About Three Important Transformations Groups

MONICA A.P. PURCARU
Transilvania University of Braşov
Department of Mathematics
Iuliu Maniu Street 50, 500091 Braşov
ROMANIA
m.purcaru@yahoo.com

MIRELA TÂRNOVEANU
Transilvania University of Braşov
Department of Mathematics
Iuliu Maniu Street 50, 500091 Braşov
ROMANIA
mi_tarnoveanu@yahoo.com

LAURA CIUPALĂ
Transilvania University of Braşov
Department of Computer Science
Iuliu Maniu Street 50, 500091 Braşov
ROMANIA
laura_ciupala@yahoo.com

Abstract: In the present paper we introduce the concepts of conformal metrical d-structure and of conformal metrical N-linear connection with respect to the conformal metrical d-structure, corresponding to an 1-form on a generalized Hamilton space. We determine the set of all conformal metrical N-linear connections in the case when the nonlinear connection is arbitrary and we find important examples and particular cases. We find the transformations group of these connections. We study the role of the torsion d-tensor fields T^i_{jk}, S^i_{jk} and S_i^{jk} in this theory, especially in the determination of the set of all semisymmetric conformal metrical N-linear connections with respect to the conformal metrical d-structure, corresponding to the same nonlinear connection N. We give the transformations group of these connections and other two important groups and we find their remarkable invariants. Finally we determine the set of all metrical N-linear connections in the case when the nonlinear connection is arbitrary, we give important examples and particular cases and for the case when the nonlinear connection is fixed we find the transformations group of these connections.

Key–Words: second order cotangent bundle, nonlinear connection, N-linear connection, metrical d-structure, conformal metrical d-structure, conformal metrical N-linear connection, semisymmetric conformal metrical N-linear connection, metrical N-linear connection, transformations group, subgroup, invariants.

1 Introduction

The geometry of the cotangent bundle (T^*M, π^*, M) has been studied by R.Miron, S.Watanabe and S.Ykeda in [5], by K.Yano and S.Ishihara in [14], by R.Miron, D.Hrimiuc, H.Shimada and S.Sabău in [4], C. Udrişte and O. Şandru in [12] etc. Contributions in development of this theory had also:

The differential geometry of the second order cotangent bundle $\left(T^{*2}M, \pi^{*^2}, M\right)$ was introduced and studied by R. Miron in [2] , R. Miron, D. Hrimiuc, H. Shimada, V.S. Sabău in [4], Gh. Atanasiu and M. Târnoveanu in [1], C. Udrişte, M.Popescu and P.Popescu in [11], C. Udrişte in [8], [9], C. Udrişte, D. Opriş in [10], C. Udrişte, I. Tevy in [13], etc.

In the present section we keep the general setting from R. Miron, D. Hrimiuc, H. Shimada, V.S. Sabău, [4] and subsequently we recall only some needed notions. For more details see [4] .

Let M be a real $n-$dimensional manifold and let $\left(T^{*^2}M, \pi^{*^2}, M\right)$ be the dual of the $2-$tangent bundle, or $2-$cotangent bundle. A point $u \in T^{*^2}M$ can be written in the form $u = (x, y, p)$, having the local coordinates $(x^i, y^i, p_i), (i = 1, 2, ..., n)$.

A change of local coordinates on the $3n$ dimensional manifold $T^{*^2}M$ is

$$\begin{cases} \bar{x}^i = \bar{x}^i\left(x^1, ..., x^n\right), \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \neq 0, \\ \bar{y}^i = \frac{\partial \bar{x}^i}{\partial x^j} \cdot y^j, \\ \bar{p}_i = \frac{\partial x^j}{\partial \bar{x}^i} \cdot p_j, (i, j = 1, 2, ..., n) . \end{cases} \quad (1)$$

We denote by $\widetilde{T^{*^2}M} = T^{*^2}M \setminus \{0\}$, where $0 : M \longrightarrow T^{*^2}M$ is the null section of the projection π^{*^2}.

Let us consider the tangent bundle of the differentiable manifold $\widetilde{T^{*^2}M}$, $\left(T\widetilde{T^{*^2}M}, \tau^{*^2}, \widetilde{T^{*^2}M}\right)$, where τ^{*^2} is the canonical projection and the vertical distribution $V : u \in \widetilde{T^{*^2}M} \longrightarrow V(u) \subset T_u\widetilde{T^{*^2}M}$, locally generated by the vector fields $\left\{ \frac{\partial}{\partial y^i}\Big|_u, \frac{\partial}{\partial p_i}\Big|_u \right\}, \forall u \in \widetilde{T^{*^2}M}$.

The following $\mathcal{F}\left(\widetilde{T^{*^2}M}\right) -$ linear mapping $J : \chi\left(\widetilde{T^{*^2}M}\right) \longrightarrow \chi\left(\widetilde{T^{*2}M}\right)$, defined by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, J\left(\frac{\partial}{\partial y^i}\right) = 0, J\left(\frac{\partial}{\partial p_i}\right) = 0, \quad \forall u \in \widetilde{T^{*^2}M} \quad (2)$$

is a tangent structure on $T^{*^2}M$.

We denote with N a nonlinear connection on the manifold $T^{*^2}M$, with the local coefficients $\left(N^j{}_i\left(x,y,p\right),N_{ij}\left(x,y,p\right)\right),\left(i,j=1,2,...,n\right).$

Hence, the tangent space of $T^{*^2}M$ in the point $u\in T^{*^2}M$ is given by the direct sum of vector spaces:

$$T_uT^{*^2}M=N\left(u\right)\oplus W_1\left(u\right)\oplus W_2\left(u\right), \quad \forall u\in T^{*^2}M. \tag{3}$$

A local adapted basis to the direct decomposition (3) is given by:

$$\left\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\right\},\left(i=1,2,...,n\right), \tag{4}$$

where:

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^j{}_i\frac{\partial}{\partial y^j}+N_{ij}\frac{\partial}{\partial p_j}. \tag{5}$$

With respect to the coordinates transformations (1), we have the rules:

$$\begin{cases}\frac{\delta}{\delta x^i}=\frac{\partial\bar{x}^j}{\partial x^i}\frac{\delta}{\delta\bar{x}^j};\\ \frac{\partial}{\partial y^i}=\frac{\partial\bar{x}^j}{\partial x^i}\cdot\frac{\partial}{\partial\bar{y}^j};\\ \frac{\partial}{\partial p_i}=\frac{\partial x^i}{\partial\bar{x}^j}\cdot\frac{\partial}{\partial\bar{p}_j}.\end{cases} \tag{5'}$$

The dual basis of the adapted basis (4) is given by:

$$\left\{\delta x^i,\delta y^i,\delta p_i\right\}, \tag{6}$$

where:

$$\begin{cases}\delta x^i=dx^i,\\ \delta y^i=dy^i+N^i{}_jdx^j,\\ \delta p_i=dp_i-N_{ji}dx^j.\end{cases} \tag{6'}$$

With respect to (1), the covector fields (6) are transformed by the rules:

$$\delta\bar{x}^i=\frac{\partial\bar{x}^i}{\partial x^j}\delta x^j,\ \delta\bar{y}^i=\frac{\partial\bar{x}^i}{\partial x^j}\delta y^j,\ \delta\bar{p}_i=\frac{\partial x^j}{\partial\bar{x}^i}\delta p_j. \tag{6''}$$

Let D be an $N-$linear connection on $T^{*^2}M$, with the local coefficients in the adapted basis: $D\Gamma\left(N\right)=\left(H^i{}_{jk},C^i{}_{jk},C_i{}^{jk}\right).$

An $N-$ linear connection D is uniquely represented, in the adapted basis (4) in the following form:

$$\begin{cases}D_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i}=H^k{}_{ij}\frac{\delta}{\delta x^k},\\ D_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i}=H^k{}_{ij}\frac{\partial}{\partial y^k},\\ D_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial p_i}=-H^i{}_{kj}\frac{\partial}{\partial p_k},\\ D_{\frac{\partial}{\partial y^j}}\frac{\delta}{\delta x^i}=C^k{}_{ij}\frac{\delta}{\delta x^k},\\ D_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}=C^k{}_{ij}\frac{\partial}{\partial y^k},\\ D_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial p_i}=-C^i{}_{kj}\frac{\partial}{\partial p_k},\\ D_{\frac{\partial}{\partial p_j}}\frac{\delta}{\delta x^i}=C_i{}^{kj}\frac{\delta}{\delta x^k},\\ D_{\frac{\partial}{\partial p_j}}\frac{\partial}{\partial y^i}=C_i{}^{kj}\frac{\partial}{\partial y^k},\\ D_{\frac{\partial}{\partial p_j}}\frac{\partial}{\partial p_i}=-C_k{}^{ij}\frac{\partial}{\partial p_k}.\end{cases} \tag{7}$$

An N-linear connection D with the local coefficients $D\Gamma\left(N\right)=\left(H^i{}_{jk},C^i{}_{jk},C_i{}^{jk}\right)$ determines the $h-,w_1-,w_2-$ covariant derivatives in the tensor algebra of d-tensor fields.

Definition 1 ([1]) *An N-linear connection on $T^{*^2}M$, is called semisymmetric if:*

$$\begin{cases}T^i{}_{jk}=\frac{1}{2}\left(-\delta^i_j\sigma_k+\delta^i_k\sigma_j\right),\\ S^i{}_{jk}=\frac{1}{2}\left(-\delta^i_j\tau_k+\delta^i_k\tau_j\right),\\ S_i^{jk}=-\frac{1}{2}\left(-\delta^j_iv^k+\delta^k_iv^j\right),\end{cases} \tag{8}$$

where $\sigma,\tau\in\chi^*\left(T^{*^2}M\right)$ and $v\in\chi\left(T^{*^2}M\right)$.

2 Conformal metrical N-linear connections in a generalized Hamilton space

Definition 2 ([4]) *A generalized Hamilton space of order two is a pair $GH^{(2)n}=\left(M,g^{ij}\left(x,y,p\right)\right)$, where:*

1° g^{ij} is a $d-$ tensor field of type $(2,0)$, symmetric and nondegenerate on the manifold $T^{*^2}M$.

2° The quadratic form $g^{ij}X_iX_j$ has a constant signature on $T^{*^2}M$.

g^{ij} is called the fundamental tensor or metric tensor of the space $GH^{(2)n}$.

In the case when $T^{*^2}M$ is a paracompact manifold then on $T^{*^2}M$ there exist the metric tensors $g^{ij}\left(x,y,p\right)$ positively defined such that $\left(M,g^{ij}\right)$ is a generalized Hamilton space.

Definition 3 ([4]) *A generalized Hamilton metric $g^{ij}\left(x,y,p\right)$ of order two (on short $GH-$metric) is* *called reductible to an Hamilton metric ($H-$metric) of order two if there exists a function $H(x,y,p)$ on $T^{*^2}M$ such that:*

$$g^{ij} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}. \qquad (9)$$

The covariant tensor field g_{ij} is obtained from the equations:

$$g_{ij}g^{jk} = \delta_i^k. \qquad (10)$$

g_{ij} is a symmetric, nondegenerate and covariant of order two, $d-$tensor field.

Definition 4 ([4]) *An $N-$linear connection D is called metrical with respect to $GH-$metric g^{ij} if:*

$$g^{ij}{}_{|k} = 0, \;\; g^{ij}|_k = 0, \;\; g^{ij}|^k = 0. \qquad (11)$$

The tensorial equations (11) imply:

$$g_{ij|k} = 0, \;\; g_{ij}|_k = 0, \;\; g_{ij}|^k = 0. \qquad (12)$$

Theorem 5 ([4])*1. There exists a unique $N-$linear connection $\overset{0}{D}\Gamma(N) = \left(\overset{0}{H^i}{}_{jk}, \overset{0}{C^i}{}_{jk}, \overset{0}{C_i}{}^{jk}\right)$ having the properties:*

1°. The nonlinear connection N is a priori given.

2°. $\overset{0}{D}\Gamma(N)$ is metrical with respect to GH-metric g^{ij} i.e.(12) are verified.

3°. The torsion tensors $\overset{0}{T^i}{}_{jk}, \overset{0}{S^i}{}_{jk}$, and $\overset{0}{S_i}{}^{jk}$ vanish.

2. The previous connection has the coefficients $\overset{0}{C^i}{}_{jk}$ and $\overset{0}{C_i}{}^{jk}$ given by

$$\begin{cases} \overset{0}{C^i}{}_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial y^j} + \frac{\partial g_{jm}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^m}\right), \\ \overset{0}{C_i}{}^{jk} = \frac{1}{2}g_{im}\left(\frac{\partial g^{mk}}{\partial p_j} + \frac{\partial g^{jm}}{\partial p_k} - \frac{\partial g^{jk}}{\partial p_m}\right), \end{cases} \qquad (13)$$

and $\overset{0}{H^i}{}_{jk}$ are generalized Christoffel symbols:

$$\overset{0}{H^i}{}_{jk} = \frac{1}{2}g^{im}\left(\frac{\delta g_{mk}}{\delta x^j} + \frac{\delta g_{jm}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^m}\right). \qquad (14)$$

The operators of Obata's type are given by:

$$\begin{cases} \Omega^{ij}_{hk} = \frac{1}{2}\left(\delta^i_h\delta^j_k - g_{hk}g^{ij}\right), \\ \Omega^{*ij}_{hk} = \frac{1}{2}\left(\delta^i_h\delta^j_k + g_{hk}g^{ij}\right). \end{cases} \qquad (15)$$

The operators of Obata's type have the same properties as the one associated with a Finsler space ([3]).

Let $\mathcal{S}_2(T^{*^2}M)$ be the set of all symmetric d-tensor fields, of the type $(0,2)$. As is easily shown, the relations for $a_{ij}, b_{ij} \in \mathcal{S}_2(T^{*^2}M)$ defined by:

$$\begin{array}{l} (a_{ij} \sim b_{ij}) \Leftrightarrow ((\exists)\lambda(x,y,p) \in \mathcal{F}(T^{*^2}M), \\ a_{ij}(x,y,p) = e^{2\lambda(x,y,p)}b_{ij}(x,y,p),) \end{array} \qquad (16)$$

is an equivalence relation on $\mathcal{S}_2(T^{*^2}M)$.

Definition 6 *The equivalent class $\hat{g}$ of $\mathcal{S}_2(T^{*^2}M)/_\sim$ to which the fundamental d-tensor field g_{ij} belongs, is called conformal metrical d-structure.*

Thus:

$$\begin{array}{l} \hat{g} = \{g' | g'_{ij}(x,y,p) = e^{2\lambda(x,y,p)}g_{ij}(x,y,p), \\ \lambda(x,y,p) \in \mathcal{F}(T^{*^2}M)\}. \end{array} \qquad (17)$$

Definition 7 *An N-linear connection, D, with local coefficients: $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$, for which there exists the 1-form ω, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$, such that:*

$$\begin{cases} g_{ij|k} = 2\omega_k g_{ij}, \;\; g_{ij}|_k = 2\dot{\omega}_k g_{ij}, \\ g_{ij}|^k = 2\ddot{\omega}^k g_{ij}, \end{cases} \qquad (18)$$

where $|_k$, $|_k$ and $|^k$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to D is called conformal metrical N-linear connection, with respect to the conformal metrical d-structure $\hat{g}$, corresponding to the 1-form ω and is denoted by: $D\Gamma(N,\omega)$.

Proposition 8 *If $D\Gamma(N,\omega) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$ are the local coefficients of a conformal metrical N-linear connection in $T^{*^2}M$, with respect to the conformal metrical structure $\hat{g}$, corresponding to the 1-form ω, then:*

$$\begin{cases} g^{ij}{}_{|k} = -2\omega_k g^{ij}, \;\; g^{ij}|_k = -2\dot{\omega}_k g^{ij}, \\ g^{ij}|^k = -2\ddot{\omega}^k g^{ij}. \end{cases} \qquad (19)$$

Proof. Using the relations (18), by covariant derivation from (10) we have the results.

Proposition 9 *The operators of Obata's type are covariant constant with respect to any conformal metrical N-linear connection, D:*

$$\begin{cases} \Omega^{ir}_{sj|l} = 0, \Omega^{ir}_{sj}|_l = 0, \Omega^{ir}_{sj}|^l = 0, \\ \Omega^{*ir}_{sj|l} = 0, \Omega^{*ir}_{sj}|_l = 0, \Omega^{*ir}_{sj}|^l = 0, \end{cases} \qquad (20)$$

where $|_l$, $|_l$ and $|^l$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to D.

Proof. Using the relations (18) and (19) by covariant derivation from (15) we have the results.

For any representative $g' \in \hat{g}$ we have:

Theorem 10 *For $g'_{ij} = e^{2\lambda} g_{ij}$, a conformal metrical N-linear connection with respect to the conformal metrical structure $\hat{g}$, corresponding to the 1-form ω, $D\Gamma(N,\omega)$, satisfies:*

$$\begin{cases} g'_{ij|k} = 2\omega'_k g'_{ij}, \;\; g'_{ij}|_k = 2\dot{\omega}'_k g'_{ij}, \\ g'_{ij}|^k = 2\ddot{\omega}'^k g'_{ij}, \end{cases} \tag{21}$$

where $\omega' = \omega + d\lambda$.

Since in Theorem 10 $\omega' = 0$ is equivalent to $\omega = d(-\lambda)$, we have:

Theorem 11 *A conformal metrical N-linear connection with respect to $\hat{g}$, corresponding to the 1-form ω, $D\Gamma(N,\omega)$, is metrical with respect to $g' \in \hat{g}$, i.e. $g'_{ij|k} = g'_{ij}|_k = g'_{ij}|^k = 0$ if and only if ω is exact.*

We shall determine the set of all conformal metrical N-linear connections, with respect to $\hat{g}$.

Let $\overset{0}{D}\ \Gamma(\overset{0}{N}) = \left(\overset{0}{H^i}_{jk}, \overset{0}{C^i}_{jk}, \overset{0}{C_i}^{jk}\right)$ be the local coefficients of a fixed $\overset{0}{N}$- linear connection $\overset{0}{D}$ on $T^{*2}M$, where $(\overset{0}{N^j}_i(x,y,p), \overset{0}{N}_{ij}(x,y,p)), (i,j = 1,2,...,n)$ are the local coefficients of the nonlinear connection $\overset{0}{N}$.

Then any N-linear connection, D, on $T^{*2}M$, with the local coefficients $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$, where $(N^j{}_i(x,y,p), N_{ij}(x,y,p)), (i,j = 1,2,...,n)$ are the local coefficients of the nonlinear connection N, can be expressed in the form ([6]):

$$\begin{cases} N^i_j = \overset{0}{N^i}_j - A^i{}_j, \\ N_{ij} = \overset{0}{N}_{ij} - A_{ij}, \\ H^i_{jk} = \overset{0}{H^i}_{jk} + A^l{}_k \overset{0}{C^i}_{jl} - A_{kl} \overset{0}{C_j}^{il} - \\ \qquad -B^i{}_{jk}, \\ C^i_{jk} = \overset{0}{C^i}_{jk} - D^i{}_{jk}, \\ C_i^{jk} = \overset{0}{C_i}^{jk} - D_i^{jk}, (i,j,k = 1,2,...,n), \end{cases} \tag{22}$$

with

$$A^k{}_{\overset{0}{i|j}} = 0, \; A_{\overset{0}{ik|j}} = 0, (i,j,k = 1,2,...,n), \tag{23}$$

where $\overset{0}{|}_k$ denotes the h-covariant derivative with respect to $\overset{0}{D}$ and $(A^i{}_j, A_{ij}, B^i{}_{jk}, D^i{}_{jk}, D_i^{jk})$ are the components of the difference tensor fields of D from $\overset{0}{D}$.

Theorem 12 *Let $\overset{0}{D}$ be a given $\overset{0}{N}$ -linear connection, with local coefficients $\overset{0}{D}\ \Gamma(\overset{0}{N}) = \left(\overset{0}{H^i}_{jk}, \overset{0}{C^i}_{jk}, \overset{0}{C_i}^{jk}\right)$. The set of all conformal metrical N-linear connections with respect to $\hat{g}$, corresponding to the 1-form ω, with local coefficients $D\Gamma(N,\omega) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ is given by:*

$$\begin{cases} N^i_j = \overset{0}{N^i}_j - X^i_j, N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^i_{jk} = \overset{0}{H^i}_{jk} + X^l{}_k \overset{0}{C^i}_{jl} - X_{kl} \overset{0}{C_j}^{il} + \\ +\frac{1}{2} g^{im}(g_{\overset{0}{mj|k}} - 2\omega_k g_{mj} + g_{mj} \overset{0}{|}_l X^l{}_k - \\ -g_{mj} \overset{0}{|}^l X_{kl}) + \Omega^{ir}_{sj} X^s{}_{rk}, \\ C^i_{jk} = \overset{0}{C^i}_{jk} + \frac{1}{2} g^{im}(g_{mj} \overset{0}{|}_k - 2\dot{\omega}_k g_{mj}) + \\ +\Omega^{ir}_{sj} Y^s{}_{rk}, \\ C_i^{jk} = \overset{0}{C_i}^{jk} + \frac{1}{2} g^{mj}(g_{mi} \overset{0}{|}^k - 2\ddot{\omega}^k g_{mi}) + \\ +\Omega^{rj}_{si} Z_r^{sk}, \; (i,j,k = 1,2,...,n), \end{cases} \tag{24}$$

with:

$$X^k{}_{\overset{0}{i|j}} = 0, \; X_{\overset{0}{ik|j}} = 0, (i,j,k = 1,2,...,n), \tag{25}$$

where $\overset{0}{|}_k$, $\overset{0}{|}_k$ and $\overset{0}{|}^k$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to $\overset{0}{D}$, $X^i_j, X_{ij}, X^i_{jk}, Y^i{}_{jk}, Z_i^{jk}$ are arbitrary d-tensor fields, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$ is an arbitrary 1-form and Ω is the operator of Obata's type given by (15).

Proof. Using the relations (18), (22), (5) by extension of the method given by R.Miron in ([3]) for the case of Finsler connections we obtain the results.

Particular cases:

1. If $X^i_j = X_{ij} = X^i{}_{jk} = Y^i{}_{jk} = Z_i^{jk} = 0$, in Theorem 12 we have:

Theorem 13 *Let $\overset{0}{D}$ be a given $\overset{0}{N}$ -linear connection on $T^{*2}M$, with local coefficients $\overset{0}{D}\ \Gamma(\overset{0}{N}) = \left(\overset{0}{H^i}_{jk}, \overset{0}{C^i}_{jk}, \overset{0}{C_i}^{jk}\right)$. Then the following $\overset{0}{N}$-linear conection K, with local coefficients $K\Gamma(\overset{0}{N},\omega) =$*

$\left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ given by (26) is conformal metrical with respect to $\hat{g}$, corresponding to the 1-form ω:

$$\begin{cases} H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + \frac{1}{2}g^{im}(g_{mj\overset{0}{|}k} - 2\omega_k g_{mj}), \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \frac{1}{2}g^{im}(g_{mj}\overset{0}{|}_k - 2\dot{\omega}_k g_{mj}), \\ C_i^{jk} = \overset{0}{C}{}_i^{jk} + \frac{1}{2}g^{jm}(g_{mi}\overset{0}{|}{}^k - 2\ddot{\omega}^k g_{mi}), \\ (i,j,k = 1,2,...,n), \end{cases} \quad (26)$$

where $\overset{0}{|}_k$, $\overset{0}{|}_k$ and $\overset{0}{|}{}^k$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to $\overset{0}{D}$, and $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$ is an arbitrary 1-form.

2. If we take a metrical $\overset{0}{N}$-linear connection as $\overset{0}{D}$ in Theorem 13, then (26) becomes:

$$\begin{cases} H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} - \delta^i_j \omega_k, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} - \delta^i_j \dot{\omega}_k, \\ C_i^{jk} = \overset{0}{C}{}_i^{jk} - \delta^j_i \ddot{\omega}^k (i,j,k = 1,2,...,n). \end{cases} \quad (27)$$

As an exemple of $\overset{0}{D}$ we take the N-linear connection given in Theorem 5.

3.

Theorem 14 *The following N-linear connection W, with local coefficients $W\Gamma(N,\omega) = \left(\overset{w}{H}{}^i{}_{jk}, \overset{w}{C}{}^i{}_{jk}, \overset{w}{C}{}_i^{jk}\right)$ is a conformal metrical N-linear connection with respect to $\hat{g}$, corresponding to the 1-form ω:*

$$\begin{cases} \overset{w}{H}{}^i{}_{jk} = \frac{1}{2}g^{im}\left(\frac{\delta g_{mk}}{\delta x^j} + \frac{\delta g_{jm}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^m}\right) - \\ -\delta^i_j \omega_k - 2\Omega^{mi}_{jk}\omega_m, \\ \overset{w}{C}{}^i{}_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial y^j} + \frac{\partial g_{jm}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^m}\right) - \\ -\partial^i_j \dot{\omega}_k - 2\Omega^{mi}_{jk}\dot{\omega}_m, \\ \overset{w}{C}{}_i^{jk} = \frac{1}{2}g_{im}\left(\frac{\partial g^{mk}}{\partial p_j} + \frac{\partial g^{jm}}{\partial p_k} - \frac{\partial g^{jk}}{\partial p_m}\right) - \\ -\partial^j_i \ddot{\omega}^k - 2\Omega^{jk}_{mi}\ddot{\omega}^m, (i,j,k = 1,2,...,n), \end{cases} \quad (28)$$

where $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$ is an arbitrary 1-form.

4. If we take a conformal metrical N-linear connection with respect to $\hat{g}$ (e.g. W) as $\overset{0}{D}$, in Theorem 12 we have:

Theorem 15 *Let $\overset{0}{D}$ be a fixed conformal metrical N-linear connection with respect to $\hat{g}$, corresponding to the 1-form ω with the local coefficients $\overset{0}{D}\ \Gamma(\overset{0}{N}, \omega) = \left(\overset{0}{H}{}^i{}_{jk}, \overset{0}{C}{}^i{}_{jk}, \overset{0}{C}{}_i^{jk}\right)$. The set of all conformal metrical N-linear connections with respect to $\hat{g}$, corresponding to the 1-form ω, with local coefficients $D\Gamma(N,\omega) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ is given by:*

$$\begin{cases} N^i{}_j = \overset{0}{N}{}^i{}_j - X^i{}_j, \\ N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + \left(\overset{0}{C}{}^i{}_{jl} + \dot{\omega}_l \delta^i_j\right) X^l{}_k - \\ - \left(\overset{0}{C}{}_j^{il} + \dot{\omega}^l \delta^i_j\right) X_{kl} + \Omega^{ir}_{sj} X^s{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \Omega^{ir}_{sj} Y^s{}_{rk}, \\ C_i^{jk} = \overset{0}{C}{}_i^{jk} + \Omega^{jr}_{si} Z_r{}^{sk}, \\ (i,j,k = 1,2,...,n), \end{cases} \quad (29)$$

with

$$X^k{}_{i\overset{0}{|}j} = 0,\ X_{ik\overset{0}{|}j} = 0, (i,j,k = 1,2,...,n), \quad (30)$$

where $\overset{0}{|}_k$, $\overset{0}{|}_k$ and $\overset{0}{|}{}^k$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to $\overset{0}{D}$, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$ is an arbitrary 1-form and $X^i{}_j, X_{ij}, X^i{}_{jk}, Y^i{}_{jk}, Z_i^{jk}$ are arbitrary d-tensor fields.

5. Finally, if we take $X^i_j = X_{ij} = 0$ in Theorem 15 we obtain:

Theorem 16 *Let $\overset{0}{D}$ be a fixed conformal metrical $\overset{0}{N}$-linear connection with respect to $\hat{g}$, corresponding to the 1-form ω, with local coefficients $\overset{0}{D}\ \Gamma(\overset{0}{N}, \omega) = \left(\overset{0}{H}{}^i{}_{jk}, \overset{0}{C}{}^i{}_{jk}, \overset{0}{C}{}_i^{jk}\right)$. The set of all conformal metrical $\overset{0}{N}$-linear connections with respect to $\hat{g}$, corresponding to the 1-form ω, corresponding to the same nonlinear connection $\overset{0}{N}$, with local coefficients $D\Gamma(\overset{0}{N}, \omega) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ is given by:*

$$\begin{cases} H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + \Omega^{ir}_{sj} X^s{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \Omega^{ir}_{sj} Y^s{}_{rk}, \\ C_i^{jk} = \overset{0}{C}{}_i^{jk} + \Omega^{jr}_{si} Z_r{}^{sk}, \\ (i,j,k = 1,2,...,n), \end{cases} \quad (31)$$

where ${X^i}_{jk}, {Y^i}_{jk}, Z_i^{jk}$ are arbitrary d-tensor fields on $T^{*2}M$ and $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$ is an arbitrary 1-form.

3 Some special classes of conformal metrical N-linear connections

We shall try to replace the arbitrary tensor fields ${X^i}_{jk}, {Y^i}_{jk}$ and Z_i^{jk} in Theorem 16, by the torsion tensor fields $T^i_{jk}, {S^i}_{jk}$ and S_i^{jk}.

We put:

$$\begin{cases} {T^{*i}}_{jk} = \frac{1}{2} g^{im}(g_{mh}{T^h}_{jk} - g_{jh}{T^h}_{mk} + \\ \quad +g_{kh}{T^h}_{jm}), \\ {S^{*i}}_{jk} = \frac{1}{2} g^{im}(g_{mh}S^h_{jk} - g_{jh}S^h_{mk} + \\ \quad +g_{kh}S^h_{jm}), \\ {S^*}_i^{jk} = \frac{1}{2} g_{im}(g^{mh}S_h^{jk} - g^{jh}S_h^{mk} + \\ \quad +g^{kh}S_h^{jm}). \end{cases} \tag{32}$$

Theorem 17 *Let $T^i_{jk}, {S^i}_{jk}$ and S_i^{jk} be three given skew symmetric tensor fields of type (1,2), (1,2) and (2,1) respectively and let ω be a given 1-form in $T^{*2}M$. Then there exists a unique conformal metrical N-linear connection with respect to $\hat{g}$, corresponding to the 1-form ω, with local coefficients $D\Gamma(N,\omega) = \left({H^i}_{jk}, {C^i}_{jk}, C_i^{jk}\right)$, having $T^i_{jk}, {S^i}_{jk}$ and S_i^{jk} as the torsion tensor fields. It is given by:*

$$\begin{cases} {H^i}_{jk} = \overset{w}{H^i}_{jk} + {T^{*i}}_{jk}, \\ {C^i}_{jk} = \overset{w}{C^i}_{jk} + {S^{*i}}_{jk}, \\ C_i^{jk} = \overset{w}{C_i^{jk}} + {S^*}_i^{jk} \end{cases} \tag{33}$$

where $W\Gamma(N,\omega) = \left(\overset{w}{H^i}_{jk}, \overset{w}{C^i}_{jk}, \overset{w}{C_i^{jk}}\right)$ are the local coefficients of conformal metrical N-linear connection with respect to $\hat{g}$, corresponding to the 1-form ω, given in (28).

Remark 18 *The conformal metrical N-linear connection with respect to $\hat{g}$, W, corresponding to the 1-form ω, with local coefficients $W\Gamma(N,\omega) = \left(\overset{w}{H^i}_{jk}, \overset{w}{C^i}_{jk}, \overset{w}{C_i^{jk}}\right)$ given in (28) is considered as the semisymmetric conformal metrical N-linear connection with the vanishing $h-$, w_1- and w_2- torsion vector fields.*

Using the Definition 1, the relations (32) become:

$$\begin{cases} {T^{*i}}_{jk} = 2\Omega^{ri}_{jk}\sigma_r, \\ {S^{*i}}_{jk} = 2\Omega^{ri}_{jk}\tau_r, \\ {S^*}_i^{jk} = 2\Omega^{jk}_{ri}\upsilon^r. \end{cases} \tag{34}$$

Using the Theorem 17 and the relations (34) we have:

Theorem 19 *The set of all semisymmetric conformal metrical N-linear connections with respect to $\hat{g}$, corresponding to the 1-form ω with local coefficients $D\Gamma(N,\omega,\sigma) = \left({H^i}_{jk}, {C^i}_{jk}, C_i^{jk}\right)$ is given by:*

$$\begin{cases} {H^i}_{jk} = \overset{w}{H^i}_{jk} + 2\Omega^{ri}_{jk}\sigma_r, \\ {C^i}_{jk} = \overset{w}{C^i}_{jk} + 2\Omega^{ri}_{jk}\tau_r, \\ C_i^{jk} = \overset{w}{C_i^{jk}} + 2\Omega^{jk}_{ri}\upsilon^r, \\ (i,j,k = 1,2,...,n), \end{cases} \tag{35}$$

where $W\Gamma(N,\omega) = \left(\overset{w}{H^i}_{jk}, \overset{w}{C^i}_{jk}, \overset{w}{C_i^{jk}}\right)$ are the local coefficients of the semisymmetric conformal metrical N-linear connection, W, given in (28) and $\sigma = \sigma_i dx^i + \tau_i \delta y^i + \upsilon^i \delta p_i$ is an arbitrary 1-form.

4 The group of transformations of conformal metrical N-linear connections

We study the transformations $D\Gamma(N,\omega) \to \bar{D}\Gamma(\bar{N},\omega')$ of the conformal metrical N-linear connections with respect to $\hat{g}$.

If we replace $\overset{0}{D}\ \Gamma(\overset{0}{N})$ and $D\Gamma(N,\omega)$ in Theorem 12, by $D\Gamma(N,\omega)$ and $\bar{D}\Gamma(\bar{N},\omega')$, respectively, two conformal metrical N- and respectively $\bar{N}$-linear connections with respect to $\hat{g}$, we obtain:

Theorem 20 *Two conformal metrical N- and respectively $\bar{N}$- linear connections with respect to $\hat{g}$: D and $\bar{D}$, with local coefficients $D\Gamma(N,\omega) = \left({H^i}_{jk}, {C^i}_{jk}, C_i^{jk}\right)$ and $\bar{D}\Gamma(\bar{N},\omega') = \left(\bar{H}^i_{jk}, \bar{C}^i_{jk}, \bar{C}_i^{jk}\right)$ respectively, are related as follows:*

$$\begin{cases} \bar{N}^i_j = N^i_j - X^i_j, \\ \bar{N}_{ij} = N_{ij} - X_{ij}, \\ \bar{H}^i_{jk} = {H^i}_{jk} + X^l_k {C^i}_{jl} - X_{kl}C_j{}^{il} - \\ -\delta^i_j p'_k + \delta^i_j \dot{\omega}_l X^l{}_k - \delta^i_j \ddot{\omega}^l X_{kl} + \Omega^{ir}_{sj} X^s{}_{rk}, \\ \bar{C}^i_{jk} = {C^i}_{jk} - \delta^i_j \dot{p}'_k + \Omega^{ir}_{sj} Y^s{}_{rk}, \\ \bar{C}_i^{jk} = C_i^{jk} - \delta^j_i \ddot{p}'^k + \Omega^{jr}_{si} Z_r{}^{sk}, \\ (i,j,k = 1,2,...,n), \end{cases} \tag{36}$$

with:

$$X^k{}_{i|j} = 0, X_{ik|j} = 0, (i, j, k = 1, 2, ..., n), \quad (37)$$

where $p' = \omega' - \omega$, $\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i + \ddot{\omega}^i \delta p_i$ *and* $\omega' = \omega'_i dx^i + \dot{\omega}'_i \delta y^i + \ddot{\omega}'^i \delta p_i$ *are two 1-forms, "$|_k$" denote the h-covariant derivative with respect to D and* $X^i{}_j, X_{ij}, X^i{}_{jk}, Y^i{}_{jk}, Z_i^{jk}$ *are arbitrary d-tensor fields.*

Proof. Using in (24) the relations (18), by direct calculation we have the results.

Conversely, given the d-tensor fields $X^i{}_j, X_{ij}, X^i{}_{jk}, Y^i{}_{jk}, Z_i^{jk}$ and one given 1-form $p' = p'_i dx^i + \dot{p}'_i \delta y^i + \ddot{p}'^i \delta p_i$ the above (36) is thought to be a transformation of a conformal metrical N-linear connection $D\Gamma(N, \omega)$ to a conformal metrical $\bar{N}$-linear connection $\bar{D}\Gamma(\bar{N}, \omega')$=$\bar{D}\Gamma(\bar{N}, \omega + p')$.

We shall denote this transformation by $t(X^i{}_j, X_{ij}, X^i{}_{jk}, Y^i{}_{jk}, Z_i^{jk}, p')$.

Thus we have:

Theorem 21 *The set $\mathcal{C}$ of all transformations* $t(X^i{}_j, X_{ij}, X^i{}_{jk}, Y^i{}_{jk}, Z_i^{jk}, p')$ *given by (36) and (37) is a transformations group of the set of all conformal metrical N-linear connections with respect to $\hat{g}$, together with the mapping product:* $t(X'^i{}_j, X'_{ij}, X'^i{}_{jk}, Y'^i{}_{jk}, Z'^{jk}_i, p'') \circ t(X^i{}_j, X_{ij}, X^i{}_{jk}, Y^i{}_{jk}, Z_i^{jk}, p') = t(X^i{}_j + X'^i{}_j, X_{ij} + X'_{ij}, X^i{}_{jk} + X'^i{}_{jk}, Y^i{}_{jk} + Y'^i{}_{jk}, Z_i^{jk} + Z'^{jk}_i, p' + p'')$.

5 The group of transformations of semisymmetric conformal metrical N-linear connections

We inquire about a subgroup of the group of transformations of conformal metrical n-linear connection: about the subgroup of transformations of the semisymmetric comformal metrical N-linear connections, corresponding to the same nonlinear connection N.

Let N be a given nonlinear connection. Then any semisymmetric conformal metrical N-linear connection, with local coefficients $\bar{D}\Gamma(N, \omega', \sigma') = \left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i^{jk}\right)$ with respect to $\hat{g}$ is given by (33) with (34).

Theorem 22 *Two semisymmetric conformal metrical N-linear connections with respect to $\hat{g}$, with local coefficients* $D\Gamma(N, \omega, \sigma) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ *and* $\bar{D}\Gamma(\bar{N}, \omega', \sigma') = \left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i^{jk}\right)$ *respectively, are related as follows:*

$$\begin{cases} \bar{H}^i{}_{jk} = H^i{}_{jk} - \delta^i_j p'_k + 2\Omega^{ri}_{jk} q_r, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} - \delta^j_i \dot{p}'_k + 2\Omega^{ri}_{jk} \dot{q}_r, \\ \bar{C}_i^{jk} = C_i^{jk} - \delta^j_i \ddot{p}'^k + 2\Omega^{ri}_{jk} \ddot{q}^r, \\ (i, j, k = 1, 2, ..., n), \end{cases} \quad (38)$$

where $p' = \omega' - \omega$, $q = \sigma' - \sigma - p'$, $p' = p'_i dx^i + \dot{p}_i \delta y^i + \ddot{p}^i \delta p_i$ *and* $q = q_i dx^i + \dot{q}_i \delta y^i + \ddot{q}^i \delta p_i$.

Proof. Using in (35) the relations (28) by direct calculation we have the results.

Conversely, given 1-forms p' and q in $T^{*^2}M$, the above (38) is thought to be a transformation of a semisymmetric conformal metrical N-linear connection D, with local coefficients $D\Gamma(N, \omega, \sigma) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$, to a semisymmetric conformal metrical N-linear connection $\bar{D}$, with local coefficients $\bar{D}\Gamma(N, \omega + p', \sigma + p' + q) = \left(\bar{H}^i{}_{jk}, \bar{C}^i{}_{jk}, \bar{C}_i^{jk}\right)$.

We shall denote this transformation by $t(p', q)$.

Thus we have:

Theorem 23 *The set $\mathcal{C}^s_N$ of all transformations $t(p', q)$ given by (38) is a transformations group of the set of all semisymmetric conformal metrical N-linear connections with respect to $\hat{g}$, having the same nonlinear connection N, together with the mapping product: $t(p', q) \circ t(p'', q') = t(p' + p'', q + q')$.*

This group, $\mathcal{C}^s_N$, is an Abelian subgroup of $\mathcal{C}$ and acts on the set of all semisymmetric conformal metrical N-linear connections, having the same nonlinear connection N, transitively.

The transformation $t(p', q) : D\Gamma(N, \omega, \sigma) \to \bar{D}\Gamma(N, \omega + p', \sigma + p' + q)$ given by (38) is expressed by the product of the following two transformations:

$$\begin{cases} \bar{H}^i{}_{jk} = H^i{}_{jk} - \delta^i_j p'_k, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} - \delta^i_j \dot{p}'_k, \\ \bar{C}_i^{jk} = C_i^{jk} - \delta^j_i \ddot{p}'^k, \\ (i, j, k = 1, 2, ..., n), \end{cases} \quad (39)$$

$$\begin{cases} \bar{H}^i{}_{jk} = H^i{}_{jk} + 2\Omega^{ri}_{jk} q_r, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} + 2\Omega^{ri}_{jk} \dot{q}_r, \\ \bar{C}_i^{jk} = C_i^{jk} + 2\Omega^{jk}_{ri} \ddot{q}^r, \\ (i, j, k = 1, 2, ..., n), \end{cases} \quad (40)$$

Definition 24 *The transformation* $t : D\Gamma(N) \to \bar{D}\Gamma(N)$, *of N-linear connection on $T^{*^2}M$, defined by*

(39) is called co-parallel transformation, where p' is a given 1-form.

Theorem 25 *The set $\mathcal{C}_N^p$ of all co-parallel transformations, t, given by (39) is an Abelian group together with the mapping product.*

Definition 26 *The transformation $t : D\Gamma(N) \to \bar{D}\Gamma(N)$, of N-linear connections, given by (40) is called Miron transformation (as the name given by M.Hashiguchi ([3]) for Finsler spaces).*

Theorem 27 *The set $\mathcal{C}_N^m$ of all Miron transformations, t, given by (40) is a transformations group, together with the mapping product.*

Theorem 28 *The group $\mathcal{C}_N^s$, of all transformations $t(p', q)$ given by (38) is the direct product of the group $\mathcal{C}_N^p$, of all co-paralel transformations and the group $\mathcal{C}_N^m$, of all Miron transformations.*

It is noted that the invariants of the group $\mathcal{C}_N^s$, will be the invariants of each of these subgroups and reciprocally.

It is directly shown that by a co-parallel transformation (39) the curvature tensor fields $R_{h\ jk}^{\ i}$, $P_{h\ jk}^{\ i}$ and $S_h^{\ ijk}$ are transformed as follows:

$$\begin{cases} \bar{R}_{h\ jk}^{\ i} = R_{h\ jk}^{\ i} - \delta_h^i p'_{jk}, \\ \bar{P}_{h\ jk}^{\ i} = P_{h\ jk}^{\ i} - \delta_h^i \dot{p}'_{jk}, \\ \bar{S}_h^{\ ijk} = S_h^{\ ijk} - \delta_h^i \ddot{p}'^{jk}, \end{cases} \tag{41}$$

where $p'_{jk}, \dot{p}'_{jk}$ and $\ddot{p}'^{jk}$ are the components of dp', expressed with respect to D.

Eliminating $p'_{jk}, \dot{p}'_{jk}$ and $\ddot{p}'^{jk}$ from (41) we have:

$$\begin{array}{l} \bar{R}^{*\ i}_{h\ jk} = R^{*\ i}_{h\ jk}, \bar{P}^{*\ i}_{h\ jk} = P^{*\ i}_{h\ jk}, \\ \bar{S}^{*\ ijk}_{h} = S^{*\ ijk}_{h}, \end{array} \tag{42}$$

where:

$$\begin{cases} R^{*\ i}_{h\ jk} = R_{h\ jk}^{\ i} - \frac{1}{n}\delta_h^i R_{s\ jk}^{\ s}, \\ P^{*\ i}_{h\ jk} = P_{h\ jk}^{\ i} - \frac{1}{n}\delta_h^i P_{s\ jk}^{\ s}, \\ S^{*\ ijk}_{h} = S_h^{\ ijk} - \frac{1}{n}\delta_h^i S_s^{\ sjk}. \end{cases} \tag{43}$$

Thus we have:

Theorem 29 *The tensor fields $R^{*\ i}_{h\ jk}, P^{*\ i}_{h\ jk}$ and $S^{*\ ijk}_{h}$, given by (43) are invariants of the group $\mathcal{C}_N^p$.*

Also we obtain:

Theorem 30 *The tensor field $C^{*\ jk}_{i}$, given by (44) is an invariant of the group $\mathcal{C}_N^p$.*

$$C^{*\ jk}_{i} = C_i^{\ jk} - \tfrac{1}{n}\delta_i^j C_s^{\ sk}. \tag{44}$$

In our previous paper [Bull Math Buc], starting from the tensor fields:

$$\begin{cases} \mathcal{K}_{h\ jk}^{\ i} = R_{h\ jk}^{\ i} - C^i_{\ hm} R^m_{(1)jk} - \\ \quad -C_h^{\ im} R_{(2)mjk}, \\ \mathcal{P}_{h\ jk}^{\ i} = P_{h\ jk}^{\ i} - C^i_{\ hm}\frac{\partial N^m_{\ j}}{\partial y^k} - \\ \quad -C_h^{\ im}\frac{\partial N_{jm}}{\partial p_k}, \end{cases} \tag{45}$$

we obtained the following important invariants of the group of semisymmetric metrical N-linear connections, having the same nonlinear connection N, $\overset{ms}{\mathcal{T}}_N$, for $n > 2$:

$$\begin{cases} H_{h\ jk}^{\ i} = \mathcal{K}_{h\ jk}^{\ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{K}_{rk} - \\ \quad -\frac{g_{rk}\mathcal{K}}{2(n-1)})\}, \\ N_{h\ jk}^{\ i} = \mathcal{P}_{h\ jk}^{\ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{P}_{rk} - \\ \quad -\frac{g_{rk}\mathcal{P}}{2(n-1)})\}, \\ M_{h\ jk}^{\ i} = S_{h\ jk}^{\ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(S_{rk} - \\ \quad -\frac{g_{rk}S}{2(n-1)})\}, \\ M_h^{\ ijk} = S_h^{\ ijk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ij}_{rh}(S^{rk} - \\ \quad -\frac{g^{rj}S'}{2(n-1)})\}, \end{cases} \tag{46}$$

where:

$$\begin{cases} \mathcal{K}_{hj} = \mathcal{K}_{h\ ji}^{\ i}, \mathcal{K} = g^{hj}\mathcal{K}_{hj}, \mathcal{P}_{hj} = \mathcal{P}_{h\ ji}^{\ i}, \\ \mathcal{P} = g^{hj}\mathcal{P}_{hj}, \mathcal{S}_{hj} = \mathcal{S}_{h\ ji}^{\ i}, \mathcal{S} = g^{hj}\mathcal{S}_{hj}, \\ \mathcal{S}^{ij} = \mathcal{S}_m^{\ ijm}, \mathcal{S}' = g_{ij}\mathcal{S}^{ij}, \end{cases} \tag{47}$$

If we replace these $\mathcal{K}_{h\ jk}^{\ i}, \mathcal{P}_{h\ jk}^{\ i}, S_{h\ ji}^{\ k}$ and $S_h^{\ ijk}$ by the tensor fields $\mathcal{K}^{*\ i}_{h\ jk}, \mathcal{P}^{*\ i}_{h\ jk}, \mathcal{S}^{*\ i}_{h\ jk}$ and $\mathcal{S}'^{*\ ijk}_{h}$ respectively, defined by:

$$\begin{cases} \mathcal{K}^{*\ i}_{h\ jk} = \mathcal{K}_{h\ jk}^{\ i} - \frac{1}{n}\delta_h^i \mathcal{K}_{m\ jk}^{\ m}, \\ \mathcal{P}^{*\ i}_{h\ jk} = \mathcal{P}_{h\ jk}^{\ i} - \frac{1}{n}\delta_h^i \mathcal{P}_{m\ jk}^{\ m}, \\ \mathcal{S}^{*\ i}_{h\ jk} = S_{h\ jk}^{\ i} - \frac{1}{n}\delta_h^i S_{m\ jk}^{\ m}, \\ \mathcal{S}'^{*\ ijk}_{h} = S_h^{\ ijk} - \frac{1}{n}\delta_h^i S_m^{\ mjk}, \end{cases} \tag{48}$$

we can obtain the invariants of the group of transformations of semisymmetric conformal metrical N-linear connections, having the same nonlinear connection N, $\mathcal{C}_N^s$:

Theorem 31 *For $n > 2$ the following tensor fields $H^{*\ i}_{h\ jk}, N^{*\ i}_{h\ jk}, M^{*\ i}_{h\ jk}$ and $M'^{*\ ijk}_{h}$ are invariants of the group $\mathcal{C}_N^s$, of transformations of semisymmetric conformal metrical N-linear connections, having the same nonlinear connection N:*

$$\begin{cases} H^{*}{}_{h}{}^{i}{}_{jk} = \mathcal{K}^{*}{}_{h}{}^{i}{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{K}^{*}_{rk} - \\ \quad -\frac{g_{rk}\mathcal{K}^{*}}{2(n-1)})\}, \\ N^{*}{}_{h}{}^{i}{}_{jk} = \mathcal{P}^{*}{}_{h}{}^{i}{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{P}^{*}_{rk} - \\ \quad -\frac{g_{rk}\mathcal{P}^{*}}{2(n-1)})\}, \\ M^{*}{}_{h}{}^{i}{}_{jk} = \mathcal{S}^{*}{}_{h}{}^{i}{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{S}^{*}_{rk} - \\ \quad -\frac{g_{rk}\mathcal{S}^{*}}{2(n-1)})\}, \\ M'^{*}{}_{h}{}^{ijk} = \mathcal{S}'^{*}{}_{h}{}^{ijk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ij}_{rh} \\ \quad (\mathcal{S}'^{*rk} - \frac{g^{rk}\mathcal{S}'^{*}}{2(n-1)})\}, \end{cases} \tag{49}$$

where:

$$\begin{cases} \mathcal{K}^{*}_{hj} = \mathcal{K}^{*}{}_{h}{}^{i}{}_{ji}, \mathcal{K}^{*} = g^{hj}\mathcal{K}^{*}_{hj}, \\ \mathcal{P}^{*}_{hj} = \mathcal{P}^{*}{}_{h}{}^{i}{}_{ji}, \mathcal{P}^{*} = g^{hj}\mathcal{P}^{*}_{hj}, \\ \mathcal{S}^{*}_{hj} = \mathcal{S}^{*}{}_{h}{}^{i}{}_{ji}, \mathcal{S}^{*} = g^{hj}\mathcal{S}^{*}_{hj}, \\ \mathcal{S}'^{*ij} = \mathcal{S}'^{*}{}_{m}{}^{ijm}, \mathcal{S}'^{*} = g_{ij}\mathcal{S}'^{*ij}. \end{cases} \tag{50}$$

Finally we give another invariant of the group $\mathcal{C}^{s}_{N}$:

Theorem 32 *The following tensor field is an invariant of the group $\mathcal{C}^{s}_{N}$:*

$$\begin{array}{l} C^{*}_{i}{}^{jk} - \frac{2}{n-1}\Omega^{kj}_{ir}C^{*}_{m}{}^{rm}, \\ (i,j,k = 1,2,...,n), \end{array} \tag{51}$$

where $C^{*}_{i}{}^{jk}$ is given by (44).

6 Metrical N-linear connections in a generalized Hamilton space

We shall determine the set of all metrical N-linear connections in the case when the nonlinear connection N is arbitrary.

Theorem 33 *Let $\overset{0}{D}$ be given $\overset{0}{N}$-linear connection on $T^{*^2}M$, with local coefficients $\overset{0}{D}\,\Gamma(\overset{0}{N}) = = (\overset{0}{H}{}^{i}{}_{jk},\ \overset{0}{C}{}^{i}{}_{jk},\ \overset{0}{C}{}_{i}^{jk})$, where the local coefficients of the nonlinear connection $\overset{0}{N}$ are: $(N^{j}{}_{i}(x,y,p), N_{ij}(x,y,p)), (i,j = 1,2,...,n)$.*

The set of all metrical N-linear connections with respect to g^{ij}, with local coefficients $D\Gamma(N) = \left(H^{i}{}_{jk}, C^{i}{}_{jk}, C_{i}^{jk}\right)$ is given by:

$$\begin{cases} N^{i}_{j} = \overset{0}{N}{}^{i}{}_{j} - X^{i}_{j}, \\ N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^{i}_{jk} = \overset{0}{H}{}^{i}{}_{jk} + X^{l}{}_{k}\overset{0}{C}{}^{i}{}_{jl} - X_{kl}\overset{0}{C}{}_{j}^{il} + \\ +\frac{1}{2}g^{im}(g_{\underset{mj|k}{0}} + g_{mj}\overset{0}{|}_{l}X^{l}{}_{k} - \\ -g_{mj}\overset{0}{|}{}^{l}X_{kl}) + \Omega^{ir}_{sj}X^{s}{}_{rk}, \\ C^{i}{}_{jk} = \overset{0}{C}{}^{i}{}_{jk} + \frac{1}{2}g^{im}g_{mj}\overset{0}{|}_{k} + \Omega^{ir}_{sj}Y^{s}{}_{rk}, \\ C_{i}^{jk} = \overset{0}{C}{}_{i}^{jk} + \frac{1}{2}g^{mj}g_{mi}\overset{0}{|}{}^{k} + \Omega^{rj}_{si}Z_{r}{}^{sk}, \\ (i,j,k = 1,2,...,n), \end{cases} \tag{52}$$

with:

$$X^{k}{}_{\underset{i|j}{0}} = 0,\ X_{\underset{ik|j}{0}} = 0,\ (i,j = 1,2,...,n), \tag{53}$$

where $\overset{0}{|}_{k}$, $\overset{0}{|}_{k}$ and $\overset{0}{|}{}^{k}$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to $\overset{0}{D}$, $X^{i}_{j}, X_{ij}, X^{i}_{jk}, Y^{i}{}_{jk}, Z_{i}^{jk}$ are arbitrary d-tensor fields and Ω is the operator of Obata's type given by (15).

Proof. Using the relations (12), (22), (5) by extension of the method given by R.Miron in ([3]) for the case of Finsler connections, we can deduce the results.

Particular cases:

1. If $X^{i}_{j} = X_{ij} = X^{i}{}_{jk} = Y^{i}{}_{jk} = Z_{i}^{jk} = 0$, in Theorem 33 we have:

Theorem 34 *Let $\overset{0}{D}$ be a given $\overset{0}{N}$-linear connection on $T^{*^2}M$, with local coefficients $\overset{0}{D}\ \Gamma(\overset{0}{N}) = \left(\overset{0}{H}{}^{i}{}_{jk}, \overset{0}{C}{}^{i}{}_{jk}, \overset{0}{C}{}_{i}^{jk}\right)$. Then the following N-linear connection $\tilde{D}$, with local coefficients $\tilde{D}\Gamma(\overset{0}{N}) = \left(\tilde{H}^{i}{}_{jk}, \tilde{C}^{i}{}_{jk}, \tilde{C}_{i}^{jk}\right)$ given by (54) is metrical:*

$$\begin{cases} N^{i}_{j} = \overset{0}{N}{}^{i}{}_{j}, \\ N_{ij} = \overset{0}{N}_{ij}, \\ \tilde{H}^{i}_{jk} = \overset{0}{H}{}^{i}{}_{jk} + \frac{1}{2}g^{im}g_{\underset{mj|k}{0}}, \\ \tilde{C}^{i}{}_{jk} = \overset{0}{C}{}^{i}{}_{jk} + \frac{1}{2}g^{im}g_{mj}\overset{0}{|}_{k}, \\ \tilde{C}_{i}^{jk} = \overset{0}{C}{}_{i}^{jk} + \frac{1}{2}g^{mj}g_{mi}\overset{0}{|}{}^{k}, \\ (i,j,k = 1,2,...,n), \end{cases} \tag{54}$$

where $\overset{0}{|}_k$, $\overset{0}{|}_k$ and $\overset{0}{|}^k$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to $\overset{0}{D}$.

2. If we take a metrical $\overset{0}{N}$-linear connection as $\overset{0}{D}$ in Theorem 33 we obtain:

Theorem 35 *The set of all metrical N-linear connections with local coefficients* $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i^{jk}\right)$ *is given by:*

$$\begin{cases} N^i_j = \overset{0}{N}{}^i{}_j - X^i_j, \\ N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + X^l{}_k \overset{0}{C}{}^i{}_{jl} - \\ \qquad -X_{kl}\overset{0}{C}{}_j^{il} + \Omega^{ir}_{sj} X^s{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \Omega^{ir}_{sj} Y^s{}_{rk}, \\ C_i^{jk} = \overset{0}{C}{}_i^{jk} + \Omega^{rj}_{si} Z_r{}^{sk}, \\ (i,j,k = 1,2,...,n) \end{cases} \tag{55}$$

with:

$$X^k{}_{\underset{i|j}{0}} = 0,\ X_{\underset{ik|j}{0}} = 0, (i,j,k = 1,2,...,n), \tag{56}$$

where $\overset{0}{|}_k$, $\overset{0}{|}_k$ and $\overset{0}{|}^k$ denote the $h-$, w_1- and w_2- covariant derivatives with respect to $\overset{0}{D}$, $X^i_j, X_{ij}, X^i_{jk}, Y^i_{jk}, Z_i^{jk}$ are arbitrary d-tensor fields and Ω is the operator of Obata's type given by (15).

3. If in Theorem 35 we consider $X^i_{jk} = X_{ij} = 0$ we obtain the set of all metrical $\overset{o}{N}$-linear connections having the same nonlinear connection $\overset{0}{N}$, given by R.Miron, D.Hrimiuc, H.Shimada and V.S. Sabău in their book ([4], Theorem 2.3, p.290).

7 The group of transformations of metrical N-linear connections

Let N be a given nonlinear connection. Then any metrical N-linear connection corresponding to the same nonlinear connection N has the local coefficients $\bar{D}\Gamma(N) = \left(\bar{H}^i_{jk}, \bar{C}^i_{jk}, \bar{C}_i^{jk}\right)$. given by:

$$\begin{cases} \bar{H}^i_{jk} = H^i_{jk} + \Omega^{ir}_{sj} X^s{}_{rk}, \\ \bar{C}^i_{jk} = C^i_{jk} + \Omega^{ir}_{sj} Y^s{}_{rk}, \\ \bar{C}_i^{jk} = C_i^{jk} + \Omega^{rj}_{si} Z^r{}_{sk}, \\ (i,j,k = 1,2,...,n), \end{cases} \tag{57}$$

where $X^i_{jk}, Y^i_{jk}, Z_i^{jk}$ are arbitrary d-tensor fields, Ω is the operator of Obata's type given by (15) and $D\Gamma(N) = \left(H^i{}_{jk}, C^i{}_{jk}, C_i{}^{jk}\right)$ are the local coefficients of a metrical N-linear connection D.

Conversely, given the tensor fields $X^i_{jk}, Y^i_{jk}, Z_i^{jk}$ the above (57) is thought to be a transformation of a metrical N-linear connection $D\Gamma(N)$ to a metrical N-linear connection $\bar{D}\Gamma(N)$.

We shall denote this transformation by $t(X^i_{jk}, Y^i_{jk}, Z_i^{jk})$.

Thus we have:

Theorem 36 *The set* $\overset{m}{\mathcal{T}}_N$ *of all transformations* $t(X^i_{jk}, Y^i_{jk}, Z_i^{jk})$ *given by (57), together with the mapping product:* $t(X'^i_{jk}, Y'^i_{jk}, Z'^{jk}_i) \circ t(X^i_{jk}, Y^i_{jk}, Z_i^{jk}) = t(X^i_{jk} + X'^i_{jk}, Y^i_{jk} + Y'^i_{jk}, Z_i^{jk} + Z'^{jk}_i)$ *is a transformations group of the set of all metrical N-linear connections, having the same nonlinear connection N.*

References:

[1] Gh. Atanasiu and M. Târnoveanu, New Aspects in the Differential Geometry of the Second Order Cotangent Bundle,Univ. Timişoara, No.90, 2005, 1-64.

[2] R. Miron, Hamilton Geometry, Seminarul de Mecanică, Univ. Timişoara, 3(1987), 1-54.

[3] R. Miron and M. Hashiguchi, Conformal Finsler Connections, *Rev.Roumaine Math.Pures Appl.*, 26, 6(1981), 861-878.

[4] R. Miron, D. Hrimiuc, H. Shimada and V. S. Sabău, The Geometry of Hamilton and Lagrange Spaces, *Kluwer Acad.Publ.*, Vol 118,FTPH, (2001).

[5] R. Miron, S. Watanabe and S. Ikeda, Cotangent Bundle Geometry, Memoriile Secţiilor ştiinţifice, Bucureşti, Acad. R.S.Romania, Seria IV,IX, I (1986), 25-46.

[6] M. Purcaru and M. Târnoveanu, On Transformations Groups of N-Linear Connections on Second Order Cotangent Bundle, *Acta Universitatis Apulensis*, Special Issue (2009), 287-296.

[7] M. Purcaru and M. Târnoveanu, Metrical Semisymmetric N-Linear Connections on a Generalized Hamilton Space, *Bull. Math. Soc. Sci. Math. Roum* (to appear).

[8] C. Udrişte, Multi-Time Optimal Control, *WSEAS Transactions on Mathematics*, Issue 12, Volume 6, December 2007, ISSN: 1109-2769.

[9] C. Udrişte, Non-classical Lagrangian dynamics and potential maps, *WSEAS Transactions on Mathematics*, Volume 7, 2008, 12-18, ISSN: 1109-2769.

[10] C. Udrişte and D. Opriş, Euler-Lagrange-Hamilton dynamics with fractional action, *WSEAS Transactions on Mathematics*, Volume 7, 2008, 19-30. ISSN: 1109-2769.

[11] C. Udrişte, M. Popescu and P. Popescu, Generalized Multitime Lagrangians and Hamiltonians, *WSEAS Transactions on Mathematics*, Volume 7, 2008, 66-72. (Scopus, Scimago) ISSN: 1109-2769.

[12] C. Udrişte and O. Şandru, Dual Nonlinear Connections, communicate to the 22^{nd} Conference on Differential Geometry and Topology, Polytechnic Institute of Bucharest, Romania, Sept., 9-13,1991.

[13] C. Udriste, I. Tevy, Multi-Time Euler-Lagrange-Hamilton Theory, *WSEAS Transactions on Mathematics* Volume 6, 2007, 701-709, ISSN: 1109-2769.

[14] K. Yano and S. Ishihara, Tangent and Cotangent Bundles. Differential Geometry, M.Dekker, Inc., New-York, 1973.