Deterministic, uncertainty and stochastic models of Kaldor-Kalecki model of business cycles

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Abstract: - This paper presents the Kaldor-Kalecki nonlinear business cycle model of the income. It will take into consideration the investment demand in the form suggested by Rodano. We will analyze the deterministic, uncertainty and stochastic Kaldor-Kalecki models. The dynamics of the mean values and the square mean values of the model's variables are set. Numerical examples are given in the end, to illustrate our theoretical results.

Key-Words: Business cycle model, deterministic model with delay, Kaldor-Kalecki model, uncertain system, stochastic system

1 Introduction

The model proposed by Kaldor [4] is one of earliest and simplest nonlinear models of business cycles. This model cannot be considered as a satisfying description of actual economies. Nevertheless, it continues to generate a considerable amount of economic, pedagogical and methodological interest, both from the point of view of the economist and of the applied dynamicist.

Kalecki introduced the idea that there is a time delay for investment before a business decision. Krawiek and Sydlowski [3] incorporated Kalecki's idea into Kaldor's model and proposing the Kaldor-Kalecki model of business cycles. In recent literature, it has been proved that information delay makes dynamic economic models unstable. In situations where delay is important, models with stochastic perturbation are framed by stochastic differential delay equations. In this paper, we will investigate the effects of the random perturbation for Kaldor-Kalecki model analyzing the steady state of the model with stochastic perturbation.

The reminder of the paper develops as follows. In Section 2, we describe a deterministic Kaldor-Kalecki model using the investment demand proposed in Rodano [1]. In Section 3, we analyze the deterministic Kaldor-Kalecki model, setting the conditions for the existence of the delay parameter value for which the model displays a Hopf bifurcation. In Section 4 we analysis the uncertainty Kaldor-Kalecki model. In Section 5, the stochastic system is presented and the locally asymptotic stability is analyzed by the variables' mean and the square mean. Numerical simulations are carried out in Section 6. Finally, concluding remarks are given in Section 7.

2 Deterministic and stochastic models of a Kaldor Kalecki business cycle with delay

In the last decade, the study of delayed differential equations that arose in business cycles has received much attention. The first model of business cycles can be traced back to Kaldor [4], who used a system of ordinary differential equations to study business cycles in 1940 by proposing nonlinear investment and saving functions so that the system may have cyclic behaviors or limit cycles, which are important from the point of view of economics. Kalecki [3] introduced the idea that there is a time delay for investment before a business decision. Krawiec and Szydlowski [3] incorporated the idea of Kalecki into the model of Kaldor by proposing the following Kaldor-Kalecki model of business cycles:

$$\begin{aligned} \frac{dY(t)}{dt} &= \alpha(I(Y(t),K(t)) - S(Y(t),K(t))) \\ \frac{dK(t)}{dt} &= I(Y(t-\tau),K(t)) - \beta K(t) \end{aligned} \tag{1}$$

where Y is the gross product, K is the capital stock, $\alpha > 0$ is the adjustment coefficient in the goods market, $\beta \in (0,1)$ is the depreciation rate of capital stock, $I(Y,K)$ and $S(Y,K)$ are investment and saving functions, and $\tau \geq 0$ is a time lag representing delay for investment due to the past investment decision.

Considering that past investment decisions also influence the change in the capital stock, in [14] we extended the model by imposing delays in both the gross product and capital stock. Thus, adding of same delay to the capital stock K in the investment function $I(Y,K)$ in the second equation of system (1), the following Kaldor-Kalecki model business cycles is obtained:

$$\begin{aligned} \frac{dY(t)}{dt} &= \alpha(I(Y(t),K(t)) - S(Y(t),K(t))) \\ \frac{dK(t)}{dt} &= I(Y(t-\tau),K(t-\tau)) - \beta K(t) \end{aligned} \tag{2}$$

As usual in a Keynesian framework, savings are assumed to be proportional to the current level of income:

$$S(Y,K) = \delta Y, \tag{3}$$

where coefficient δ, $\delta \in (0,1)$ represents the propensity to save.

As usual, the investment demand is assumed to be an increasing and sigmoid-shaped function of the income. Without loss of generality, in the following we shall consider the form proposed in Rodano [1]:

$$I(Y,K) = \delta u + r\left(\frac{\delta u}{\beta} - K\right) + f(Y-u) \tag{4}$$

where $\frac{\delta u}{\beta}$ is the “normal” level of capital stock. In (4), two short-run investment components are considered: the first one is proportional to the difference between normal capital stock and current stock, according to a coefficient r $(r > 0)$, usually explained by the presence of adjustment costs; the second one is an increasing, but not linear, function of the difference between current income and its normal level.

This second component of the short-run investment function is a convenient specification of the sigmoid-shaped relationship between investment and income proposed by Kaldor. We note that this analytic specification does not compromise the generality of the results.

From (2) with (3) and (4) we obtain the following system:

$$\begin{aligned} \frac{dY(t)}{dt} &= \alpha(\delta u + r(\frac{\delta u}{\beta} - K(t)) + f(Y(t)-u) - \delta Y(t)) \\ \frac{dK(t)}{dt} &= \delta u + r(\frac{\delta u}{\beta} - K(t-\tau)) + f(Y(t-\tau)-u) - \beta K(t) \end{aligned} \tag{5}$$

System (5) with the initial conditions:

$$Y(\theta) = \phi_1(\theta),\ K(\theta) = \phi_2(\theta),\ \theta \in [-\tau, 0] \tag{6}$$

and $\phi_1, \phi_2 : [-\tau, 0] \to R$ of C^1 class functions, represent a system of differential equations with delay.

3 The analysis of Kaldor-Kalecki deterministic model

The equilibrium point of system (5) is the solution of the following system:

$$\begin{aligned} \delta u + r\left(\frac{\delta u}{\beta} - K\right) + f(Y-u) - \delta Y = 0 \\ \delta u + r\left(\frac{\delta u}{\beta} - K\right) + f(Y-u) - \beta K = 0 \end{aligned} \tag{7}$$

From (7) we obtain the equilibrium point $Y_0 = u,\ K_0 = \frac{\delta u}{\beta}$. By carrying out the translation $u_1(t) = Y(t) - u,\quad u_2(t) = K(t) - \frac{\delta u}{\beta}$ from (5) we get the system:

$$\begin{aligned} \frac{du_1(t)}{dt} &= -\alpha\delta u_1(t) - \alpha r u_2(t) + \alpha f(u_1(t)) \\ \frac{du_2(t)}{dt} &= -\beta u_2(t) - r u_2(t-\tau) + f(u_1(t-\tau)), \\ u_1(\theta) &= \phi_1(\theta) - u,\ u_2(\theta) = \phi_2(\theta) - \frac{\delta u}{\beta},\ \theta \in [-\tau, 0]. \end{aligned} \tag{8}$$

We assume that $f(x)$ is a nonlinear C^4 function with $\rho_1 = f'(0) \neq 0$.

The linearized system of (8) is given by:

$$\begin{aligned} \frac{dy_1(t)}{dt} &= \alpha(\rho_1 - \delta) y_1(t) - \alpha r y_2(t) \\ \frac{dy_2(t)}{dt} &= -\beta y_2(t) + \rho_1 y_1(t-\tau) - r y_2(t-\tau). \end{aligned} \tag{9}$$

The characteristic function for (9) is given by:

$$f(\lambda, \tau) = \lambda^2 + (\beta - \alpha(\rho_1 - \delta))\lambda + (\lambda r + \alpha\delta r)e^{-\lambda\tau} \tag{10}$$

From (11) we have:

Proposition 1:

If $\tau = 0$, the characteristic equation $f(\lambda, 0) = 0$ has the roots with negative real part, if and only if, $\rho_1 < \min\left\{\delta + \frac{\beta + r}{\alpha},\ \delta + \frac{\delta r}{\alpha}\right\}$.

In this case system (8) is asymptotically stable.

Proposition 2:

If $\delta < \frac{\beta(\beta + r)}{\alpha r}$, and $\tau \in (0, \tau_0)$, the characteristic equation $f(\lambda, \tau) = 0$ has the roots with negative real part, where τ_0 is given by:

$$\tau_0 = \frac{1}{\omega_0} arctg\left(\frac{\omega_0(\omega_0^2 + \alpha\rho_1(\beta - \alpha\delta) + \alpha^2\delta^2)}{\omega_0^2(\alpha\rho_1 - \beta) + \alpha^2\beta\delta(\rho_1 - \delta)}\right) \quad (11)$$

and ω_0 is a positive root of the equation:

$$\omega^4 + (\alpha^2(\rho_1 - \delta)^2 + \beta^2 - r^2)\omega^2 + \alpha^2\beta^2(\rho_1 - \delta)^2 - \alpha^2\delta^2 r^2 = 0. \quad (12)$$

Proof:

For $\tau \neq 0$, let $\lambda = i\omega$ be the root of the equation $f(\lambda, \tau) = 0$. By replacement, we obtain:

$$\begin{aligned} &\omega^2 + \alpha\beta(\rho_1 - \delta) = \alpha\delta r\cos\theta + \omega r\sin\theta \\ &(\alpha(\rho_1 - \delta) - \beta)\omega = \omega r\cos\theta - \alpha\delta r\sin\theta \end{aligned} \quad (13)$$

where $\theta = \omega\tau$. From (13), by squaring each relation and their addition, we obtain equation (12).

From condition $\delta < \frac{\beta(\beta + r)}{\alpha r}$ we get $\rho_1 < \delta + \frac{\delta + r}{\alpha\beta} < \delta + \frac{\beta + r}{\alpha}$.

Thus $\beta^2(\rho_1 - \delta)^2 - \delta^2 r^2 < 0$. Equation (12) has a positive root ω_0. Relation (13) yields:

$$tg(\omega_0\tau_0) = \frac{\omega_0(\omega_0^2 + \alpha\rho_1(\beta - \alpha\delta) + \alpha^2\delta^2)}{\omega_0^2(\alpha\rho_1 - \beta) + \alpha^2\beta\delta(\rho_1 - \delta)}. \quad (14)$$

For

$$\tau_0 = \frac{1}{\omega_0} arctg \frac{\omega_0(\omega_0^2 + \alpha\rho_1(\beta - \alpha\delta) + \alpha^2\delta^2)}{(\omega_0^2(\alpha\rho_1 - \beta) + \alpha^2\beta\delta(\rho_1 - \delta))}, \quad (15)$$

the normal form, the limit cycle and the Lyapunov coefficient that characterizes this cycle, for the system (8) are obtained with the method from [9], [10], [11].

4 The analysis of the uncertainty Kaldor-Kalecki model

In what follows we present some concepts of the uncertainty theory introduced by Liu in [6], [7], [8]. Classical measure and probability measure belong to the class of “completely additive measure”, i.e., the measure of union of disjoint events is just the sum of the measures. In contrast, capacity, belief measure, plausibility measure, fuzzy measure, possibility measure and necessity measure belong to the class of “completely nonadditive measures”. Since this class of measures does not assume the self-duality property, all of those measures are inconsistent with the law of contradiction and law of excluded middle, which dominate human thinking logic.

Uncertainty theory was founded by Liu [6] in 2007 and refined by Liu [8] in 2010. Nowadays uncertainty theory has become a branch of mathematics based on normality, monotonicity, self duality, countable subadditivity, and product measure axioms.

The first fundamental concept in uncertainty theory is uncertain measure that is used to measure the belief degree of an uncertain event.

The second one is uncertain variable that is used to represent imprecise quantities.

The third one is uncertainty distribution that is used to describe uncertain variables in an incomplete but easy-to-use way.

Uncertainty theory is thus deduced from those three foundation stones, and provides a mathematical model to deal with uncertain phenomena.

Let Γ be a nonempty set. A collection L of subsets of Γ is called a L-algebra if:

(a) $\Gamma \in L$;

(b) if $\Lambda \in L$, then $\Lambda^c \in L$;

(c) if $\Lambda_1, \Lambda_2, \Lambda_3, ... \in L$, then $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup ... \in L$.

Each element Λ in the σ-algebra L is called an event. Uncertain measure is a function from L to $[0, 1]$. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ a number M{ Λ } which indicates the belief degree that Λ will occur. In order to ensure that the number M{ Λ } has certain mathematical properties, Liu [6] proposed the following four axioms:

Axiom 1. (Normality Axiom)
M{ Γ } = 1 for the universal set Γ.

Axiom 2. (Monotonicity Axiom)
$M\{\Lambda_1\} \le M\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$.

Axiom 3. (Self-Duality Axiom)
$M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event Λ.

Axiom 4. (Countable Subadditivity Axiom)
For every countable sequence of events $\{\Lambda_i\}$, we have $M\{\bigcup_{i=1}^{\infty}\Lambda_i\} \le \sum_{i=1}^{\infty} M(\Lambda_i)$.

Let Γ be a nonempty set, L a σ-algebra over Γ, and M an uncertain measure. Then the triplet (Γ, L, M) is called an uncertainty space.

An uncertain variable is a measurable function psi from an uncertainty space (Γ, L, M) to the set of real numbers, i.e., for any Borel set B of real numbers, the set:
$\{\Psi \in B\} = \{\Lambda \in L \mid \Psi(\Gamma) \in B\}$ is an event.

The uncertainty distribution θ of an uncertain variable Ψ is defined by:

$$\theta(x) = M\{\Psi \le x\}.$$

An uncertain variable Ψ is called normal if it has a normal uncertainty distribution:

$$\theta(x) = (1 + e^{\frac{\pi(e-x)}{\sigma\sqrt{3}}})^{-1}$$

denoted by $N(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$.

Let Ψ be an uncertain variable. Then the expected value of Ψ is defined by:

$$E(\Psi) = \int_0^{\infty} M(\Psi \ge r)dr - \int_{-\infty}^{0} M(\Psi \le r)dr$$

provided that at least one of the two integrals is finite.

The normal uncertain variable Ψ has e as expected value.

Let Ψ be an uncertain variable with finite expected value *e*. Then the variance of Ψ is defined by $V[\Psi] = E[(\Psi - e)^2]$.

Let T be an index set and let (Γ, L, M) be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, L, M)$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set:
$\{X_t \in B\} = \{\Lambda \in L \mid X_t(\Lambda) \in B\}$ is an event.

An uncertain process X_t is said to have independent increments if
$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_k} - X_{t_{k-1}}$
are independent uncertain variables where t_0 is the initial time and $t_1, t_2, ... t_k$ are any times with $t_0 < t_1 < ... < t_k$.

An uncertain process C_t is said to be a canonical process if:

(i) $C_0 = 0$, and almost all sample paths are Lipschitz continuous;

(ii) C_t has stationary and independent increments;

(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is $\theta(x) = (1 + e^{\frac{\pi x}{t\sqrt{3}}})^{-1}$.

Uncertain differential equation was proposed by Liu [6] in 2008 as a type of differential equation driven by canonical process. After that, an existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [2] in 2010.

Suppose C_t is a canonical process, and f and g are some given functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \quad (16)$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies (16) identically in t.

It is almost impossible to find analytic solutions for general uncertain differential equations. This fact provides a motivation to design numerical methods to solve uncertain differential equations.

Let a be a number with $0 < a < 1$. An uncertain differential equation (16) is said to have an a-path $X(t,a)$ if it solves the corresponding ordinary differential equation:

$$\frac{dX(t,a)}{dt} = f(t, X(t,a)) + g(t, X(t,a))\Phi(a)^{-1}$$

where $\Phi(a)^{-1}$ is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,

$$\Phi(a)^{-1} = \frac{\sqrt{3}}{3}\ln\frac{a}{1-a} \quad (17)$$

The uncertainty Kaldor-Kaleki model is given by

$$\begin{aligned} dY(t) &= \alpha(\delta u + r(\frac{\delta u}{\beta} - K(t)) + f(Y(t)-u) - \delta Y(t))dt - \\ &\quad - c_1(Y(t)-Y_0)dC(t) \\ dK(t) &= (\delta u + r(\frac{\delta u}{\beta} - K(t-\tau)) + f(Y(t)-u))dt - \\ &\quad - c_2(K(t)-K_0)dC(t) \end{aligned} \quad (18)$$

By carrying out the translation $u_1(t) = Y(t) - u, \quad u_2(t) = K(t) - \frac{\delta u}{\beta}$, from (18) we get the system:

$$\begin{aligned} du_1(t) &= (-\alpha\delta u_1(t) - \alpha r u_2(t) + \alpha f(u_1(t)))dt - c_1 u_1(t)dC(t) \\ du_2(t) &= (-\beta u_2(t) - r u_2(t-\tau) + f(u_1(t-\tau)))dt - c_2 u_2(t)C(t) \end{aligned} \quad (19)$$

For $0 < a < 1$, $a-$path, $U(t,a) = (u_1(t,a),\ u_2(t,a))$ for (19) is given by:

$$\begin{aligned} \frac{du_1(t,a)}{dt} &= -\alpha\delta u_1(t,a) - \alpha r u_2(t,a) + \alpha f(u_1(t,a)) - \\ &\quad - c_1 u_1(t,a)\Phi^{-1}(a) \\ \frac{du_2(t)}{dt} &= -\beta u_2(t,a) - r u_2(t-\tau,a) + f(u_1(t-\tau,a)) - \\ &\quad - c_2 u_2(t,a)\Phi^{-1}(a) \end{aligned} \quad (20)$$

where

$$\Phi^{-1}(a) = \frac{\sqrt{3}}{\pi}\ln\frac{a}{1-a}.$$

The linearized system of (20) is given by:

$$\begin{aligned} \frac{dy_1(t,a)}{dt} &= (\alpha(\rho_1 - \delta) - c_1\Phi^{-1}(a))y_1(t,a) - \alpha r(y_2(t,a)) \\ \frac{dy_2(t,a)}{dt} &= -(\beta + c_2\Phi^{-1}(a))y_2(t,a) + \rho_1 y_1(t-\tau,a) - \\ &\quad - r y_2(t-\tau,a) \end{aligned} \quad (21)$$

The characteristic function for (21) is given by:

$$\begin{aligned} g(\lambda,\tau,a) &= f(\lambda,\tau) + \Phi^{-1}(a)(c_1+c_2)\lambda + \\ &\quad + \Phi^{-1}(a)c_1 r e^{-\lambda\tau} + \Phi^{-1}(a)(c_1 + c_2 + \\ &\quad + c_1\beta - c_2\alpha(\rho_1 - \delta) + c_1 c_2\Phi^{-1}(a)) \end{aligned} \quad (22)$$

where

$f(\lambda,\tau)$ is given by (10).

The characteristic function (22) is given by:

$$\begin{aligned} g(\lambda,\tau,a) &= \lambda^2 + a_1(a)\lambda + (a_2(a)\lambda + a_3(a))e^{-\lambda\tau} + \\ &\quad + a_4(a) \end{aligned} \quad (23)$$

where

$$\begin{aligned} a_1(a) &= \beta - \alpha(\rho_1 - \delta) + \Phi^{-1}(a)(c_1 + c_2) \\ a_2(a) &= r, \quad a_3(a) = \alpha\delta r + \Phi^{-1}(a)c_1 r \\ a_4(a) &= \Phi^{-1}(a)(c_1 + c_2 + c_1\beta - c_2\alpha(\rho_1 - \delta) + c_1 c_2\Phi^{-1}(a) \end{aligned} \quad (24)$$

From (23) we have:

Proposition 3:

a) If $a = 0.5$, then $g(\lambda, \tau, 0.5) = f(\lambda,\tau)$.

b) If $\tau = 0$ and $\rho_1 < \min\{\delta + \frac{\beta + r}{\alpha}, \delta + \frac{\delta r}{\alpha}\}$ then the characteristic equation $g(\lambda,0,a) = 0$ has the roots with negative real part for $a \in [0.51,1]$ and has one root with positive real part for $a \in [0,0.5)$.

c) If $\delta < \frac{\beta(\beta + r)}{\alpha r}$ and $\tau(a) \in (0, \tau_0(a))$, $a \in [0.5,1]$ the characteristic equation $g(\lambda,0,a) = 0$ has the roots with negative real part, where $\tau_0(a)$ is given by

$$\tau_0(a) = \frac{1}{\omega_0(a)} arctg\frac{\omega_0(a)(a_2(a)\omega_0(a)^2 + a_1(a)a_3(a) - a_2(a)a_4(a)}{\omega_0(a)^2(a_3(a) - a_1(a)a_2(a) - a_3(a)a_4(a))} \quad (25)$$

and $\omega_0(a)$ is a positive root of the equation:

$$\begin{aligned} &\omega(a)^4 + (a_1(a)^2 - 2a_4(a) - a_2(a)^2)\omega(a)^2 + \\ &a_4(a)^2 - a_3(a)^2 = 0 \end{aligned} \quad (26)$$

Proof: The results are obtained like as from the proof of proposition 2.

For $\tau_0(a)$ given by (25), the limit cycle and the Lyapunov coefficient that characterizes this cycle, for the system (20) are obtained with the method from [9], [10], [11].

5 The analysis of the stochastic Kaldor-Kalecki model

Let (Ω, F, P) be the given probability space and $w(t) \in R$ be a scalar Wiener process on Ω, having independent stationary Gaussian increments with $w(0)=0$, $E(w(t)-w(s))=0$ and $E(w(t)w(s))=\min(t,s)$, where E is the mathematical expectation. The sample trajectories of $w(t)$ are continuous, nowhere differentiable and have infinite variations on any finite time interval [4].

For dynamical system (5), we are interested in knowing the effect of the noise perturbation on the equilibrium point (Y_0, K_0). The stochastic disturbance model of system (5) is given by a system of stochastic differential equations with delay in the following way:

$$dY(t)=\alpha(\delta u+r(\frac{\delta u}{\beta}-K(t))+f(Y(t)-u)-\delta Y(t))dt- \quad (27)$$
$$-\sigma_1(Y(t)-Y_0)dw(t)$$
$$dK(t)=(\delta u+r(\frac{\delta u}{\beta}-K(t-\tau))+f(Y(t)-u))dt-$$
$$-\sigma_2(K(t)-K_0)dw(t)$$

where $\sigma_1>0,\ \sigma_2>0.$

The solution of (27) is a stochastic process denoted by $Y(t)=Y(t,\omega),\ K(t)=K(t,\omega),\ \omega\in\Omega$. From the Chebyshev inequality, the possible range of Y, K at a time t is "almost" determined by its mean and variance at time t. So, the first and second moments are important for investigating the solution behavior.

Let the stochastic systems given by (27). Linearizing (27) around the equilibrium (Y_0, K_0) yields the linear stochastic differential delay equation:

$$dy(t)=(Ay(t)+By(t-\tau))dt-Cy(t)dw(t) \quad (28)$$

where $y(t)=(y_1(t), y_2(t))^T$ and

$$A=\begin{pmatrix}\alpha(\rho_1-\delta) & -\alpha r\\ 0 & -\beta\end{pmatrix},\ B=\begin{pmatrix}0 & 0\\ -\rho_1 & -r\end{pmatrix},\ C=\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}. \quad (29)$$

Let $Y(t)$ be the fundamental solution of the system:

$$y(t)=Ay(t)+By(t-\tau). \quad (30)$$

The solution of (24) is a stochastic process given by:

$$y(t,\phi)=y_\phi(t)-\int_0^t Y(t-s)Cy(t-\tau,\phi)dw(s) \quad (31)$$

where $y_\phi(t)$ is the solution given by:

$$y_\phi(t)=Y(t)\phi(0)+\int_{-\tau}^0 Y(t-\tau-s)\phi(s)ds \quad (32)$$

and $\phi:[-\tau,0]\to R^2$ is a family of continuous functions.

The existence and uniqueness theorem for the stochastic differential delay equations has been established in [5].

The solution $y(t,\phi)$ is a stochastic process with distribution at any time t determined by the initial function $\phi(\theta)$. From the Chebyshev inequality, the possible range of y, at time t is "almost" determined by its mean and variance at time t. Thus, the first and second moments of the solutions are important for the investigation of the solution's behavior.

We have used E to denote the mathematical expectation and we denote $y(t,\phi)$ by $y(t)$. From (16) we obtain:

Proposition 4:

The moment of the solution of (24) is given by:

$$\frac{dE(y(t))}{dt}=AE(y(t))+BE(y(t-\tau)). \quad (33)$$

The mean of the solution for (28) behaves precisely like the solution of the unperturbed deterministic equation (10).

The proof results from taking into account the mathematical expectation of both sides of (28) as well as the properties of the Ito calculus.

To examine the stability of the second moment of $y(t)$ for linear stochastic differential delay equation (28) we use Ito's rule to obtain the stochastic differential of $y(t)y(t)^T$ where $y(t)=(y_1(t)\ \ y_2(t)\ \ y_3(t))^T$:

$$\begin{aligned}\frac{dR(t,t)}{dt}&=E\{dy(t)dy^T(t)+y(t)dy^T(t)\}\\ &=E\{Ay(t)y^T(t)+y(t)y^T(t)A^T+By(t-\tau)y^T(t)+\\ &\quad+y(t)y^T(t-\tau)B^T+Cy(t)y^T(t)C\}\end{aligned} \quad (34)$$

Let $R(t,s)=E\{y(t)y^T(s)\}$ be the covariance matrix of the process $y(t)$ so that $R(t,t)$ satisfies:

$$\frac{dR(t,t)}{dt} = AR(t,t) + R(t,t)A^T + BR(t-\tau,t) + + R(t,t-\tau)B^T + CR(t,t)C \quad (35)$$

From (34) and A, B, C given by (29), we get:

Proposition 5:

1. The differential system (35) is given by:

$$\begin{aligned}\frac{dR_{11}(t,t)}{dt} &= (2\alpha(\rho_1-\delta)+\sigma_1^2)R_{11}(t,t) - 2\alpha r R_{12}(t,t)\\ \frac{dR_{22}(t,t)}{dt} &= (-2\beta+\sigma_2^2)R_{22}(t,t) - 2rR_{22}(t,t-\tau) + \\ &\quad + 2\rho_1 R_{12}(t-\tau,t) \\ \frac{dR_{12}(t,t)}{dt} &= -\alpha r R_{22}(t,t) + \alpha(\rho_1-\delta-\beta-\sigma_1\sigma_2)R_{12}(t,t) + \\ &\quad + \rho_1 R_{11}(t,t-\tau) - rR_{12}(t,t-\tau)\end{aligned} \quad (36)$$

2. The characteristic function of (30) is given by:

$$\begin{aligned}h(\lambda,\tau) &= (2\lambda - 2\alpha(\rho_1-\delta)-\sigma_1^2)(2\lambda+\beta-\sigma_2^2+2re^{-\lambda\tau})\\ &\quad (2\lambda-\alpha(\rho_1-\delta)-\sigma_1\sigma_2+\beta+re^{-\lambda\tau}) + \\ &\quad + 2\alpha r\rho_1(4\lambda+\beta-\sigma_1^2-\sigma_2^2-2\alpha(\rho_1-\delta)+2re^{-\lambda\tau})\end{aligned} \quad (37)$$

Proof:

1. The system (36) is obtained from (31) by taking into account that $R_{ij}(t,s) = R_{ji}(t,s)$, $i, j = 1,2$.
2. Let, $R_{ij}(t,s) = e^{\lambda(t+s)}K_{ij}$, $i, j = 1,2$. where K_{ij} are constants. Replacing $R_{ij}(t,s)$ in (35) and setting the condition that the system we obtain should accept nontrivial solution, we get $h(\lambda,\tau) = 0$.

The stability of the second moment is done by analyzing the roots of the characteristic equation $h(\lambda,\tau) = 0$.

Proposition 6:

If $\sigma_1 = \sigma_2$ then the characteristic function (34) is given by:

$$h(\lambda,\tau) = (\lambda + a_1 + re^{-\lambda\tau})(4\lambda^2 + 4a_1\lambda + a_2 + (c_1\lambda + c_2)e^{-\lambda\tau}) \quad (38)$$

where

$$\begin{aligned}a_1 &= \beta - \alpha(\rho_1-\delta)+\sigma_1\\ a_2 &= \sigma_1^2(2\alpha(\rho_1-\delta)+\sigma_1^2)\\ c_1 &= 4r\\ c_2 &= 4\alpha\delta r - 2r\sigma_1^2\end{aligned} \quad (39)$$

Proposition 7:

If $\tau = 0$, the characteristic equation $h(\lambda,0) = 0$ has roots with negative real parts if and only if $a_1 + r < 0$, $4a_1 + c_1 > 0$, $a_2 + c_2 > 0$, where a_1, a_2, c_1, c_2 are given by (39).

In this case the equation system that described the square mean is asymptotically stable.

Proposition 8:

If $\tau \neq 0$ and the equation:

$$16\omega^4 + (16a_1^2 - 8c_2 - c_1^2) + a_2^2 - c_2^2 = 0 \quad (40)$$

has a positive root ω_2, then for $\tau \in (0, \tau_2)$ the characteristic equation $h(\lambda,\tau) = 0$ has roots with negative real parts.

Therefore, the square mean values are asymptotically stable and

$$\tau_2 = \frac{1}{\omega_2} arctg \frac{(4a_1c_2 - a_2c_1 + 4c_1\omega_2)\omega_2}{4\omega_2(c_2 - a_1c_1) - a_2c_2}.$$

For $\tau = \tau_2$ the system (36) has a limit cycle.

The proof is similar as the one in the case of Proposition 1.

6 Numerical simulations

For the numerical simulation, the following values were taken into consideration: $\alpha = 0.8$, $\beta = 0.2$, $\delta = 0.3$, $r = 2$, $u = 3$, and the function $f(x) = \frac{0.3}{(1+e^{-4x})} - 0.5$. The value of τ_0 given by (16) is $\tau_0 = 0.77$.

Therefore, for $\tau \in (0, \tau_0)$ the differential system (5) is asymptotically stable. At the same time, the mean values of system (29) are asymptotically stable.

For $\alpha = 0.8$, $\beta = 0.02$, $\delta = 0.3$, $r = 2$, $u = 3$, $c_1 = 5$, $c_2 = 2$ and the function $f(x) = \frac{0.3}{1+e^{-4x}} - 0.5$, the equation $g(\lambda, 0, 0.2870691525) = 0$, has the roots $\lambda_1 = 3.932662454$, $\lambda_2 = -2.024189682$ The equation $g(\lambda, 0, 0.5249160926) = 0$, has the roots $\lambda_1 = -0.7741029691$, $\lambda_2 = -1.8764029491$.

For $a \in [0.5, 1]$ figures Fig.1, Fig.2, Fig.3, represents the orbits of $(t, y_1(t,a))$, $(t, y_2(t,a))$, $(y_1(t,a), y_2(t,a))$ of systems (21).

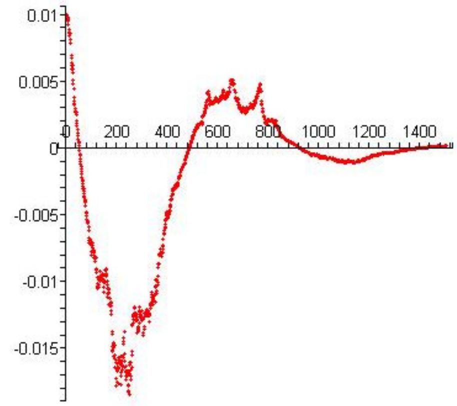

Fig.1 The orbit $(t, y_1(t,a))$

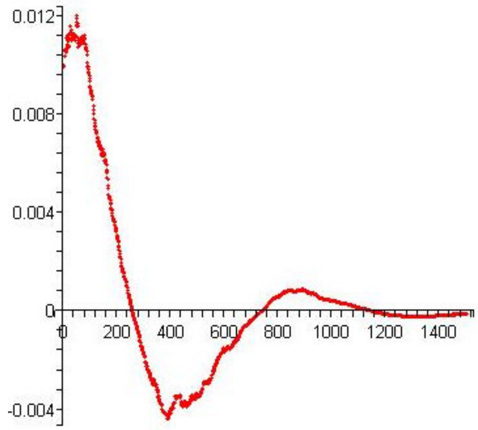

Fig.2 The orbit $(t, y_2(t,a))$

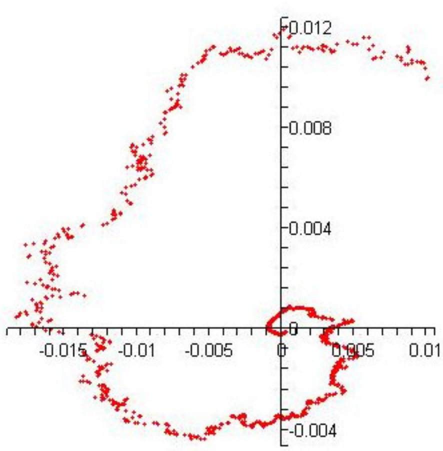

Fig.3 The orbit $(y_1(t,a), y_2(t,a))$

If $\sigma_1 = \sigma_2 = 0.283239697$ and $\tau \in (0, \tau_2)$ where $\tau_2 = 0.7585959136$ the system of the square mean's values is asymptotically stable.

Because $\tau_2 < \tau_1$ for $\tau \in (0, \tau_2)$, the variances given by:

$$D(y_1(t)) = E(y_1(t))^2 - E(y_1(t)^2)$$

$$D(y_2(t)) = E(y_2(t))^2 - E(y_2(t)^2)$$

are asymptotically stable.

Figures Fig.4, Fig.5 represent the orbits of the mean values $(t, E(y_1(t))$ and $(t, E(y_2(t))$:

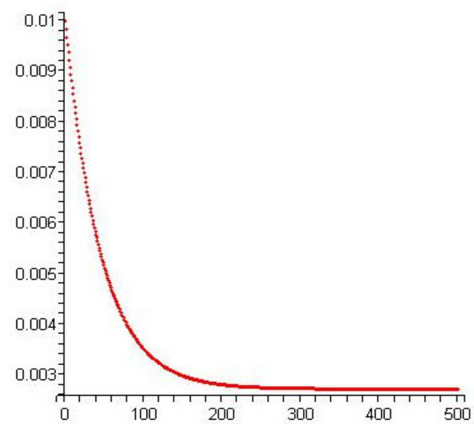

Fig.4 The orbit $(t, E(y_1(t))$

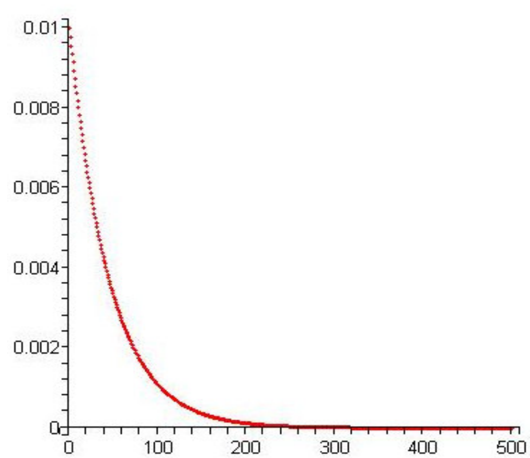

Fig.5 The orbit $(t, E(y_2(t))$

Figures Fig.6, Fig.7 represent the orbits of the square mean's values $(t, R(y_1(t)^2)$ and $(t, R(y_2(t)^2)$:

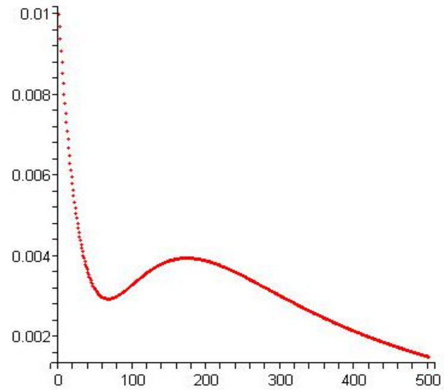

Fig.6 The orbit $(t, R(y_1(t)^2)$

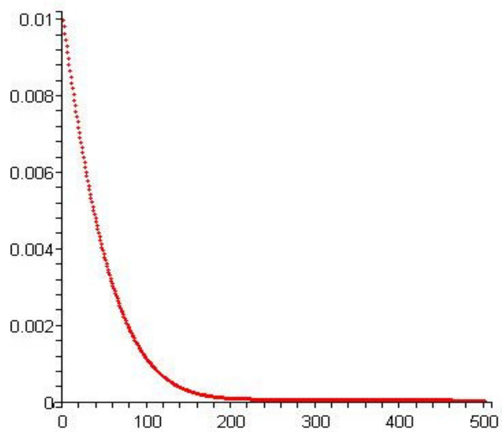

Fig.7 The orbit $(t, R(y_2(t)^2)$

Figures Fig.8, Fig.9 represent the orbits of the dispersions' $(t, D(y_1(t))$ and $(t, D(y_2(t))$:

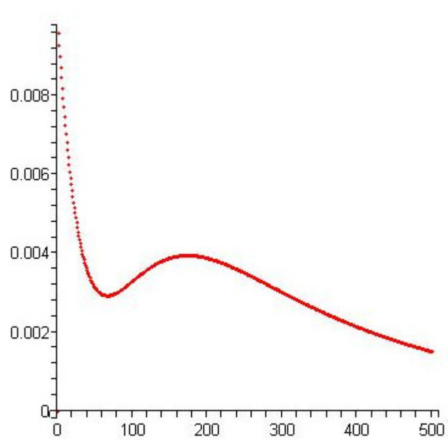

Fig.8 The orbit $(t, D(y_1(t))$

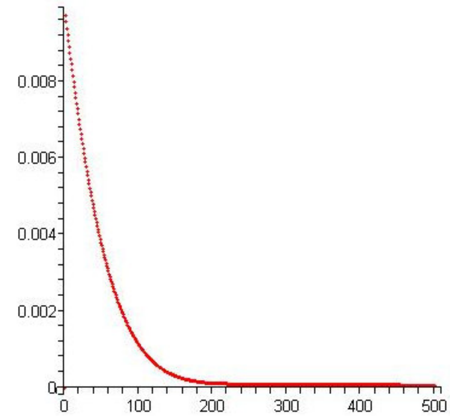

Fig.9 The orbit $(t, D(y_2(t))$

Figures Fig.10, Fig.11 represent the orbits of the mean values $(t, y_1(t,\omega))$ and $(t, y_2(t,\omega))$:

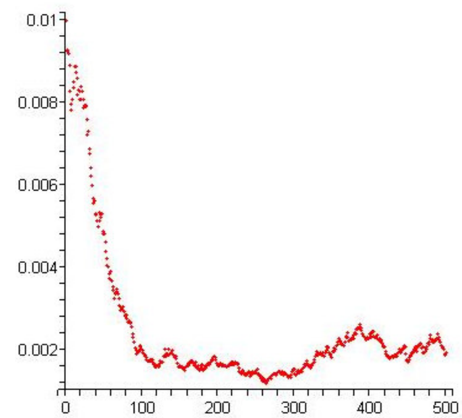

Fig.10 The orbit $(t, y_1(t,\omega))$

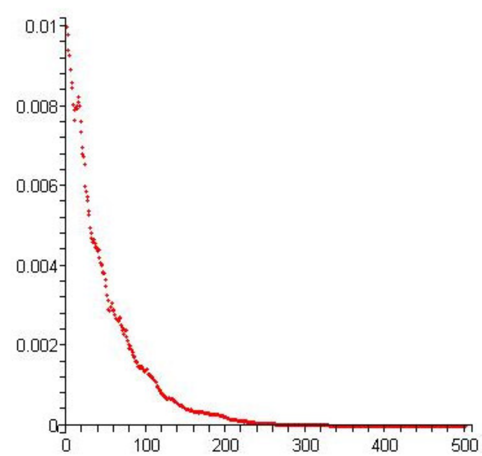

Fig.11 The orbit $(t, y_2(t,\omega))$

A similar analysis can be carried out for the following functions:

$f_1(x) = 0.2\tan(x)$, $f_2(x) = 0.2\arctan h(x)$,

$f_3(x) = 0.2\sin(x)$, $f_4(x) = \dfrac{e^x}{1+e^x}$.

7 Conclusion

The analysis of a Kaldor-Kalecki business cycle model in this paper allowed us to obtain some new dynamic scenarios which may be interesting both for the applied dynamicist and the economist.

The paper has analyzed the Kaldor-Kalecki model and the steady state of model with uncertainty and stochastic perturbation. For the uncertainty model we have analyzed a -path solutions associated to the model. For the stochastic model, we have analyzed the square mean and the depression of the model's variables.

We have determined the values of the delay for which Kaldor-Kalecki system is asymptotically stable and for which the system displays a limit cycle. We have determined the values of τ for which the square mean's values and the variances are stable.

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