The analysis of mathematical models associated to some economic growth processes with endogenous population

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Abstract: The aim of this paper is to extend the study done by [5]. Here, we will analyze two mathematical models associated to some economic growth processes with endogenous population depending on the current fertility. The first model is a version of the Ramsey model in continuous and infinite time with the Cobb-Douglas production function. In the second model we consider the AK production function. The mathematical models of these economical growth processes lead to two optimal control problems with an infinite horizon. The necessary conditions for optimality are given for both economic problems. Using the optimality conditions we prove the existence, the uniqueness and the stability of the steady states.

Key–Words: mathematical models applied in economies, endogenous growth, endogenous population, optimization problems.

1 Introduction

The Ramsey growth model is a neoclassical model of economic growth based on the work of Ramsey, whose idea was to determine the saving rate endogenously, through a dynamic maximization process.

In this paper, based on the Ramsey growth model [1], we consider two versions of this model with endogenous population. In the standard Ramsey growth model, the growth rate of population is constant and exogenous, yielding an exponential behavior of the population size over time. This type of time behavior is unrealistic. A more realistic approach would be to consider a logistic law for the population growth as in [2] or a growth rate of population which depend on the current level of per capita income as in [9]. In [2], Brida and Accinelli analyzed how the Ramsey model is affected by the choice of a logistic growth of population, considering that the society's welfare is measured by a utility function of per capita consumption. As in [2], Guerrini in [10] analyzed how the Ramsey model is affected by the choice of a logistic growth of population, assuming that the society's welfare over time is measured by weighting the utility index of per capita consumption by numbers, i.e multiplying the utility function of the representative men by the total population. In [9], Fanti analyzed the Solow-type model with endogenous population depending on the current income, taking into account a Malthusian relation between fertility and income. They consider that the population growth rate is a function of the current level of per capita income. In this paper we consider two Ramsey type models with endogenous population depending on the current income as in [9]. In the first model we consider that the output is determined with a Cobb-Douglas production function and in the second model we assume that the output is determined with a AK production function. These economical growth models lead to two optimal control problems. In order to position the present paper in the current literature on economic growth, we note that our models are Ramsey-type models of balanced growth with endogenous population. The present paper is organized as follows. In Section 2, we present the model that we use in this article with the Cobb-Douglas production function. In Section 3, we give necessary conditions for the optimal solution of the economical growth problem with the endogenous population depending on the current fertility and the Cobb-Douglas production function. In Section 4, we determine the steady state of the optimal control problem and we show that it is a saddle point. Also, we examine the qualitative dynamic behavior of the optimal solution. In Section 5, we present the model with the AK production function. In Section 6, we give necessary conditions for the optimal solution of the economical growth problem with endogenous population depending on the current fertility and with the AK production function. Also, we determine the steady state of the optimal control problem and we show that there is a saddle point. We examine the qualitative dynamic behavior of the optimal solution, as well. Some conclusions are given in section 7.

2 The economical growth model with the Cobb-Douglas production function

In this paper, based on [1], [3], [4], [7], [13], [11], we consider an economical growth model with endogenous population depending on the current income. The economy consists of a fixed number of identical infinitely lived households that, for simplicity, is normalized to one. The representative household is populated by identical and infinitely lived agents. The size of population (identified with the size of labour force) at moment t is denoted by $L(t)$, which grows at a rate that depends on the current fertility. Time is taken to be continuous. Also, we assume the economy closed (i.e. all of the stock capital must be owned by someone in economy and the net foreign debt is zero.)

The representative household has access to a technology described by a neoclassical production function. Thus, we consider that the output is determined by the following Cobb-Douglas production function ([12])

$$Y(t) = K^{\alpha}(t)L^{1-\alpha}(t) \tag{1}$$

where $Y(t)$ and $K(t)$ denote the aggregate output and the aggregate capital stock spent producing goods and $\alpha \in (0, 1)$.

The output can be used either for consumption or investment.

Therefore, the household's budget constraint is

$$Y(t) = I(t) + C(t) \tag{2}$$

where $C(t)$ is the aggregate consumption, $I(t)$ is the gross investment.

Capital is accumulated in the economy by the investment. The capital accumulation equation is given by

$$\dot{K}(t) = K^{\alpha}(t)L^{1-\alpha}(t) - C(t) - \delta K(t), \tag{3}$$

where $\delta \in (0, 1)$ is the depreciation rate of the capital stock.

In what follows, expressing all the model variables in per capita units, we obtain new variables of the model:

$y(t) = \frac{Y(t)}{L(t)}$ – the output per unit of labour,

$c(t) = \frac{C(t)}{L(t)}$ – the consumption per unit of labour,

$k(t) = \frac{K(t)}{L(t)}$ – the capital stock per unit of labour,

$k_0 = \frac{K_0}{L(0)}$ – the initial capital stock per unit of labour.

Contrary to most subsequent developments, where the growth rate of the labour force, $n_s = \frac{\dot{L}}{L}$, was treated as exogenously determined, in this paper we consider that it is endogenous.

Taking into account the Malthusian relation between fertility and income, in this paper we will consider the growth rate of the labour force as a function of the current level of per capita income:

$$n_s = n_s(y),$$

which becomes in the Cobb-Douglas case

$$n_s = n_s(k^{\alpha}).$$

Following Fanti and Manfredi [9], we assume that the function n_s is linear and increasing. Thus we consider

$$n_s = nk^{\alpha}$$

and we have

$$\frac{\dot{L}(t)}{L(t)} = nk^{\alpha}(t) \tag{4}$$

where $n > 0$ is a constant parameter, tuning the reaction of the growth rate of the labour force to change in per-capita income.

Proposition 1 *The capital accumulation equation (3) in per capita terms is given by*

$$\dot{k}(t) = k^{\alpha}(t) - c(t) - nk^{\alpha+1}(t) - \delta k(t), \tag{5}$$

and the initial capital stock is $k(0) = k_0 > 0$.

Proof. If we differentiate the ratio $k(t) = \frac{K(t)}{L(t)}$ with respect to t, we obtain

$$\dot{k}(t) = \frac{\dot{K}(t)}{L(t)} - \frac{K(t)}{L(t)}\frac{\dot{L}(t)}{L(t)}. \tag{6}$$

Using the relation (4) in the equation (6), we have

$$\frac{\dot{K}(t)}{L(t)} = \dot{k}(t) + nk^{\alpha+1}(t). \tag{7}$$

Dividing the capital accumulation equation by the labour force size, $L(t)$ we obtain

$$\frac{\dot{K}(t)}{L(t)} = \left(\frac{K(t)}{L(t)}\right)^{\alpha} - \delta\frac{K(t)}{L(t)} - \frac{C(t)}{L(t)} \tag{8}$$

From (7) and (8), we have the capital accumulation equation in per capita terms given by

$$\dot{k}(t) = k^{\alpha}(t) - c(t) - nk^{\alpha+1}(t) - \delta k(t).$$

■

In this economy, the objective of a social planner is to choose at each moment in time the level of consumption $c(t)$ so as to maximize the household's global utility taking into account the budget constraint for the household, relation (5), and the initial stock of capital k_0.

The household's global utility is defined as

$$U = \int_0^{\infty} e^{-\rho t}u(c(t))dt \tag{9}$$

where $u(\cdot)$ is the instantaneous utility function which depend on per capita consumption, $c(t)$.

The function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is of class C^2 and satisfies

$$u(0) = 0, \ \ u'(c) > 0, \ \ u''(c) < 0, \forall c \geq 0,$$

$$\lim_{c\to 0} u'(c) = \infty, \lim_{c\to\infty} u'(c) = 0$$

and the parameter $\rho > 0$ is the time preference rate.

The capital initial stock that is available for a household is K_0. Thus, the capital initial stock per worker is k_0.

Therefore, we can formulate the optimization problem such as

$$\max_{c(t)} \int_0^{\infty} e^{-\rho t}u(c(t))dt \tag{10}$$

subject to

$$\begin{aligned} \dot{k}(t) &= k^{\alpha}(t) - c(t) - nk^{\alpha+1}(t) - \delta k(t) \quad (11)\\ k(0) &= k_0. \quad (12) \end{aligned}$$

3 Determination of optimality conditions for the economic problem with the Cobb-Douglas production function

The economic problem is to choose in every moment t, the size of consumption so as to maximize the global utility taking into account the budget constraint for household and the capital initial stock k_0. This economic problem leads us to the following mathematical optimization problem (P) :

Problem P. Determine (k^*, c^*) which maximize the following functional

$$\int_0^{\infty} e^{-\rho t}u(c(t))dt \tag{13}$$

with $k \in AC([0,\infty), \mathbf{R}_+)$, $c \in \mathcal{X} = \{c : [0,\infty) \to \mathbf{R}_+, c - measurable\}$, which verifies:

$$\begin{aligned} \dot{k}(t) &= k^{\alpha}(t) - c(t) - nk^{\alpha+1}(t) - \delta k(t), \quad (14)\\ k(0) &= k_0, \quad (15) \end{aligned}$$

where $AC([0,\infty), \mathbf{R}_+)$ is the class of absolutely continuous functions.

This problem can be solved using Pontryagin's maximum principle as in [14], [8]. The state variable in this problem is $k(t)$ and the control variable is $c(t)$.

We denote by $\mu(t)$ the co-state variable corresponding to $k(t)$.

We will continue to determine the necessary conditions for optimality problem P. Thus, we define the function of Hamilton-Pontryagin given by

$$H(k, c, \mu, t) = e^{-\rho t}u(c) + \mu(k^{\alpha} - c - nk^{\alpha+1} - \delta k).$$

Theorem 1 *Let $(k^*(t), c^*(t))$ be an optimal solution which solves problem* $\mathbf{P}$. *Then, there exists the adjoint absolutely continuous function $q(t)$ such that for all $t \in [0,\infty)$, the relations*

$$q(t) = u'(c^*(t)) \tag{16}$$

$$\dot{q}(t) = q(t)(\rho + \delta + n(\alpha+1)k^{*\alpha}(t) - \alpha k^{*\alpha-1}(t)) \tag{17}$$

hold.

Proof. Let $(k^*(t), c^*(t))$ be an optimal solution for **P**. The Hamilton function associated to problem (P) is

$$H(k(t), c(t), \mu(t), t) = e^{-\rho t}u(c(t)) + \\ +\mu(t)(k^{\alpha}(t) - c(t) - nk^{\alpha+1}(t) - \delta k(t)). \quad (18)$$

From the Pontryagin's principle there exists the adjoint absolutely continuous function $\mu(t)$ such that

$$\dot{\mu}(t) = -\frac{\partial H}{\partial k} = -\mu(t)(\alpha k^{*\alpha-1}(t) - n(\alpha+1)k^{*\alpha}(t) - \delta)) \quad (19)$$

and $c^*(t)$ is value $c \in [0, \infty)$ that maximizes

$$H(k^*(t), c, \mu(t), t) = e^{-\rho t}u(c) + \\ +\mu(t)(k^{*\alpha}(t) - c - nk^{*\alpha+1}(t) - \delta k^*(t)). \quad (20)$$

Using the transformation $\mu(t) = e^{-\rho t}q(t)$, the Hamilton function becomes

$$H(k^*(t), c, q(t), t) = e^{-\rho t}[u(c) + \\ +q(t)(k^{*\alpha}(t) - c - nk^{*\alpha+1}(t) - \delta k^*(t)]. \quad (21)$$

The first and second derivatives of function H with respect to c are given by

$$H'_c(k^*(t), c, q(t), t) = e^{-\rho t}\left(u'(c) - q(t)\right) \quad (22)$$

$$H''_{cc}(k^*(t)), c, q(t), t) = e^{-\rho t}u''(c). \quad (23)$$

From (23) and the properties of the utility function, we obtain that H is a concave function of c.

Because $c^*(t) \in [0, \infty)$ maximizes (21) and H is a concave function of c we have

$$H'_c(k^*(t), c^*(t), q(t), t) = 0, \quad (24)$$

thus

$$q(t) = u'(c^*(t)). \quad (25)$$

Using again the transformation $\mu(t) = e^{-\rho t}q(t)$, relation (19) becomes

$$\dot{q}(t) = q(t)(\rho + \delta + n(\alpha+1)k^{*\alpha}(t) - \alpha k^{*\alpha-1}(t)). \quad (26)$$

■

Remark 2 *If we differentiate the condition (16) with respect to t and we use (17), then we obtain*

$$\dot{c}(t) = \frac{u'(c(t))}{u''(c(t))}(\rho + \delta + n(\alpha+1)k^{\alpha}(t) - \alpha k^{\alpha-1}(t)). \quad (27)$$

Remark 3 *The optimal trajectory of problem* $(\mathbf{P})$ *is a solution of the following system of differential equations*

$$\dot{k}(t) = k^{\alpha}(t) - c(t) - nk^{\alpha+1}(t) - \delta k(t) \quad (28)$$

$$\dot{c}(t) = \frac{u'(c(t))}{u''(c(t))}(\rho + \delta + n(\alpha + 1)k^{\alpha}(t) - \alpha k^{\alpha-1}(t)). \quad (29)$$

4 Qualitative analysis of the optimal solution for the economic problem with the Cobb-Douglas production function

Generally, system (28)-(29) is not analytically solvable, but we can state some qualitative properties of the solution. First, we determine the steady states (k^*, c^*) of the differential equation system (28)-(29).

We recall that in the steady state both per capita capital stock $k(t)$ and the level of consumption per capita $c(t)$ are constant.

Note that our analysis is restricted to the interior steady states only, i.e. we exclude the economically meaningless solutions such as $k^* = 0, c^* = 0$.

Proposition 4 *(Stationary state). The nonlinear differential equations system (28)-(29) has a unique steady state* (k^*, c^*)*, where the capital-labour ratio* k^* *is the root of the following equation*

$$\alpha k^{\alpha-1} - n(\alpha + 1)k^{\alpha} = \rho + \delta \quad (30)$$

and the per capita consumption c^* *is given by*

$$c^* = (k^*)^{\alpha} - n(k^*)^{\alpha+1} - \delta k^*.$$

Proof. To determinate the steady state of the above system we choose the stationary solutions $k(t) = k^*$, $c(t) = c^*$.

From $\dot{c}(t) = 0$ we obtain

$$\alpha k^{\alpha-1} - n(\alpha + 1)k^{\alpha} = \rho + \delta. \quad (31)$$

Next, we consider

$$g(k) = \alpha k^{\alpha-1} - n(\alpha + 1)k^{\alpha}$$

and

$$g'(k) = \alpha(\alpha - 1)k^{\alpha-2} - n(\alpha + 1)\alpha k^{\alpha-1}.$$

Since $k > 0$, $n > 0$ and $\alpha \in (0, 1)$ we get $g'(k) < 0$, $\lim_{k\to 0} g(k) = \infty$ and $\lim_{k\to\infty} g(k) = -\infty$.

Therefore, the conditions $\lim_{k\to 0} g(k) = \infty$, $\lim_{k\to\infty} g(k) = -\infty$ and $g'(k) < 0$ ensure that equation (31) has a unique positive solution k^*.

Hence, the steady-state of per capita capital stock k^* exists, and it is unique.

Consequently, using (28), there is a unique c^* satisfying the identity

$$c^* = (k^*)^\alpha - n(k^*)^{\alpha+1} - \delta k^*.$$

■

Proposition 5 *The steady state of the nonlinear differential equations system*

$$\begin{aligned}\dot{k}(t) &= k^\alpha(t) - c(t) - nk^{\alpha+1}(t) - \delta k(t)\\ \dot{c}(t) &= \frac{u'(c(t))}{u''(c(t))}(\rho + \delta + n(\alpha+1)k^\alpha(t) - \alpha k^{\alpha-1}(t)).\end{aligned}$$

is a saddle point with a one dimensional stable manifold.

Proof. To investigate the stability of the steady state, we linearize system (28)-(29) in the steady state.

For there, we denote

$$p = \frac{u'(c^*)}{u''(c^*)}(n(\alpha+1)\alpha(k^*)^{\alpha-1} - \alpha(\alpha-1)(k^*)^{\alpha-2}) < 0.$$

Thus, we obtain

$$\begin{aligned}\dot{k}(t) &\simeq (\alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta)(k(t) - k^*) -\\ &- (c(t) - c^*) \quad (32)\\ \dot{c}(t) &\simeq p(k(t) - k^*) \quad (33)\end{aligned}$$

The Jacobian matrix of the linearized system in the steady state is

$$J = \begin{pmatrix} \alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta & -1 \\ p & 0 \end{pmatrix}$$

In order to characterize the local stability of the system, we need to compute the eigenvalues of the Jacobian matrix.

The eigenvalues of the above matrix are the solutions of the characteristic equation

$$\lambda^2 - \lambda(\alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta) + p = 0. \quad (34)$$

From equation (34) we have

$$\Delta = (\alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta)^2 - 4p > 0$$

$$p = \frac{u'(c^*)}{u''(c^*)}(n(\alpha+1)\alpha(k^*)^{\alpha-1} - \alpha(\alpha-1)(k^*)^{\alpha-2}) < 0.$$

Because the determinant of the equation is positive, $\Delta > 0$ and the product is negative, $p < 0$, then the equation (34) has two real roots with contrary signs

$$\lambda_{1,2} = \frac{\left(\alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta\right) \pm \sqrt{\Delta}}{2}. \quad (35)$$

Using (30), it results that relation (35) which gives the eigenvalues, can be written as

$$\lambda_{1,2} = \frac{\rho \pm \sqrt{\Delta}}{2}$$

The eigenvalues of the linearized system, being the real numbers with contrary signs, lead to the fact that the steady state (k^*, c^*) is a saddle point.

Because the steady state (k^*, c^*) is a saddle point, there are two manifolds passing through the steady state: a stable manifold W_s and an instable manifold W_i. ■

Theorem 2 *i) The eigenvector, corresponding to the eigenvalue*

$$\lambda_1 = \frac{\left(\alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta\right) - \sqrt{\Delta}}{2},$$

tangent in the steady state (k^*, c^*) to the stable manifold W_s is given by $v = \alpha(1, v_2)$, $\alpha \in \mathbb{R}$

$$v_2 = \frac{\left(\alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta\right) + \sqrt{\Delta}}{2};$$

ii) The eigenvector, corresponding to the eigenvalue

$$\lambda_2 = \frac{\left(\alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta\right) + \sqrt{\Delta}}{2},$$

tangent in the steady state (k^*, c^*) to the stable manifold W_i is given by $\omega = \alpha(1, \omega_2)$, $\alpha \in \mathbb{R}$ where

$$\omega_2 = \frac{\left(\alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta\right) - \sqrt{\Delta}}{2}.$$

Proof. The matrix of the linearized system is given by :

$$J = \begin{pmatrix} \alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta & -1 \\ p & 0 \end{pmatrix}.$$

Its eigenvalues are the roots of the characteristic equation

$$\lambda^2 - trJ\lambda + \det J = 0$$

where

$$trJ = \alpha(k^*)^{\alpha-1} - n(\alpha+1)(k^*)^\alpha - \delta$$

and

$$\det J = p.$$

Thus, the eigenvalues are given by

$$\lambda_1=\frac{\left(\alpha\,(k^*)^{\alpha-1}-n(\alpha+1)\,(k^*)^{\alpha}-\delta\right)-\sqrt{\Delta}}{2}<0$$

and

$$\lambda_2=\frac{\left(\alpha\,(k^*)^{\alpha-1}-n(\alpha+1)\,(k^*)^{\alpha}-\delta\right)+\sqrt{\Delta}}{2}>0.$$

Next, we shall determine the eigenvector $v = (v_1, v_2)^T$ associated to the eigenvalue λ_1. This vector is tangent in the steady state (k^*, c^*) to the optimal trajectory.

The eigenvector is the solution of the equation

$$Jv = \lambda_1 v,\ \ v = (v_1, v_2)^T.$$

Therefore,

$$\begin{aligned}(\alpha(k^*)^{\alpha-1}-n(\alpha+1)(k^*)^{\alpha}-\delta-\lambda_1)v_1-v_2 &= 0\\ pv_1-\lambda_1 v_2 &= 0.\end{aligned}$$

Normalizing $v_1 = 1$, we obtain $v = (1, v_2)$, where

$$v_2=\frac{\left(\alpha\,(k^*)^{\alpha-1}-n(\alpha+1)\,(k^*)^{\alpha}-\delta\right)+\sqrt{\Delta}}{2}>0.$$

The slope of the stable manifold W_s in the steady state (k^*, c^*) will be given by v_2.

In the same way for the eigenvalue λ_2 we obtain the associated eigenvector $\omega = (1, \omega_2)$, where

$$\omega_2=\frac{\left(\alpha\,(k^*)^{\alpha-1}-n(\alpha+1)\,(k^*)^{\alpha}-\delta\right)-\sqrt{\Delta}}{2}<0.$$

The slope of the instable manifold W_i in the steady state (k^*, c^*) will be given by ω_2. ■

5 The economic growth model with the AK production function

In this section, we present the mathematical model of the economic process described in section 2, where we consider the AK production function and the population growth rate that depend on the current income.

We will consider the same setup as in Section 2, but the production function is a linear one that depend only on the capital stock. Thus, we assume that the output is determined by the AK production function, given by

$$Y(t) = AK(t) \tag{36}$$

where $Y(t)$ and $K(t)$ denote the aggregate output and the aggregate capital stock spent producing goods and $A > 0$.

In this case the capital accumulation equation is given by

$$\dot{K}(t) = AK(t) - C(t) - \delta K(t), \tag{37}$$

where $C(t)$ is the aggregate consumption and $\delta \in (0, 1)$ is the depreciation rate of the capital stock.

Let $L(t)$ be the population size at moment t, (identified with the labour force size). Denoting by $y(t) = \frac{Y(t)}{L(t)}$ the output per capita, the production function can be expressed in intensive form as

$$y(t) = Ak.$$

The same as in Section 2, the growth rate of the labour force, $n_s = \frac{\dot{L}}{L}$, is assumed endogenous. Taking into account the Malthusian relation between fertility and income, we will consider the growth rate of the labour force as a function of the current level of per capita income:

$$n_s = n_s(y),$$

which becomes in the AK case

$$n_s = n_s(Ak).$$

Following Fanti and Manfredi [9], we assume that the function n_s is a linear one:

$$n_s = nAk$$

where $n > 0$ is a constant parameter, tuning the reaction of the growth rate of the labour force to change in per-capita income.

The initial capital stock that is available for a household is K_0 and the initial capital stock per capita is k_0.

Let $c(t) = \frac{C(t)}{L(t)}$ and $k(t) = \frac{K(t)}{L(t)}$ denote consumption and capital stock per capita.

Therefore, capital accumulation equation (37) in per capita terms is given by

$$\dot{k}(t) = Ak(t) - c(t) - nk^2(t) - \delta k(t), \tag{38}$$

and the initial capital stock is $k(0) = k_0 > 0$.

In this economy, the objective of a social planner is to choose at each moment in time the level of consumption $c(t)$ so as to maximize the household's global utility taking into account: the budget constraint for the household, relation (38), and the initial capital stock k_0.

The household's global utility is defined as

$$U = \int_0^\infty e^{-\rho t} u(c(t))dt \qquad (39)$$

where $u(\cdot)$ is the instantaneous utility function which depends on per capita consumption, $c(t)$.

The function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^2 class function and satisfies

$$u(0) = 0,\ \ u'(c) > 0,\ \ u''(c) < 0, \forall c \geq 0,$$

$$\lim_{c\to 0} u'(c) = \infty,\ \lim_{c\to\infty} u'(c) = 0$$

and the parameter $\rho > 0$ is the time preference rate.

Therefore, we can formulate the optimization problem such as

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(c(t))dt \qquad (40)$$

subject to

$$\begin{aligned} \dot{k}(t) &= (A-\delta)k(t) - nk^2(t) - c(t) \qquad (41)\\ k(0) &= k_0. \qquad (42) \end{aligned}$$

6 Qualitative analysis of the optimal solution for the economic problem with the AK production function

In this section, the economic problem is to choose in every moment t, the consumption size so as to maximize the global utility taking into account the budget constraint for household and the initial capital stock k_0. This leads to the following mathematical optimization problem (P) :

Problem P. Determine (k^*, c^*) which maximizes the following functional

$$\int_0^\infty e^{-\rho t} u(c(t))dt \qquad (43)$$

in the class of functions $k \in AC([0,\infty), \mathbb{R}_+)$, and $c \in \mathcal{X}$, where
$\mathcal{X} = \{c : [0,\infty) \to [0, A]\,, c\text{−measurable}, A < \infty\}$ which verifies:

$$\begin{aligned} \dot{k}(t) &= (A-\delta)k(t) - nk^2(t) - c(t) \qquad (44)\\ k(0) &= k_0. \qquad (45) \end{aligned}$$

In our problem **P,** k is the state variable and c is the control variable.

Definition 6 *A trajectory* $(k(t), c(t))$ *is called an admissible trajectory, with initial capital* k_0*, for problem* (P) *if it verifies the relations (44)-(45).*

Definition 7 *An admissible trajectory,* $(k^*(t), c^*(t))$ *, is called optimal trajectory if:*

$$\int_0^\infty e^{-\rho t} u(c(t))dt \leq \int_0^\infty e^{-\rho t} u(c^*(t))dt.$$

for every admissible trajectory $(k(t), c(t))$ *of problem* (P)*.*

As in [4], [6] we have the following theorem:

Theorem 3 *If* $(c(t), k(t))$ *is an optimal trajectory of problem* (P)*, then it verifies the Euler-Lagrange equation:*

$$-\frac{d}{dt}\left[u'\left(\phi(k(t)) - \dot{k}(t)\right)e^{-\rho t}\right] =$$

$$= u'(\phi(k(t)) - \dot{k}(t))\phi'(k(t)) \cdot e^{-\rho t},$$

where

$$\phi(k(t)) = (A-\delta)k(t) - nk^2(t)$$

and

$$c(t) = \phi(k(t)) - \dot{k}(t).$$

Using the above Euler Lagrange equation we obtain

$$\begin{aligned} &[-u''(c(t))\,\dot{c}(t) + \rho u'(c(t))]e^{-\rho t}\\ &= u'(c(t))\phi'(k(t))e^{-\rho t} \end{aligned} \qquad (46)$$

which is equivalent with

$$\begin{aligned} &-u''(c(t))\,\dot{c}(t) + \rho u'(c(t))\\ &= u'(c(t))\,(A - \delta - 2nk(t)). \end{aligned} \qquad (47)$$

Finally, we have the evolution equation of consumption given by:

$$\dot{c}(t) = \frac{u'(c(t))}{u''(c(t))}\,(\rho + \delta - A + 2nk(t)).$$

Remark 8 *The optimal trajectory of problem* $(\mathbf{P})$ *is the solution of the following system*

$$\begin{aligned} \dot{k}(t) &= (A-\delta)k(t) - nk^2(t) - c(t) & (48) \\ \dot{c}(t) &= \frac{u'(c(t))}{u''(c(t))}(\rho+\delta-A+2nk(t)). & (49) \end{aligned}$$

In what follows, we will make a qualitative analysis of the optimal solution using an exponential utility function. Thus, we will assume that the instantaneous utility function is an exponential function, given by

$$u(c(t)) = \frac{1}{\theta}(1-e^{-\theta c(t)}), \qquad (50)$$

the global utility function is given by

$$\frac{1}{\theta}\int_0^\infty e^{-\rho t}(1-e^{-\theta c(t)})dt,$$

and the parameter θ is positive.

Next, we can rewrite system (48)-(49) taking into account the exponential utility function, thus we have:

$$\begin{aligned} \dot{k}(t) &= (A-\delta)k(t) - nk^2(t) - c(t) & (51) \\ \dot{c}(t) &= -\frac{1}{\theta}(\rho+\delta-A+2nk(t)). & (52) \end{aligned}$$

Proposition 9 *The system of differential equations*

$$\begin{aligned} \dot{k}(t) &= (A-\delta)k(t) - nk^2(t) - c(t) \\ \dot{c}(t) &= -\frac{1}{\theta}(\rho+\delta-A+2nk(t)) \end{aligned}$$

exhibits saddle-path stability.

Proof. In order to determinate the steady state of the above system we choose the stationary solutions $k(t) = k^*$, $c(t) = c^*$. From $\dot{c}(t) = 0$ we obtain

$$k^* = \frac{A-\rho-\delta}{2n}. \qquad (53)$$

Using k^* in equation $\dot{k}(t) = 0$ we can determine

$$c^* = (A-\delta)k^* - nk^{*2}. \qquad (54)$$

The point (k^*, c^*) represents a steady state for the system of differential equations.

In order to investigate the stability of the steady state (k^*, c^*) we linearize the system in the steady state (k^*, c^*) and we obtain

$$\begin{aligned} \dot{k}(t) &\approx (A-\delta-2nk^*)(k(t)-k^*) - (c(t)-c^*) \\ \dot{c}(t) &\approx \frac{-2n}{\theta}(k(t)-k^*) \end{aligned}$$

The matrix of the linearized system is

$$\begin{pmatrix} A-\delta-2nk^* & -1 \\ \frac{-2n}{\theta} & 0 \end{pmatrix}$$

The eigenvalues are solutions of equation

$$\lambda^2 - \lambda(A-\delta-2nk^*) - \frac{2n}{\theta} = 0. \qquad (55)$$

The discriminant of the equation (55) is given by

$$\begin{aligned} \Delta &= (A-\delta-2nk^*)^2 + \frac{8n}{\theta} \\ &= \rho + \frac{8n}{\theta} > 0 \end{aligned}$$

and the product of the roots is given by

$$p = \frac{-2n}{\theta} < 0.$$

Because equation (55) has the positive discriminant and the negative product, it results that equation (55) has two real roots of opposite signs

$$\lambda_{1,2} = \frac{(A-\delta-2nk^*) \pm \sqrt{\Delta}}{2}. \qquad (56)$$

By (53), relation (56), which gives the eigenvalues, can be written thus

$$\lambda_{1,2} = \frac{\rho \pm \sqrt{\rho^2 + \frac{8n}{\theta}}}{2}.$$

The eigenvalues of the linearized system being real numbers with contrary signs, the steady state (k^*, c^*) is the saddle point.

Because the steady state (k^*, c^*) is a saddle point, there are two manifolds passing through the steady state: a stable manifold W_s and an instable manifold W_i. The dynamic equilibrium follows the stable manifold ■

Remark 10 *By comparing with the* AK *standard model [1], which has no transitional dynamics, the* AK *economic growth model with endogenous population has the transitional dynamics. The transitional dynamics are similar to those of Ramsey's standard model.*

7 Conclusions

In this paper we have analyzed two economic growth models with endogenous population. In the first model we have considered a Cobb-Douglas production function and in the second model we have assumed that the output is determined by a AK production function.

The mathematical models of these economical growth processes lead to two optimal control problems with an infinite horizon. The necessary conditions for optimality are given for both economic problems. Using the optimality conditions we have proved the existence, the uniqueness and the stability of the steady states.

In the model with endogenous population depending on the current income and the Cobb-Douglas production function, the transitional dynamics are similar to those of Ramsey's standard model.

In the AK model with endogenous population depending on the current income, the transitional dynamics are similar to those of the Ramsey's standard model.

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References

[1] R.J. Barro, X. Sala-i-Martin, *Economic Growth,* McGraw-H ill, New York, 1995.

[2] J. G. Brida, and E. Accinelli, *The Ramsey model with logistic population growth,* Economics Buletin, 3, 2008, pp.1-8.

[3] O. Bundău, Optimal conditions for the control problem associated to a Ramsey model with taxes and exponential utility, *Proceedings of the 10th WSEAS Int. Conf. on Mathematics and Computers in Business and Economics, Prague*, Czech Republic, ISSN 1790-5109, 2009, pp.141-146.

[4] O. Bundău, Optimal control applied to a Ramsey model with taxes and exponential utility, *Journal Wseas Transaction on Mathematics*, ISSN 1109-2769, Issue 12, volume 8, December, 2009, pp. 689-698.

[5] O. Bundău, M. Neamţu, Optimal conditions for the control problem associated to a Ramsey model with endogenous population, *Proceedings of the 11th WSEAS Int. Conf. on Mathematics and Computers in Business and Economics*, Iaşi, Romania, ISSN 1790-2769, 2010, pp.175-180.

[6] O. Bundău, A. Juratoni, Optimal Conditions for the Control Problem Associated to an Economic Growth Process Exposed to a Risk Factor, *Journal Wseas Transaction on Mathematics*, ISSN 1109-2769, Issue 7, Volume 9, July 2010, pp. 519-528.

[7] O. Bundău, F. Pater and A. Juratoni, The Analysis of a Mathematical Model Associated to an Economic Growth Process, *Proc. Conf. Numerical Analysis and Applied Mathematics*, Creta 18-22 sep, A.I.P, vol 1168, ISBN 978-0-7354-0709, 2009.

[8] E. Burmeister, A.–R. Dobell, *Matematical theories of economic growth,* Great Britan, 1970.

[9] L. Fanti and P. Manfredi, The Solow's model with endogenous population: a neoclassical growth cycle model, *Journal of Economic Development*, Vol. 28, Nr.2, 2003, pp. 103-115.

[10] M. Ferrara and L. Guerrini, The Ramsey model with logistic population growth and Benthamite felicity function, *Proceedings of the 10th WSEAS International Conference on Mathematics and Computers in Business and Economics*, Prague, Czech Republic, March 23-25, 2009, pp. 231-234.

[11] M. Ferrara and L. Guerrini, The Green Solow Model with Logistic Population Change, *Proceedings of the 10th WSEAS International Conference on Mathematics and Computers in Business and Economics*, Prague, Czech Republic, March 23-25, 2009, pp. 17-20.

[12] M. Ferrara and L. Guerrini, The Ramsey Model with Logistic Population Growth and Benthamite Felicity Function Revisited, *Journal Wseas Transaction on Mathematics*, ISSN 1109-2769, Issue 3, Volume 8, March, 2009, pp.97-106.

[13] M. Neamţu, M. Pirtea, G.Mircea, D. Opriş, Stochastic, Fuzzy, Hybrid Delayed Dynamics Heterogeneous Competitions with Product Differentiation, *Journal Wseas Transaction on*

Mathematics, ISSN 1109-2769, Issue 7, Volume 9, July, 2010, pp. 581-591.

[14] A. Seierstad, K. Sydsæter, *Optimal Control Theory with Economic Applications,* North-Holland, Amsterdam, 1987.