (35). From equation (35), we can rewrite the corresponding singular eigenvalue problem (35) in the following equivalent form,
\[
x^2\phi''(x) + x\frac{p'(x)}{p(x)}\phi'(x) + x^2\frac{k(x)}{p(x)}\phi(x) = 0 \text{ on } I,
\]
\[
\phi(0) = 0 = \phi(1).
\]
(36)

To ensure the existence of eigenfunctions $\phi_n$ and eigenvalues $\lambda_n$, we need an additional condition on functions $k$ and $p$.

(E) The limit of $\frac{xp'(x)}{p(x)}$ and $\frac{x^2k(x)}{p(x)}$ are finite as $x$ converges to 0 and $\frac{xp'(x)}{p(x)}$ and $\frac{x^2k(x)}{p(x)}$ are analytic at $x = 0$.

We note that theorem 5.7.1 [2] implies that eigenfunctions $\phi_n$ and eigenvalues $\lambda_n$ of a corresponding singular eigenvalue problem (36) exist. Moreover completeness of eigenfunctions $\phi_n$ of a singular eigenvalue problem (36) results from the next hypothesis.

(F) $\int_0^1 H(x, \xi)k(x)\phi(x)d\xi dx$ is finite where $H$ is the corresponding Green’s function to problem (36).

Well-known properties of eigenfunctions $\phi_n$ and eigenvalues $\lambda_n$ are shown in next lemma referred to [xx].

Lemma 6.1

6.1.1. $\int_0^1 k(x)\phi_n(x)\phi_n(x)dx = \left\{\begin{array}{ll} 1 & \text{for } m = m, \\
0 & \text{for } m \neq n.
\end{array}\right.$

6.1.2. All eigenvalues are real and positive.

6.1.3. Eigenfunctions are complete with the weight function $k$.

6.1.4. $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ and $\lim_{n \to \infty} \lambda_n = \infty$.

6.1.5. $\int_0^1 p(x)\phi_n'(x)\phi_n'(x)dx = \left\{\begin{array}{ll} \lambda_n & \text{for } n = m, \\
0 & \text{for } n \neq m.
\end{array}\right.$

6.1.6. For any $n \in \mathbb{N}$, $\phi_n \in C^\infty((0,1])$.

The Green’s function $G$ corresponding to the degenerate parabolic initial-boundary value problem (34) is determined by the following problem: let $x, \xi$ be in $I$ and $t, \tau$ in $(0, T)$,
\[
k(x)G_t - (p(x)G_x)_x = \delta(x - \xi)\delta(t - \tau),
\]
\[
G(0, t, \xi, \tau) = 0 = G(1, t, \xi, \tau),
\]
\[
G(x, t, \xi, \tau) = 0 \text{ for } t > \tau,
\]
(37)

where $\delta$ is the Dirac delta function. As obtaining equation (25), the corresponding Green’s function
of (37) defined by
\[ G(x,t,\xi,\tau) = \sum_{n=1}^{\infty} \phi_n(\xi) \phi_n(x) e^{-\lambda_n(t-t)} \text{ for } x, \xi \in \overline{I} \text{ and } 0 \leq \tau < t \leq T. \]

The following lemma is due to properties of \( G \) corresponding to problem (34).

**Lemma 6.2.** Assume that \( \lambda_n = O(n^s) \) for some \( s > 1 \) as \( n \to \infty \).

6.2.1. \( G \) is continuous for \( x, \xi \in \overline{I} \) and \( 0 \leq \tau < t \leq T \).

6.2.2. \( G \) is positive for \( x, \xi \in \overline{I} \) and \( 0 \leq \tau < t \leq T \).

As equation (26), the equivalent integral equation to the extended degenerate parabolic problem (34) is given by
\[ u(x,t) = \int_{0}^{1} k(\xi) G(x,t,\xi,\tau) u_0(\xi) d\xi + \int_{0}^{1} k(\xi) G(x,t,\xi,\tau) f(u(\xi,\tau)) d\xi d\tau. \]

Next theorem shows the existence of an unique solution of the extended degenerate parabolic problem (34) before blow-up occurs

**Theorem 6.3** There exists a finite time \( T_1 > 0 \) such that the extended degenerate parabolic problem (34) has a unique continuous solution \( u \) on the finite time interval \( [0,T_1] \) for any \( x \in \overline{I} \).

**Proof.** The proof of this theorem is similar to that of theorem 3.3.

Let \( \tilde{T}_{\text{max}} \) be the supremum of all \( T_1 \) such that the solution \( u \) of the extended degenerate parabolic problem (34) is bounded. The following theorem says that the solution of the extended problem (34) blows up in finite time if \( \tilde{T}_{\text{max}} \) is finite.

**Theorem 6.4** if \( \tilde{T}_{\text{max}} \) is finite, then \( \sup_{(x,t) \in \overline{Q}_{\text{max}}} |u(x,t)| \) is unbounded as \( t \) converges to \( \tilde{T}_{\text{max}} \).

**Proof.** The proof of this theorem is similar to that of theorem 3.4.

Furthermore we give the additional properties of a solution \( u \) of the extended degenerate parabolic problems (34), that is, positivity and increasing in \( t \) of \( u \). In order to obtain results, we need the following lemma.

**Lemma 6.5** Let \( v \) be a classical solution of the following problem:
\[ v_t - \frac{1}{k(x)} (p(x)v)_x - B(x,t)v(x,t) \]
\[ \geq B(x,t)v(x,t) \text{ for } (x,t) \in Q_T, \]
\[ v(0,t) = v_1(t) \text{ for } t \in (0,T), \]
\[ v(x,0) = u_0(x) \geq 0 \text{ for } x \in \overline{I}, \]

where \( B(x,t) \) is a nonnegative and bounded function on \( \overline{Q}_T \). Then \( v(x,t) \geq 0 \) for any \( (x,t) \in \overline{Q}_T \).

**Proof.** Let \( \eta \) be any positive constant. Let
\[ w(x,t) = v(x,t) + \eta (1 + x^2) e^{\tau} \]
where \( \eta \) is some positive constant with \( \eta > \max_{0 \leq \tau < t} p(x) + \max_{(x,t) \in [0,1]} k(x) \).

Let us consider
\[ k(x)w_t - (p(x)w)_x - k(x)B(x,t)w(x,t) = k(x)w_t - (p(x)w)_x - k(x)B(x,t)v(x,t) + \eta (1 + x^2) e^{\tau} \]
\[ -2 \eta e^{\tau} [x'(x) + p(x)] - k(x)B(x,t)\eta(1 + x^2) e^{\tau} \]
\[ \geq \eta e^{\tau} \left( c(1 + x^2) - 2 [x'(x) + p(x)] - k(x)B(x,t)(1 + x^2) \right) \]
\[ \geq 2 \eta e^{\tau} \left[ c - \max_{(x,t) \in [0,1]} p(x) \right] \]
\[ \geq k(x) \max_{(x,t) \in [0,1]} B(x,t) \]

By the definition of \( c \), we have that for any \( (x,t) \in \overline{Q}_T \),
\[ k(x)w_t - (p(x)w)_x - k(x)B(x,t)w(x,t) \geq 0 \]

We see that \( w(x,t) \geq 0 \) for \( (x,t) \in [0,1] \times (0,T) \cup \overline{I} \times \{ 0 \} \). We would like to show that \( w(x,t) > 0 \) for any \( (x,t) \in \overline{Q}_T \). Suppose that there exists a point \( (x_1,t_1) \) with \( w(x_1,t_1) \leq 0 \). We define the set \( A \) by
\[ A = \{ t \text{ such that } w(x,t) \leq 0 \text{ for some } x \in I \}. \]

Thus, the set \( A \) is non-empty. Let \( \tilde{t} = \inf A \). Since \( w(x,0) = u_0(x) + \eta(1 + x^2) > 0 \) for \( x \in I \), we obtain that \( \tilde{t} > 0 \). Furthermore, since \( A \) is closed, by the definition of \( \tilde{t} \), there exists a point \( x_2 \) in \( I \) such that \( w(x_2,\tilde{t}) = 0 \), \( w(x_2,t) \leq 0 \) and \( w(x,\tilde{t}) > 0 \).

Moreover, we also get that \( w_{x_2}(x_2,\tilde{t}) \geq 0 \) because \( w \) attains its local minimum at the point \( x_2 \). Then we have that
\[ 0 \geq k(x)w_{x_2}(x_2,\tilde{t}) \]
\[ \geq k(x)w_{x_2}(x_2,\tilde{t}) - p(x_2)w_{x_2}(x_2,\tilde{t}) - p'(x_2)w(x_2,\tilde{t}) \]
\[ -k(x)B(x_2,\tilde{t})w(x_2,\tilde{t}) > 0. \]

Therefore, we get a contradiction. This shows that \( w(x,t) > 0 \) for any \( (x,t) \in \overline{Q}_T \). Since \( \eta \) is arbitrary, we let \( \eta \to 0^+ \) and then we obtain the desired result.

**Lemma 6.6** Let \( u \) be a continuous solution of the
extended degenerate parabolic problem (34). Then \( u(x,t) \geq u_0(x) \) and \( u_0(x,t) \geq 0 \) for any \((x,t) \in \overline{Q}_T\).

To ensure that a solution \( u \) of the extended degenerate parabolic problem (34) blows up in finite time, we give the condition to guarantee the occurrence for blow-up in finite time. Let \( \phi(\cdot, >0) \) be the first eigenfunction of the singular eigenvalue problem (35) and \( \lambda_1(\cdot, >0) \) its corresponding eigenvalue. Moreover we suppose

\[
\int_0^1 k(x)\phi(x)dx = 1.
\]

We construct the function \( H \) by

\[
H(t) = \int_0^1 k(x)u(x,t)\phi(x)dx.
\]

Notice that since \( \lim_{s \to \infty} \frac{f(s)}{s} \to \infty \), there exists a positive constant \( z_0 \) such that

\[
f(s) - \lambda s > 0 \quad \text{for any } s \geq z_0.
\]

**Theorem 6.5** Assume that

\[
\int_0^s \frac{ds}{f(s) - \lambda s} < \infty.
\]

Then, for any initial function \( u_0 \) with

\[
H(0) = \int_0^1 k(x)u_0(x)\phi(x)dx \geq z_0,
\]

the solution \( u \) of the extended degenerate parabolic problem (34) blows up in finite time.

**Proof.** By modifying the proof of theorem 4.1, this theorem is proven.

The last theorem concern the blow-up set of the extended degenerate parabolic problem (34).

**Theorem 6.6** The blow-up set of a solution \( u \) of the extended degenerate parabolic problem (34) is \( \overline{T} \).

**Proof.** The proof of this theorem is similar to that of theorem 5.1.

### 7 Conclusion

In this work, we obtain four main results for the degenerate parabolic problem (1) which are the theorem 3.3, 3.4, 4.1 and 5.1. The first main result, the theorem 3.3, says that there is a finite time \( T \) with \( T > 0 \) such that the degenerate parabolic problem (1) has a unique solution \( u \) on the time interval \([0,T]\) for any \( x \) in \( \overline{T} \). Theorem 3.3 can be proven by the Banach fixed point theorem. Let \( T_{\max} \) be the supremum of all \( T \) such that the solution \( u \) of the degenerate parabolic problem (1) is bounded. Theorem 3.4 shows that the solution \( u \) of our degenerate parabolic problem (1) blows up if \( T_{\max} \) is finite. In fact, \( T_{\max} \) may be not finite. This is the reason why theorem 4.1 is constructed. Theorem 4.1 is the sufficient condition to blow-up in finite time. The last main result of problem (1), theorem 5.1, indicates that the blow-up set of the degenerate parabolic problem (1) is \( \overline{T} \). We finally extend our degenerate parabolic problem (1) into the general form: \( k(x)u_t - (p(x)u_x)_x = k(x)f(u(x,t)) \) where \( k \) and \( p \) are given functions. Under some conditions, we also obtain the same results as the previous problem, that is, theorem 6.3-6.6.

### Acknowledgment

Authors would like to thank the Department of Mathematics, Faculty of Science, Mahidol University for financial support during the preparation of this paper.

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