# A New Diagonally Implicit Runge-Kutta-Nyström Method for Periodic IVPs

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*Abstract:* - A new diagonally implicit Runge-Kutta-Nyström (RKN) method is developed for the integration of initial-value problems for second-order ordinary differential equations possessing oscillatory solutions. Presented is a method which is three-stage fourth-order with dispersive order six and 'small' principal local truncation error terms and dissipation constant. The analysis of phase-lag, dissipation and stability of the method are also given. This new method is more efficient when compared with current methods of similar type for the numerical integration of second-order differential equations with periodic solutions, using constant step size.

Key-Words: - Runge-Kutta-Nyström methods; Diagonally implicit; Phase-lag; Oscillatory solutions

#### **1** Introduction

This paper deals with numerical method for secondorder ODEs, in which the derivative does not appear explicitly,

$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y'_0,$$
(1)

for which it is known in advance that their solution is oscillating. Such problems often arise in different areas of engineering and applied sciences such as celestial mechanics, quantum mechanics, elastodynamics, theoretical physics and chemistry, and electronics. An *m*-stage Runge-Kutta-Nyström (RKN) method for the numerical integration of the IVP is given by

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^m b_i k_i$$
  
$$y'_{n+1} = y'_n + h \sum_{i=1}^m b_i' k_i$$
(2)

where

$$k_i = f\left(x_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^m a_{ij} k_j\right)$$
  $i = 1, ..., m.$ 

The RKN parameters  $a_{ij}, b_j, b'_j$  and  $c_j$  are assumed to be real and *m* is the number of stages of the method. Introduce the *m*-dimensional vectors c, b and b' and  $m \times m$  matrix *A*, where  $c = [c_1, c_2, \dots, c_m]^T$ ,  $b = [b_1, b_2, \dots, b_m]^T$ ,  $b' = [b'_1, b'_2, \dots, b'_m]^T$ ,  $A = [a_{ij}]$ respectively. RKN methods can be divided into two broad classes: explicit  $(a_{jk} = 0, k \ge j)$  and implicit  $(a_{jk} = 0, k > j)$ . The latter contains the class of diagonally implicit RKN (DIRKN) methods for which all the entries in the diagonal of *A* are equal. RKN method of algebraic order *r* can be expressed in Butcher notation by the table of coefficients



Generally problems of the form (1) which have periodic solutions can be divided into two classes. The first class consists of problems for which the solution period is known a priori. The second class consists of problems for which the solution period is initially unknown. Several numerical methods of various types have been proposed for the integration of both classes of problems. See Stiefel and Bettis [3], van der Houwen and Sommeijer [12], Gautschi [16] and others.

When solving (1) numerically, attention has to be given to the algebraic order of the method used, since this is the main criterion for achieving high accuracy. Therefore, it is desirable to have a lower stage RKN method with maximal order. This will lessen the computational cost. If it is initially known that the solution of (1) is of periodic nature then it is essential to consider phase-lag (or dispersion) and amplification (or dissipation). These are actually two types of truncation errors. The first is the angle between the true and the approximated solution, while the second is the distance from a standard cyclic solution. In this paper we will derive a new diagonally implicit RKN method with three-stage fourth-order with dispersion of high order.

A number of numerical methods for this class of problems of the explicit and implicit type have been extensively developed. For example, van der Houwen and Sommeijer [12], Simos, Dimas and Sideridis, [15], and Senu, Suleiman and Ismail [18] have developed explicit RKN methods of algebraic order up to five with dispersion of high order for solving oscillatory problems. For implicit RKN methods, see for example van der Houwen and Sommeijer [13], Sharp, Fine and Burrage [14], Imoni, Otunta and Ramamohan [17] and Al-Khasawneh, Ismail and Suleiman [23]. The secondorder ODEs of (1) can be reduced to the system of first-order ODEs, then solving using Runge-Kutta method, see for example Ismail, Che Jawias, Suleiman and Jaafar [24], Razali, Ahmad, Darus, and Rambely [25], Ismail, Che Jawias, Suleiman and Jaafar [26] and Podisuk and Phummark [28]. But in this paper we will develop RKN method for solving problem (1) directly.

In this paper a dispersion relation is imposed and together with algebraic conditions to be solved explicitly. In the following section the construction of the new diagonally implicit RKN method is described. Its coefficients are given using the Butcher tableau notation. Finally, numerical tests on second order differential equation problems possessing an oscillatory solutions are performed.

# 2 Analysis of Phase-Lag and Stability

In this section we shall discuss the analysis of phase-lag for RKN method. The first analysis of phase-lag was carried out by Bursa and Nigro [10]. Then followed by Gladwell and Thomas [5] for the linear multistep method, and for explicit and implicit Runge-Kutta(-Nystrom) methods by van der Houwen and Sommeijer [12], [13]. The phase be divided analysis can in two parts; inhomogeneous and homogeneous components. Following van der Houwen and Sommeijer [12], inhomogeneous phase error is constant in time, meanwhile the homogeneous phase errors are accumulated as n increases. Thus, from that point of view we will confine our analysis to the phaselag of homogeneous component only.

The phase-lag analysis of the method (2) is investigated using the homogeneous test equation

$$y'' = (i\lambda)^2 y(t).$$
(3)

Alternatively the method (2) can be written as

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^m b_i f(t_n + c_i h, Y_i)$$
  

$$y'_{n+1} = y'_n + h \sum_{i=1}^m b'_i f(t_n + c_i h, Y_i)$$
(4)

where

$$Y_{i} = y_{n} + c_{i}hy'_{i} + h^{2}\sum_{j=1}^{m} a_{ij}f(t_{n} + c_{i}h, Y_{j}).$$

By applying the general method (2) to the test equation (3) we obtain the following recursive relation as shown by Papageorgiou, Famelis and Tsitouras [4]

$$\begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix} = D\begin{bmatrix} y_n \\ hy'_n \end{bmatrix}, \quad z = \lambda h,$$

where

$$D(H) = \begin{pmatrix} 1 - Hb^{T} (I + HA)^{-1} e & 1 - Hb^{T} (I + HA)^{-1} c \\ -Hb'^{T} (I + HA)^{-1} e & 1 - Hb'^{T} (I + HA)^{-1} c \end{pmatrix}$$
(5)

where  $H = z^2$ ,  $e = (1 \cdots 1)^T$ ,  $c = (c_1 \cdots c_m)^T$ . Here D(H) is the stability matix of the RKN method and its characteristic polynomial

$$\xi^{2} - \operatorname{tr}(D(z^{2}))\xi + \det(D(z^{2})) = 0, \qquad (6)$$

is the stability polynomial of the RKN method. Solving difference system (5), the computed solution is given by

$$y_n = 2 |c|| \rho|^n \cos(\omega + n\phi).$$
(7)

The exact solution of (1) is given by

$$y(t_n) = 2 \mid \sigma \mid \cos(\chi + nz).$$
(8)

Eq. (7) and (8) led us to the following definition.

**Definition 1.** (Phase-lag). Apply the RKN method (2) to (1). Then we define the phase-lag  $\varphi(z) = z - \phi$ . If  $\varphi(z) = O(z^{u+1})$ , then the RKN method is said to have phase-lag order u. Additionally, the quantity  $\alpha(z) = 1 - |\rho|$  is called amplification error. If  $\alpha(z) = O(z^{v+1})$ , then the RKN method is said to have dissipation order v.

Let us denote

$$R(z^2) = \operatorname{trace}(D)$$
 and  $S(z^2) = \det(D)$ .

From Definition 1, it follows that

$$\varphi(z) = z - \cos^{-1}\left(\frac{R(z^2)}{2\sqrt{S(z^2)}}\right), \quad |\rho| = \sqrt{S(z^2)}.$$

Let us denote  $R(z^2)$  and  $S(z^2)$  in the following form

$$R(z^{2}) = \frac{2 + \alpha_{1}z^{2} + \dots + \alpha_{m}z^{2m}}{(1 + \hat{\lambda}z^{2})^{m}},$$
(9)

$$S(z^{2}) = \frac{1 + \beta_{1}z^{2} + \dots + \beta_{m}z^{2m}}{(1 + \hat{\lambda}z^{2})^{m}},$$
(10)

where  $\hat{\lambda} = 2\lambda^2$  is diagonal element in the Butcher tableau. Here the necessary condition for the fourthorder accurate diagonally implicit RKN method (2) to have hase-lag order six in terms of  $\alpha_i$  and  $\beta_i$  is given by

$$\alpha_3 - \beta_3 = 8\lambda^6 - 12\lambda^4 + \frac{\lambda^2}{2} - \frac{1}{360} .$$
 (11)

Notice that the fourth-order method is already dispersive order four and dissipative order five. Furthermore dispersive order is even and dissipative

order is odd. The following quantity is used to determine the dissipation constant of the formula.

$$1 - |\rho| = \left(3\lambda^{2} - \frac{1}{2}\beta_{1}\right)z^{2} - \left(\frac{15}{2}\lambda^{4} + \frac{1}{2}\beta_{2} - \frac{3}{2}\beta_{1}\lambda^{2} - \frac{1}{8}\beta_{1}^{2}\right)z^{4} - \left(-\frac{35}{2}\lambda^{6} - \frac{3}{2}\beta_{2}\lambda^{2} + \frac{15}{4}\beta_{1}\lambda^{4} - \frac{1}{4}\beta_{1}\beta_{2} + \frac{3}{8}\beta_{1}^{2}\lambda^{2} + \frac{1}{2}\beta_{3} + \frac{1}{16}\beta_{1}^{3}\right)z^{6} + O(z^{8}).$$
(12)

We next discuss the stability properties of method for solving (1) by considering the scalar test problem (3) applied to the method (2), from which the expression given in (5) is obtain. Eliminating  $y'_n$  and  $y'_{n+1}$  in (5), we obtain a difference equation of the form

$$y_{n+2} - R(H)y_{n+1} + S(H)y_n = 0.$$
 (13)

The characteristic equation associated with equation (13) is given as in (6). Chawla and Sharma [11] have discussed the interval of periodicity and absolute stability of Nyström method. Since our concerned here is with the analysis of high order dispersive RKN method, we therefore drop the necessary condition of periodicity interval i.e  $S(H) \equiv 1$ . Hence the method derived will be with empty interval of periodicity. We now consider the interval of absolute stability of RKN method. We therefore need the characteristic equation (6) to have roots with modulus less than one so that approximate solution will converge to zero as n tends to infinity. For convenience, we note the following definition adopted for method (5).

**Definition 2.** An interval  $(-H_a, 0)$  is called the interval of absolute stability of the method (5) if, for all  $H \in (-H_a, 0)$ ,  $|\xi_{1,2}| < 1$ .

#### **3** Contruction of the Method

In the following we shall derive a three-stage fourthorder accurate diagonally implicit RKN method with dispersive order six and dissipative order five, by taking into account the dispersion relation in Section 2. The RKN parameters must satisfy the following algebraic conditions to find fourth-order accuracy as given in Hairer and Wanner [2].

order 1

order 2

 $\sum b'_i = 1$ 

$$\sum b_i = \frac{1}{2}, \quad \sum b'_i c_i = \frac{1}{2}$$
 (15)

(14)

order 3

$$\sum b_i c_i = \frac{1}{6}, \quad \frac{1}{2} \sum b'_i c_i^2 = \frac{1}{6}$$
(16)

order 4

$$\frac{1}{2}\sum b_i c_i^2 = \frac{1}{24}, \quad \frac{1}{6}\sum b_i' c_i^3 = \frac{1}{24}, \quad \sum b_i' a_{ij} c_j = \frac{1}{24}.$$
(17)

For most methods the  $c_i$  need to satisfy

$$\frac{1}{2}c_i^2 = \sum_{j=1}^m a_{ij} \quad (i = 1,...,m).$$
(18)

In Sharp, Fine and Burrage [14] stated that fourthorder method with dispersive order eight do not exist. Therefore, the method of algebraic order four (r=4) with dispersive order six (u=6) and dissipative order five (v=5) is now considered. From algebraic conditions (14)-(18), it formed eleven equations with thirteen unknowns to be solved. We let  $b_1 = 0$  and  $\lambda$  be a free parameter. Therefore the following solution of one-parameter family is obtain

$$a_{31} = [288\lambda^3 - 24\lambda - 72\lambda^2 - 24\lambda^2\sqrt{3} + 3 - \sqrt{3} + 12\lambda\sqrt{3}]/[12(12\lambda - 3 + \sqrt{3})],$$

$$\begin{split} a_{21} &= -2\,\lambda^2 + \frac{1}{6} - \frac{\sqrt{3}}{12}, \\ a_{32} &= -\frac{1 + 96\,\lambda^3 - 8\,\lambda - 24\,\lambda^2}{2(12\,\lambda - 3 + \sqrt{3})}, b_1 = 0, \\ b_2 &= \frac{1}{4} + \frac{\sqrt{3}}{12}, b_3 = \frac{1}{4} - \frac{\sqrt{3}}{12}, \\ b_1' &= 0, b_2' = \frac{1}{2}, b_3' = \frac{1}{2}, c_1 = 2\,\lambda, \\ c_2 &= \frac{1}{2} - \frac{\sqrt{3}}{6}, c_3 = \frac{1}{2} + \frac{\sqrt{3}}{6}. \end{split}$$

From the above solution, we are going to derive a method with dispersion of order six. The six order dispersion relation (11) need to be satisfied and this can be written in terms of RKN parameters which corresponds to the above family of solution yields the following equation

$$(2880\lambda^{4}\sqrt{3} + (960 - 1440\sqrt{3})\lambda^{3} + (120 - 40\sqrt{3})\lambda^{2} + (120\sqrt{3} - 192)\lambda - 11\sqrt{3} + 18)/240(12\lambda - 3 + \sqrt{3}) = 0,$$

and solving for  $\lambda$  yields

# -0.1015757589,0.09374433416, 0.2097189023 and 0.1056624327.

The first two values will give us a nonempty stability interval while the others will produce the methods with empty stability interval. Taking the first two values of  $\lambda$ , give us two fourth-order diagonally implicit RKN methods with dispersive order six. For  $\lambda = -0.1015757589$ , will be produced the method which has PLTE

$$||\tau^{(5)}||=1.875825 \times 10^{-3}$$
 and  $||\tau'^{(5)}||_2=1.697439 \times 10^{-3}$ 

for  $y_n$  and  $y'_n$  respectively. The stability interval is approximately (-8.10,0). Fig. 1 is the stability region of the new method. We denote this method as DIRKNNew method (see Table 1). The coefficients of DIRKNNew are generated using computer algebra package Maple and the Maple environment variable Digits controlling the number of significant digits which is set to 10.



Fig. 1 Stability region for DIRKNNew method

Method	и	d	$\left\  \tau^{(r+1)} \right\ _2$	$\  \tau'^{(r+1)} \ _2$
DIRKNNew	6	$1.19 \times 10^{-4}$	$1.88 \times 10^{-3}$	$1.70 \times 10^{-3}$
DIRKNImoni	4	-	$3.75 \times 10^{-2}$	$3.22 \times 10^{-2}$
DIRKNHS	4	$1.43 \times 10^{-4}$	$6.35 \times 10^{-4}$	$1.59 \times 10^{-4}$
DIRKNSharp	6	$1.02 \times 10^{-2}$	$1.85 \times 10^{-3}$	$6.26 \times 10^{-4}$
DIRKNRaed	4	$-1.80 \times 10^{-2}$	$3.13 \times 10^{-2}$	$1.71 \times 10^{-2}$
Notations :	и –	- Dispersion	order, d -	Dissipation
constant, $\ \tau'$	$(r+1) \parallel$	<sub>2</sub> – Error	coefficient	for $y_n$ ,
$\  \tau'^{(r+1)} \ _2$ – Error coefficient for $y'_n$				

Table 2:	Summary	of the	DIRKN	methods
	2			

Table 1 The DIRKNNew method



Table 2 compares the properties of our method with the methods derived by van der Houwen and Sommeijer [20], Sharp, Fine and Burrage [14], Imoni, Otunta and Ramamohan [17] and Al-Khasawneh, Ismail and Suleiman [23].

## 4 Problem Tested

In order to evaluate the effectiveness of the new embedded method, we solved several model problems which have oscillatory solutions. The code developed uses constant step size mode and results are compared with the methods proposed in [14], [17], [20] and [23]. Figures 1–5 show the numerical results for all methods used. These codes have been denoted by:

- **DIRKNNew** : A new three-stage fourth-order dispersive order six derived in this paper.
- **DIRKNHS** : A three-stage fourth-order dispersive order four derived by van der Houwen and Sommeijer [20].
- **DIRKNSharp** : A three-stage fourth-order dispersive order six as in Sharp, Fine and Burrage [14].
- **DIRKNImoni** : A three-stage fourth-order derived by Imoni, Otunta and Ramamohan [17].
- DIRKNRaed : A four-stage fourth-order drived by Al-Khasawneh, Ismail, Suleiman [23].

For purposes of illustration, we will compare our results with those derived by using four methods; DIRKN three-stage fourth-order derived by van der Houwen and Sommeijer [20] and Imoni, Otunta and Ramamohan [17], three-stage fourthorder dispersive order six derived by Sharp, Fine and Burrage [14], and Al-Khasawneh, Ismail and Suleiman [23].

**Problem 1**(*Homogenous*)

$$\frac{d^2 y(t)}{dt^2} = -100 y(t), \quad y(0) = 1, \quad y'(0) = -2$$

 $0 \leq t \leq 10$  .

Exact solution  $y(t) = -\frac{1}{5}\sin(10t) + \cos(10t)$ 

#### Problem 2

$$\frac{d^2 y(t)}{dt^2} = -y(t) + t, \quad y(0) = 1, \quad y'(0) = 2$$

 $0 \leq t \leq 15\pi$  .

Exact solution  $y(t) = \sin(t) + \cos(t) + t$ 

Source : Allen and Wing [19]

Problem 3(Inhomogeneous system)

$$\frac{d^2 y_1(t)}{dt^2} = -v^2 y_1(x) + v^2 f(t) + f''(t),$$
  

$$y_1(0) = a + f(0), \quad y_1'(0) = f'(0),$$
  

$$\frac{d^2 y_2(t)}{dt^2} = -v^2 y_2(t) + v^2 f(t) + f''(t),$$
  

$$y_2(0) = f(0), \quad y_2'(0) = va + f'(0),$$

 $0 \le t \le 20 \; .$ 

Exact solution is

 $y_1(t) = a\cos(vt) + f(t)$ ,  $y_2(t) = a\sin(vt) + f(t)$ , f(t) is chosen to be  $e^{-0.05t}$  and parameters v and a are 20 and 0.1 respectively.

Source : Lambert and Watson [7]

**Problem 4** (*An almost Periodic Orbit problem*)

$$\frac{d^2 y_1(t)}{dt^2} + y_1(t) = 0.001\cos(t),$$
  

$$y_1(0) = 1, \quad y_1'(0) = 0$$
  

$$\frac{d^2 y_2(t)}{dt^2} + y_2(t) = 0.001\sin(t),$$
  

$$y_2(0) = 0, \quad y_2'(0) = 0.9995,$$

 $0 \leq t \leq 1000$  .

Exact solution  $y_1(t) = \cos(t) + 0.0005t\sin(t)$ ,  $y_2(t) = \sin(t) - 0.0005t\cos(t)$ 

Source : Stiefel and Bettis [3]

#### Problem 5

$$y_1'' = \frac{-y_1}{\left(\sqrt{y_1^2 + y_2^2}\right)^3} \quad y_1(0) = 1, \quad y_1'(0) = 0$$
$$y_2'' = \frac{-y_2}{\left(\sqrt{y_1^2 + y_2^2}\right)^3} \quad y_2(0) = 0, \quad y_2'(0) = 1,$$

 $0 \le t \le 10 \; .$ 

Exact solution  $y_1(t) = \cos(t)$ ,  $y_2(t) = \sin(t)$ 

Source : Dormand et al. [27]

### **5** Numerical Results

In this section we evaluate the accuracy and the effectiveness of the new DIRKN method derived in the previous section when they are applied to the numerical solution of several model problems.

The results for the five problems above are tabulated in Tables 3-7. One measure of the accuracy of a method is to examine the Emax(T), the maximum error which is defined by

Emax(T) = max || 
$$y(t_n) - y_n$$
 ||,  
where  $t_n = t_0 + nh$ ,  $n = 1, 2, ..., \frac{T - t_0}{h}$ .

Tables 3-7 show the absolute maximum error for DIRKNNew. DIRKNImoni. DIRKNHS. DIRKNSharp and DIRKNRaed methods when solving Problems 1-5 with three different step values for long period integration. From numerical results in Table 3-7, we observed that the new method is more accurate compared with DIRKNImoni. DIRKNHS and DIRKNRaed methods which do not relate to the dispersion order of the method. Also the new method is more accurate compared with DIRKNSharp method although the dispersion order is the same but the dissipation constant for our method is smaller than the DIRKNSharp method (see Table 2).

h	Method	T=100	T=1000	T=4000
0.002	5DIRKNNew	6.6480(-10)	1.0432(-7)	7.7282(-7)
	DIRKNImon	i1.5646(-2)	1.4622(-1)	4.7069(-1)
	DIRKNHS	1.2561(-7)	1.3689(-6)	5.8314(-6)
	DIRKNShar	p3.0150(-7)	3.0229(-6)	1.2120(-5)
	DIRKNRaed	9.2774(-6)	9.2904(-5)	3.7129(-4)
0.005	DIRKNNew	1.4042(-9)	1.2816(-8)	5.2981(-7)
	DIRKNImon	i2.0121(-2)	1.8480(-1)	5.6322(-1)
	DIRKNHS	6.6977(-7)	6.6966(-6)	2.7338(-5)
	DIRKNShar	p2.5569(-6)	2.5624(-5)	1.0255(-4)
	DIRKNRaed	1.4811(-4)	1.4849(-3)	5.9392(-3)
0.01	DIRKNNew	1.2746(-7)	1.2641(-6)	5.0385(-6)
	DIRKNImon	i5.9680(-2)	4.6223(-1)	9.2860(-1)
	DIRKNHS	3.2305(-5)	3.2361(-4)	1.2955(-3)
	DIRKNShar	p3.1342(-4)	3.1448(-3)	1.2637(-2)
	DIRKNRaed	2 3699(-3)	2 3786(-2)	9 5368(-2)

Table 3: Comparing Our Results with the Methods in the Literature for Problem 1

Table 4: Comparing Our Results with the Methods

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h	Method	T=100	T=1000	T=4000
0.06	5DIRKNNew	2.4734(-8)	2.4734(-8)	5.1346(-8)
	DIRKNImoni	5.2936(-3)	5.3213(-2)	2.0104(-1)
	DIRKNHS	6.8021(-7)	6.8361(-6)	2.7394(-5)
	DIRKNSharp	4.0017(-6)	4.1061(-5)	1.6419(-4)
	DIRKNRaed	5.8594(-5)	5.8706(-4)	2.3509(-3)
0.12	5DIRKNNew	3.9303(-7)	3.9303(-7)	1.8558(-6)
	DIRKNImoni	1.0214(-2)	1.0110(-1)	3.6319(-1)
	DIRKNHS	1.0871(-5)	1.0930(-4)	4.3835(-4)
	DIRKNSharp	1.3006(-4)	1.3398(-3)	5.3657(-3)
	DIRKNRaed	8.0270(-4)	8.0329(-3)	3.2192(-2)
0.25	DIRKNNew	6.2304(-6)	1.2593(-5)	6.4522(-5)
	DIRKNImoni	1.9124(-2)	1.8683(-1)	6.2958(-1)
	DIRKNHS	1.7332(-4)	1.7444(-3)	7.0007(-3)
	DIRKNSharp	4.4802(-3)	4.6441(-2)	1.9520(-1)
	DIRKNRaed	1.2897(-2)	1.2969(-1)	5.3226(-1)

Table 5: Comparing Our Results with the Methods in the Literature for Problem 3

h	Method	T=100	T=1000	T=4000
0.002	5DIRKNNew	2.7829(-7)	2.7617(-6)	1.0977(-5)
	DIRKNImoni	5.9756(-3)	4.6003(-2)	9.1504(-2)
	DIRKNHS	3.9675(-7)	3.9897(-6)	1.6028(-5)
	DIRKNSharp	1.8995(-6)	1.9004(-5)	7.6048(-5)
	DIRKNRaed	2.9121(-5)	2.9123(-4)	1.1650(-3)
0.005	DIRKNNew	2.4806(-8)	2.4849(-7)	9.9021(-7)
	DIRKNImoni	1.1371(-2)	7.0105(-2)	9.9201(-2)
	DIRKNHS	6.3468(-6)	6.3496(-5)	2.5414(-4)
	DIRKNSharp	6.1529(-5)	6.1776(-4)	2.4938(-3)
	DIRKNRaed	4.6623(-4)	4.6689(-3)	1.8754(-2)

0.01	DIRKNNew	8.0340(-7)	8.0370(-6)	3.2133(-5)
	DIRKNImoni	1.9988(-2)	9.0402(-2)	1.0002(-1)
	DIRKNHS	1.0142(-4)	1.0156(-3)	4.0582(-3)
	DIRKNSharp	2.0819(-3)	2.2852(-2)	1.2662(-1)
	DIRKNRaed	7.5063(-3)	7.7409(-2)	2.5731(-1)

Table 6: Comparing Our Results with the Methods in the Literature for Problem 4

h	Method	T=100	T=1000	T=4000	
0.06	5DIRKNNew	2.9059(-8)	2.9059(-8)	6.6009(-8)	
	DIRKNImoni	3.9398(-3)	4.0219(-2)	2.1108(-1)	
	DIRKNHS	5.6025(-7)	5.8308(-6)	3.2036(-5)	
	DIRKNSharp	3.4938(-6)	3.6382(-5)	1.9990(-4)	
	DIRKNRaed	4.1138(-5)	4.2777(-4)	2.3486(-3)	
0.12	5DIRKNNew	3.9716(-7)	3.9716(-7)	1.8843(-6)	
	DIRKNImoni	7.3190(-3)	7.3627(-2)	3.7216(-1)	
	DIRKNHS	7.6595(-6)	7.9664(-5)	4.3794(-4)	
	DIRKNSharp	9.3794(-5)	9.7492(-4)	5.3673(-3)	
	DIRKNRaed	5.6329(-4)	5.8856(-3)	3.2103(-2)	
0.25	DIRKNNew	6.3791(-6)	1.2074(-5)	6.4943(-5)	
	DIRKNImoni	1.3995(-2)	1.3655(-1)	6.7067(-1)	
	DIRKNHS	1.2220(-4)	1.2725(-3)	7.0099(-3)	
	DIRKNSharp	3.2209(-3)	3.3895(-2)	1.9396(-1)	
	DIRKNRaed	9.0479(-3)	9.4159(-2)	5.1301(-1)	

Table 7: Comparing Our Results with the Methods in the Literature for Problem 5

h	Method	T=100	T=1000	T=4000
0.002	5DIRKNNew	5.5401(-11)	1.0163(-8)	7.5573(-8)
	DIRKNImoni	5.5528(-2)	1.9949(0)	1.9949(0)
	DIRKNHS	7.4271(-11)	9.8619(-9)	7.4034(-8)
	DIRKNSharp	5.7702(-11)	1.9349(-8)	2.1818(-7)
	DIRKNRaed	1.9249(-6)	1.9649(-4)	3.1493(-3)
0.005	DIRKNNew	4.0657(-11)	7.0058(-10)	6.3443(-8)
	DIRKNImoni	9.1042(-2)	1.9928(0)	1.9928(0)
	DIRKNHS	3.0398(-10)	4.8264(-9)	3.3611(-8)
	DIRKNSharp	3.1897(-9)	2.8667(-7)	4.6199(-6)
	DIRKNRaed	3.8351(-6)	3.9272(-4)	6.2956(-3)
0.01	DIRKNNew	7.1980(-10)	2.8538(-9)	9.2242(-9)
	DIRKNImoni	1.8191(-1)	1.9897(0)	1.9897(0)
	DIRKNHS	4.8285(-9)	8.5076(-8)	7.7698(-7)
	DIRKNSharp	9.5779(-8)	9.1298(-6)	1.4554(-4)
	DIRKNRaed	7.3809(-6)	7.7803(-4)	1.2501(-2)

Notation : 1.2345(-4) means  $1.2345 \times 10^{-4}$ 

Figs. 2-6 show the decimal logarithm of the maximum global error for the solution (MAXE) versus the function evaluations. From Figs. 1-5, we observed that DIRKNNew performed better compared to DIRKNRaed and DIRKNImoni for integrating second-order differential equations

possessing an oscillatory solution in terms of function evaluations. In terms of global error DIRKNNew produced smaller error compared to DIRKNRaed, DIRKNHS, DIRKNSharp and DIRKNImoni.



Fig. 2 Efficiency curves for Problem 1 for  $h = 1/4^i$ , i = 2,...,6



Fig. 3 Efficiency curves for Problem 2 for  $h=1/2^i$ , i=1,...,5



Fig. 4 Efficiency curves for Problem 3 for  $h = 0.6/4^i$ , i = 2,...,6



Fig. 5 Efficiency curves for Problem 4 for  $h = 1/2^i$ , i = 2,...,6



Fig. 6 Efficiency curves for Problem 5 for  $h=1/4^i$ , i=1,...,5

### 6. Conclusion

In this paper we have derived a new three-stage fourth-order diagonally implicit RKN method with dispersive order six and 'small' dissipation constant and principal local truncation errors. We have also performed various numerical tests. From the results tabulated in Tables 3-7 and Fig. 2 to Fig. 6, we conclude that the new method is more efficient for integrating second-order equations possessing an oscillatory solution when compared to the current DIRKN methods derived by van der Houwen and Sommeijeir [20], Sharp, Fine and Burrage [14], Imoni, Otunta and Ramamohan [17] and Al-Khasawneh, Ismail and Suleiman [23].

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