Deterministic and stochastic Cournot duopoly games with tax evasion.

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Abstract: *In this paper the static model of the Cournot duopoly with tax evasion is analyzed. A study for the local stability of the stationary states is carried out in the case of the dynamic model with tax evasion and delay. Also, the stochastic approach is taken into consideration. Numerical simulations are performed for the above mentioned processes. Finally, conclusions regarding the economic processes are provided.*

Key–Words: Cournot competition, rent seeking games, stochastic differential equations, product differentiation, time delay, Hopf bifurcation, tax evasion

1 Introduction

During the last decades revenues from indirect tax have become increasingly important in many economies. Substantial attention has been devoted to evasion of indirect taxes. It is well known that indirect tax evasion, especially evasion of VAT, may erode a substantial part of tax revenues [3], [5], [9].

In [4] a model with tax evasion is presented. The authors consider n firms which enter the market with a homogenous good. These firms have to pay an ad valorem sales tax, but may evade a certain amount of their tax duty. The aims of the firms are to maximize their profits. The equilibrium point is determined and an economic analysis is made.

Based on [2], [4], [14]-[17], in our paper we analyze three economic models with tax evasion: the static model of Cournot duopoly with tax evasion in Section 2, the rent seeking game with tax evasion and time delay in Section 3 and the stochastic model with tax evasion and time delay in Section 4.

In Section 2, in the static model the purpose of the firms is to maximize their profits. We determine the firms' outputs and the declared revenues which maximize the profits, as well as the conditions for the model's parameters in which the maxim profits are obtained.

In Section 3, we formulate a new dynamic model, based on the model from Section 2, in which the time delay is introduced. That means, the two firms do not enter the market at the same time. One of them is the leader firm and the other is the follower firm. The second one knows the leader's output in the previous moment $t-\tau$, $\tau \geq 0$.

Using classical methods [6], [10], we investigate the local stability of the stationary state by analyzing the corresponding transcendental characteristic equation of the linearized system. By choosing the delay as a bifurcation parameter we show that this model exhibits a limit cycle.

The stochastic model is presented in Section 4. In order to analyze the locally asymptotic stability of the solution, the first and second moments are discussed.

Finally numerical simulations, some conclusions and future research possibilities are offered.

2 The static model of Cournot duopoly with tax evasion

The static model of Cournot duopoly is described by an economic game, where two firms enter the market with a homogenous consumption product. The firms do not cooperate, that means there is no collusion. Each firm's output decision affects the good's price. Moreover, firms compete in quantities and choose the quantities simultaneously.

The elements which describe the model are: the quantities which enter the market from the two firms $x_i \geq 0$, $i=\overline{1,2}$; the declared revenues z_i, $i=\overline{1,2}$; the inverse demand function $p:\mathbb{R}_+ \to \mathbb{R}_+$ (p is a derivable function with $p'(x)<0$, $\lim\limits_{x\to a_1} p(x)=0$, $\lim\limits_{x\to 0} p(x)=b_1$, $\left(a_1 \in \overline{\mathbb{R}}, b_1 \in \overline{\mathbb{R}}\right)$; the penalty function $F:\mathbb{R}_+\rightarrow\mathbb{R}_+$ (F is a derivable function with $F'(x)>0$, $F''(x)>0$, $F(0)=0$); the cost functions $C_i:\mathbb{R}_+\rightarrow\mathbb{R}_+$ (C_i are derivable functions with $C_i'(x_i)>0, C_i''\geq 0, i=\overline{1,2}$).

The government levies an ad valorem tax on each firm's sales at the rate $t_1\in(0,1)$. To increase their disposable income the firms evade taxes by underreporting their true income.

The true tax base of firm i is $x_ip(x_1+x_2)$. Firm i declares only $z_i\leq x_ip(x_1+x_2)$ as tax base to the tax authority. Accordingly, evaded revenues of firm i are given by $x_ip(x_1+x_2)-z_i$.

To combat tax evasion the government audits taxpayers randomly and depending on the tax evasion rate detects the evasion. The joint probability of being audited and detected is $q\in[0,1]$. With probability $1-q$ tax evasion remains undetected and the tax bill of firm i amounts to t_1z_i. In case of detection, firm i has to pay taxes on the full amount of revenues, $x_ip(x_1+x_2)$, and, in addition, a penalty $F(x_ip(x_1+x_2)-z_i)$. The penalty is increasing and convex in evaded revenues $x_ip(x_1+x_2)-z_i$. Moreover, it is assumed that $F(0)=0$, namely law-abiding firms go unpunished.

Therefore, the tax paid by firm i is then either $t_1x_ip(x_1+x_2)+F(x_ip(x_1+x_2)-z_i)$ with probability q, or t_1z_i with probability $1-q$.

The profit functions of the two firms are: $P_i:\mathbb{R}_+^2\rightarrow\mathbb{R}_+$, $i=\overline{1,2}$, given by:

$$\begin{aligned}P_i(x_1,x_2,z_1,z_2)=&(1-q)\left[x_ip(x_1+x_2)-C_i(x_i)-\right.\\&\left.-t_1z_i\right]+q[(1-t_1)x_ip(x_1+x_2)-C_i(x_i)\\&-F(x_ip(x_1+x_2)-z_i)].\end{aligned}\tag{1}$$

The first bracketed term in (1) equals the profit of firm i if evasion activities remain undetected. The second term in (1) represents the profit of firm i in case tax evasion is detected.

An essential assumption of this model is that each firm aims to maximize profit, based on the expectation that its own output and declared revenues decision will not have an effect on the decisions of its rivals. Therefore, the firm's purpose is to maximize (1) with respect to output x_i and declared revenue z_i. This represents a mathematical optimization problem which is given by:

$$\underset{\{x_i,z_i\}}{max}\,P_i,\quad i=\overline{1,2}.\tag{2}$$

From the hypothesis about the functions $p,F,C_i,i=\overline{1,2}$, we have:

Proposition 1 *The solution of problem (2) is given by the solution of the following system:*

$$\begin{aligned}&\frac{\partial P_i}{\partial x_i}=\left[1-qt_1-qF'(x_ip(x_1+x_2)-z_i)\right]\cdot\\&\left[p(x_1+x_2)+x_ip'(x_1+x_2)\right]-C_i'(x_i)=0,\\&\frac{\partial P_i}{\partial z_i}=-(1-q)t_1+qF'(x_ip(x_1+x_2)-z_i)=0,\\&\quad i=\overline{1,2}.\end{aligned}\tag{3}$$

In what follows, we will consider the penalty function as $F(x)=\frac{1}{2}st_1x^2$, $s\geq 1$ and the cost functions as $C_i(x_i)=c_ix_i$, $c_i>0$, $i=1,2$.

From (3) we can deduce:

Proposition 2 *The solution of system (3) satisfies the relations:*

$$\begin{aligned}&p(x_1+x_2)+x_ip'(x_1+x_2)=\frac{c_i}{1-t_1},i=1,2,\\&z_i=x_ip(x_1+x_2)-\frac{1-q}{qs},i=1,2.\end{aligned}\tag{4}$$

From Proposition 2 we obtain the following two propositions:

Proposition 3 *If $p(x)=a-bx$, $a>0$, $b>0$ and $(c_1,c_2)\in D=\{(c_1,c_2)\in\mathbb{R}^2,\ c_1>0,\ c_2>0,\ c_2-2c_1+a(1-t_1)>0,\ c_1-2c_2+a(1-t_1)>0\}$, then the solution of system (4) is given by :*

$$\begin{aligned}&x_1^*=\frac{c_2-2c_1+a(1-t_1)}{3b(1-t_1)},\\&x_2^*=\frac{c_1-2c_2+(1-t_1)}{3b(1-t_1)},\\&z_1^*=\frac{(a(1-t_1)-2c_1+c_2)(a(1-t_1)+c_1+c_2)}{9b(1-t_1)^2}-\\&-\frac{1-q}{qs},\\&z_2^*=\frac{(a(1-t_1)-2c_2+c_1)(a(1-t_1)+c_1+c_2)}{9b(1-t_1)^2}-\\&-\frac{1-q}{qs}.\end{aligned}\tag{5}$$

Proposition 4 *If $p(x)=\frac{1}{x}$ and $\frac{1-q}{qs+q-1}c_1\leq c_2\leq\frac{qs+q-1}{1-q}c_1$, then the solution of system (4) is given by :*

$$\begin{aligned}&x_1^*=\frac{c_2(1-t_1)}{(c_1+c_2)^2},\qquad x_2^*=\frac{c_1(1-t_1)}{(c_1+c_2)^2},\\&z_1^*=\frac{c_2}{c_1+c_2}-\frac{1-q}{qs},z_2^*=\frac{c_1}{c_1+c_2}-\frac{1-q}{qs}.\end{aligned}\tag{6}$$

3 The dynamic rent seeking game with tax evasion and time delay

The dynamic model describes the variation in time of output $x_i(t)$, $i = 1, 2$ taking into account the marginal profits $\dfrac{\partial P_i}{\partial x_i}$, $i = 1, 2$. Assume that each agent adjusts its declared revenue $z_i(t)$, $i = 1, 2$ proportionally to the marginal profits $\dfrac{\partial P_i}{\partial z_i}$, $i = 1, 2$. Then, the dynamic model is given by the following differential system of equations:

$$\begin{aligned}
&\dot{x}_1(t) = k_1\frac{\partial P_1}{\partial x_1} = k_1\{[1 - qt_1 - qF'(x_1 p(x_1 + x_2) -\\
&- z_1)] \cdot [p(x_1 + x_2) + x_1 p'(x_1 + x_2)] - C_1(x_1)\},\\
&\dot{x}_2(t) = k_2\frac{\partial P_2}{\partial x_2} = k_2\{[1 - qt_1 - qF'(x_2 p(x_1 + x_2) -\\
&- z_2)] \cdot [p(x_1 + x_2) + x_2 p'(x_1 + x_2)] - C_2(x_2)\},\\
&\dot{z}_1(t) = k_3\frac{\partial P_1}{\partial z_1} = k_3[-(1-q)t_1 + qF'(x_1 p(x_1 + x_2) -\\
&- z_1)],\\
&\dot{z}_2(t) = k_4\frac{\partial P_2}{\partial z_2} = k_4[-(1-q)t_1 + qF'(x_2 p(x_1 + x_2) -\\
&- z_2)],
\end{aligned} \tag{7}$$

with the initial conditions $x_i(0) = x_{i0}$, $z_i(0) = z_{i0}$, $i = \overline{1,2}$ and $h_i > 0$, $k_i > 0$, $i = \overline{1,2}$.

For $F(x) = \dfrac{1}{2}st_1x^2$, $s \geq 1$ and $C_i(x_i) = c_i x_i$, $c_i > 0$, $i = \overline{1,2}$ system (7) becomes:

$$\begin{aligned}
&\dot{x}_1(t) = k_1\{[1 - qt_1 - qst_1(x_1(t)p(x_1(t)+\\
&+ x_2(t)) - z_1(t))] \cdot [p(x_1(t) + x_2(t)) +\\
&+ x_1(t)p'(x_1(t) + x_2(t))] - c_1\}\\
&\dot{x}_2(t) = k_2\{[1 - qt_1 - qst_1(x_2(t)p(x_1(t)+\\
&+ x_2(t)) - z_2(t))] \cdot [p(x_1(t) +\\
&+ x_2(t)) + x_2(t)p'(x_1(t) + x_2(t))] - c_2\}\\
&\dot{z}_1(t) = k_3[-(1-q)t_1 + qst_1(x_1(t) \cdot\\
&\cdot p(x_1(t) + x_2(t)) - z_1(t))]\\
&\dot{z}_2(t) = k_4[-(1-q)t_1 + qst_1(x_2(t) \cdot\\
&\cdot p(x_1(t) + x_2(t)) - z_2(t))]\\
&x_i(0) = x_{i0},\ z_i(0) = z_{i0},\ i = 1, 2.
\end{aligned} \tag{8}$$

System (8) has the stationary state $(x_1^*, x_2^*, z_1^*, z_2^*)$ given by Proposition 3.

In what follows we analyze the rent seeking games with tax evasion and delay. For $\tau = 0$ we obtain the model from [4]. For $\tau = 0$ and $t_1 = 0$ we obtain the model from [2]. We consider the model (8) where we introduce the time delay τ. We suppose the first firm is the leader and the second firm is the follower. The follower knows the quantity of the leader firm, $x_1(t - \tau)$, which entered the market at the moment $t - \tau$, $\tau > 0$.

The differential system that describes this model is given by:

$$\begin{aligned}
&\dot{x}_1(t) = k_1\{[1 - qt_1 - qst_1(x_1(t)p(x_1(t)+\\
&+ x_2(t)) - z_1(t))] \cdot [p(x_1(t) + x_2(t)) +\\
&+ x_1(t)p'(x_1(t) + x_2(t))] - c_1\}\\
&\dot{x}_2(t) = k_2\{[1 - qt_1 - qst_1(x_2(t)p(x_1(t-\tau)+\\
&+ x_2(t)) - z_2(t))] \cdot [p(x_1(t-\tau) + x_2(t))+\\
&+ x_2(t)p'(x_1(t-\tau) + x_2(t))] - c_2\}\\
&\dot{z}_1(t) = k_3[-(1-q)t_1 + qst_1(x_1(t)p(x_1(t)+\\
&+ x_2(t)) - z_1(t))]\\
&\dot{z}_2(t) = k_4[-(1-q)t_1 + qst_1(x_2(t)p(x_1(t)+\\
&+ x_2(t)) - z_2(t))]\\
&x_1(\theta) = \phi(\theta), \theta \in [-\tau, 0], x_2(0) = x_{20},\\
&z_i(0) = z_{i0}, i = 1, 2.
\end{aligned} \tag{9}$$

For $p(x) = a - bx$ the stationary state of system (9) is given by (5). For $p(x) = \dfrac{1}{x}$ the stationary state of system (9) is given by (6).

Let $(x_1^*, x_2^*, z_1^*, z_2^*)$ be the stationary state of the system (9). The linearized system of (9) is given by:

$$\dot{y}(t) = Ay(t) + By(t - \tau), \tag{10}$$

where $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T$ and $A = (a_{ij})$, $B = (b_{ij})$, $i, j = 1, 2, 3, 4$, with:

$$\begin{aligned}
&a_{11} = k_1(-qst_1\frac{c_1^2}{(1-t_1)^2} + (1 - t_1)(2p'(x_1^* + x_2^*) +\\
&+ x_1^* p''(x_1^* + x_2^*))),\\
&a_{12} = k_1(-qst_1 x_1^* p'(x_1^* + x_2^*)\frac{c_1}{1 - t_1} + (1 - t_1)\cdot\\
&\cdot (p'(x_1^* + x_2^*) + x_1^* p''(x_1^* + x_2^*))),
\end{aligned}$$

$$a_{13}=\frac{k_1qst_1c_1}{1-t_1}, a_{14}=0,$$
$$a_{21}=0, a_{23}=0, a_{24}=\frac{qst_1c_2k_2}{1-t_1},$$
$$a_{22}=k_2(-qst_1\frac{c_2^2}{(1-t_1)^2}+(1-t_1)(2p'(x_1^*+x_2^*)+$$
$$+x_2^*p''(x_1^*+x_2^*))),$$
$$a_{31}=\frac{k_3qst_1c_1}{1-t_1}, a_{34}=0, a_{33}=-k_3qst_1,$$
$$a_{32}=k_3qst_1x_1^*p'(x_1^*+x_2^*),$$
$$a_{41}=k_4qst_1x_2^*p'(x_1^*+x_2^*), a_{42}=k_4qst_1\frac{c_2}{1-t_1},$$
$$a_{43}=0, a_{44}=-k_4qst_1,$$
$$b_{21}=k_2(-qst_1x_2^*p'(x_1^*+x_2^*)\frac{c_2}{1-t_1}+$$
$$+(1-t_1)\left(p'(x_1^*+x_2^*)+x_2^*p''(x_1^*+x_2^*)\right)),$$
$$b_{ij}=0, i=1,3,4, j=2,3,4. \tag{11}$$

The characteristic function of (10) is given by:

$$f(\lambda,\tau)=\lambda^4+n_{43}\lambda^3+n_{42}\lambda^2+n_{41}\lambda+n_{40}+$$
$$+e^{-\lambda\tau}(n_{22}\lambda^2+n_{21}\lambda+n_{20}), \tag{12}$$

where

$$\lambda^4+n_{43}\lambda^3+n_{42}\lambda^2+n_{41}\lambda+n_{40}=$$
$$=(\lambda-a_{11})(\lambda-a_{22})(\lambda-a_{33})(\lambda-a_{44})-$$
$$-a_{42}a_{24}(\lambda-a_{11})(\lambda-a_{33})-a_{12}a_{24}a_{41}(\lambda-a_{33})-$$
$$-a_{31}a_{13}(\lambda-a_{22})(\lambda-a_{44})-a_{41}a_{12}a_{24}(\lambda-a_{33})+$$
$$+a_{31}a_{13}a_{24}a_{42}-a_{41}a_{32}a_{13}a_{24}, \tag{13}$$

$$n_{22}\lambda^2+n_{21}\lambda+n_{20}=-a_{12}b_{21}(\lambda-a_{33})(\lambda-a_{44})-$$
$$-a_{13}a_{32}b_{21}(\lambda-a_{44})$$

Proposition 5 *1. For $\tau=0$, the characteristic equation (12) is given by:*

$$f(\lambda,0)=\lambda^4+m_{43}\lambda^3+m_{42}\lambda^2+m_{41}\lambda+m_{40}, \tag{14}$$

where $m_{43}=n_{43}$, $m_{42}=n_{42}+n_{22}$, $m_{41}=n_{41}+n_{21}$, $m_{40}=n_{40}+n_{20}$.

The stationary state of system (9) is asymptotically stable if and only if the following conditions:

$$D_1=m_{43}>0, D_2=m_{43}m_{42}-m_{41}>0,$$
$$D_3=m_{41}D_2-m_{43}^2m_{40}>0, D_4=m_{41}D_3>0 \tag{15}$$

hold.

The proof is obtained from the Routh-Hurwitz criterion for $f(\lambda,0)=0$.

If $\tau\neq 0$ then the roots of $f(\lambda,\tau)=0$ depend on τ. Considering τ as parameter, we determine τ_0 so that $\lambda=i\omega$ is a root of $f(\lambda,\tau)=0$. Substituting $\lambda=i\omega$ into equation (12) we obtain:
$\omega^4-in_{43}\omega^3-n_{42}\omega^2+in_{41}\omega+n_{40}+\left(-n_{22}\omega^2+in_{21}\omega+n_{20}\right)(cos\omega\tau-isin\omega\tau)=0.$

From the above equation we have:

$$\cos\omega\tau=\frac{(n_{42}\omega^2-\omega^4-n_{40})(-n_{22}\omega^2+n_{20})}{(-n_{22}\omega^2+n_{20})^2+n_{21}^2\omega^2}+$$
$$+\frac{n_{21}\omega(n_{43}\omega^3-n_{41}\omega)}{(-n_{22}\omega^2+n_{20})^2+n_{21}^2\omega^2},$$

and

$$\sin\omega\tau=\frac{(n_{42}\omega^2-\omega^4-n_{40})n_{21}\omega}{(-n_{22}\omega^2+n_{20})^2+n_{21}^2\omega^2}+$$
$$+\frac{(n_{22}\omega^2-n_{20})(n_{43}\omega^3-n_{41}\omega)}{(-n_{22}\omega^2+n_{20})^2+n_{21}^2\omega^2}.$$

Taking into account that $\sin^2\omega\tau+\cos^2\omega\tau=1$, the following equation is obtained:

$$\omega^8+r_6\omega^6+r_4\omega^4+r_2\omega^2+r_0=0 \tag{16}$$

where

$r_6=n_{43}^2-2n_{42}$, $r_4=n_{42}^2+2n_{40}-2n_{43}n_{41}-n_{22}^2$,

$r_2=n_{41}^2-2n_{42}n_{40}+2n_{22}n_{20}-n_{21}^2$, $r_0=n_{40}^2-n_{20}^2$.

If ω_0 is a positive root of (16) then there is a Hopf bifurcation and the value of τ_0 is given by:

$$\tau_0=\frac{1}{\omega_0}arctg\frac{a_1a_4\omega_0+a_2a_3}{-a_1a_3+a_2a_4\omega_0}, \tag{17}$$

where $a_1=\omega_0^4-n_{42}\omega_0^2+n_{40}$, $a_2=-n_{43}\omega_0^3+n_{41}\omega_0$, $a_3=n_{22}\omega_0^2-n_{20}$, $a_4=n_{21}\omega_0$.

We can conclude with the following theorem:

Theorem 6 *(i) If ω_0 is a positive root of (16) and $Re(\frac{d\lambda}{d\tau})_{\lambda=i\omega_0,\tau=\tau_0}\neq 0$, where τ_0 is given by (17), then a Hopf bifurcation occurs at the stationary state $(x_1^*,x_2^*,z_1^*,z_2^*)$ as τ passes through τ_0.*

(ii) If conditions (15) hold and $n_0>0$, then the stationary state is asymptotically stable for any $\tau>0$.

4 The stochastic model

Let $(\Omega, \mathcal{F}, P)$ be the given probability space, and $w(t) \in R$ be a scalar Wiener process defined on Ω having independent stationary Gaussian increments with $w(0) = 0$, $E(w(t) - w(s)) = 0$ and $E(w(t) - w(s)) = min(t, s)$. The symbol E denotes the mathematical expectation [11]. The sample trajectories of $w(t)$ are continuous, nowhere differentiable and have infinite variation on any finite time interval [8].

For the dynamical system (9), we are interested in finding the effect of the noise perturbation on the steady state. Let the stochastic disturbance model of (9) given by a system of stochastic differential equations with delay:

$$\begin{aligned}
&dx_1(t) = k_1\{[1 - qt_1 - qst_1(x_1(t)p(x_1(t) + x_2(t))-\\
&- z_1(t))] \cdot [p(x_1(t) + x_2(t)) + x_1(t)p'(x_1(t) +\\
&+ x_2(t))] - c_1\}dt - k_1\sigma_1(x_1(t) - x_1^*)dw(t)\\
&dx_2(t) = k_2\{[1 - qt_1 - qst_1(x_2(t)p(x_1(t-\tau) +\\
&+ x_2(t)) - z_2(t))] \cdot [p(x_1(t-\tau) + x_2(t)) + x_2(t) \cdot\\
&\cdot p'(x_1(t-\tau) + x_2(t))] - c_2\}dt-\\
&- k_2\sigma_2(x_2(t) - x_2^*)dw(t)\\
&dz_1(t) = k_3[-(1-q)t_1 + qst_1(x_1(t)p(x_1(t) + x_2(t))-\\
&- z_1(t))]dt - k_3\sigma_3(z_1(t) - z_1^*)dw(t)\\
&dz_2(t) = k_4[-(1-q)t_1 + qst_1(x_2(t)p(x_1(t) + x_2(t))-\\
&- z_2(t))]dt - k_4\sigma_4(z_2(t) - z_2^*)dw(t)\\
&x_1(0) = \phi(\theta), \theta \in [-\tau, 0], x_2(0) = x_{20},\ z_i(0) = z_{i0},\\
&i = 1, 2, \sigma_i > 0, i = 1, 2, 3, 4.
\end{aligned} \tag{18}$$

The solution of (18) is a stochastic process denoted by $x_1(t) = x_1(t, \omega)$, $x_2(t) = x_2(t, \omega)$, $z_1(t) = z_1(t, \omega)$, $z_2(t) = z_2(t, \omega)$, $\omega \in \Omega$. From the Chebyshev inequality the possible rang of x_1, x_2, z_1, z_2 at a time t is "almost" determined by its mean and variance at time t. So, the first and the second moments are important for investing the solution behavior.

Linearizing (18) around the stationary state (x_1^*, x_2^*, z_1^*, z_2^*), yields the linear stochastic differential delay equation:

$$dy(t) = (Ay(t) + By(t-\tau))dt - Cy(t)dw(t), \tag{19}$$

where $y(t) = (y_1(t), \quad y_2(t), \quad y_3(t), \quad y_4(t))^T$, A and B are given by (11) and $C = diag(-k_1\sigma_1, -k_2\sigma_2, -k_3\sigma_3, -k_4\sigma_4)$.

Let $y(t)$ be the fundamental solution of the system:

$$\dot{y}(t) = Ay(t) + By(t-\tau). \tag{20}$$

The solution of (19) is a stochastic precess given by:

$$y(t, \Phi) = y_{\Phi(t)} - \int_0^t Y(t-s)Cy(t-\tau, \Phi)dw(s) \tag{21}$$

where $y_\Phi(t)$ is the solution given by:

$$y_\Phi(t) = Y(t)\Phi(0) + \int_{-\tau}^0 Y(t-\tau-s)\Phi(s)ds \tag{22}$$

and $\Phi : [-\tau, 0] \to \mathbb{R}^4$ is the family of continuous functions.

The existence and uniqueness theorem for the stochastic differential delay equation has been established in [8].

We denoted $y(t, \Phi)$ by $y(t)$ and E the mathematical expectation. From (19) we obtain:

Proposition 7 *The moments of the solution of (19) are given by:*

$$\frac{dE(y(t))}{dt} = AE(y(t)) + BE(y(t-\tau)). \tag{23}$$

To examine the stability of the second moments of $y(t)$ for linear stochastic differential delay equation (19) we use It$\hat{o}$ rule to given the stochastic differential of $y(t)y^T(t)$.

$$\begin{aligned}
&\frac{d}{dt}E(y(t)y^T(t)) = E(dy(t)y^T(t) + y(t)dy^T(t)) =\\
&= E(Ay(t)y^T(t) + y(t)y^T(t)A^T +\\
&+ By(t-\tau)y^T(t) + y(t)y^T(t-\tau)B^T + Cy(t)y^T(t)C).
\end{aligned} \tag{24}$$

Let $R(t, s) = E(y(t)y^T(s))$ be the covariance matrix of the process $y(t)$ so that $R(t, t)$ satisfies:

$$\begin{aligned}
&\frac{dR(t,t)}{dt} = AR(t,t) + R(t,t)A^T + BR(t-\tau, t) +\\
&+ R(t, t-\tau)B^T + CR(t,t)C.
\end{aligned} \tag{25}$$

If $R(t, s) = (R_{ij})_{i,j=\overline{1,4}}$, from (25) we get:

Proposition 8 *The differential system (25) is given by:*

$$
\begin{aligned}
&\frac{R_{11}(t,t)}{dt} = (2a_{11} + k_1^2\sigma_1^2)R_{11}(t,t) + 2a_{12}R_{12}(t,t) + \\
&+ 2a_{13}R_{13}(t,t),\\
&\frac{R_{22}(t,t)}{dt} = (2a_{22} + k_2^2\sigma_2^2)R_{22}(t,t) + 2a_{24}R_{24}(t,t) + \\
&+ 2b_{21}R_{12}(t-\tau,t),\\
&\frac{R_{33}(t,t)}{dt} = (2a_{33} + k_3^2\sigma_3^2)R_{33}(t,t) + 2a_{31}R_{13}(t,t) + \\
&+ 2a_{32}R_{23}(t,t),\\
&\frac{R_{44}(t,t)}{dt} = (2a_{44} + k_4^2\sigma_4^2)R_{44}(t,t) + 2a_{41}R_{14}(t,t) + \\
&+ 2a_{42}R_{24}(t,t),\\
&\frac{R_{12}(t,t)}{dt} = (a_{11} + a_{22} + k_1k_2\sigma_1\sigma_2)R_{12}(t,t) + \\
&+ a_{12}R_{22}(t,t) + b_{21}R_{11}(t,t-\tau) + a_{24}R_{14}(t,t) + \\
&+ a_{13}R_{23}(t,t),\\
&\frac{R_{13}(t,t)}{dt} = (a_{11} + a_{33} + k_1k_3\sigma_1\sigma_3)R_{13}(t,t) + \\
&+ a_{31}R_{11}(t,t) + a_{13}R_{33}(t,t) + a_{32}R_{12}(t,t) + \\
&+ a_{12}R_{23}(t,t),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&\frac{R_{14}(t,t)}{dt} = (a_{11} + a_{44} + k_1k_4\sigma_1\sigma_4)R_{14}(t,t) + \\
&+ a_{41}R_{11}(t,t) + a_{42}R_{12}(t,t) + a_{12}R_{24}(t,t) + \\
&+ a_{13}R_{34}(t,t),\\
&\frac{R_{23}(t,t)}{dt} = (a_{22} + a_{33} + k_2k_3\sigma_2\sigma_3)R_{23}(t,t) + \\
&+ a_{32}R_{22}(t,t) + b_{21}R_{13}(t-\tau,t) + a_{24}R_{34}(t,t) + \\
&+ a_{31}R_{12}(t,t),\\
&\frac{R_{24}(t,t)}{dt} = (a_{22} + a_{44} + k_2k_4\sigma_2\sigma_4)R_{24}(t,t) + \\
&+ a_{42}R_{22}(t,t) + b_{21}R_{14}(t-\tau,t) + a_{24}R_{44}(t,t) + \\
&+ a_{41}R_{12}(t,t),\\
&\frac{R_{34}(t,t)}{dt} = (a_{33} + a_{44} + k_3k_4\sigma_3\sigma_4)R_{34}(t,t) + \\
&+ a_{41}R_{13}(t,t) + a_{31}R_{14}(t,t) + a_{32}R_{24}(t,t) + \\
&+ a_{42}R_{23}(t,t).
\end{aligned}
$$

The proof is based on the fact that $R_{ij}(t,s) = R_{ji}(s,t)$, $i,j = 1,2,3,4$.

Consider the following matrices:

$$
A_{11}(\lambda) = \\
= \begin{pmatrix}
p_{11}(\lambda) & 0 & 0 & 0 & -2a_{12}\\
0 & p_{22}(\lambda) & 0 & 0 & -2b_{21}e^{-\lambda\tau}\\
0 & 0 & p_{33}(\lambda) & 0 & 0\\
0 & 0 & 0 & p_{44}(\lambda) & 0\\
-b_{21}e^{-\lambda\tau} & -a_{12} & 0 & 0 & p_{12}(\lambda)
\end{pmatrix},
$$

$$
A_{12} = \begin{pmatrix}
-2a_{13} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2a_{24} & 0\\
-2a_{31} & 0 & -2a_{32} & 0 & 0\\
0 & -2a_{11} & 0 & -2a_{42} & 0\\
0 & -2a_{24} & -2a_{13} & 0 & 0
\end{pmatrix},
$$

$$
A_{21} = \begin{pmatrix}
-a_{31} & 0 & -a_{13} & 0 & -a_{32}\\
-a_{41} & 0 & 0 & 0 & -a_{42}\\
0 & -a_{32} & 0 & 0 & -a_{31}\\
0 & -a_{42} & 0 & -a_{24} & -a_{41}\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{27}
$$

$$
A_{22}(\lambda) = \\
= \begin{pmatrix}
p_{13}(\lambda) & 0 & -a_{12} & 0 & 0\\
0 & p_{14}(\lambda) & 0 & -a_{12} & -a_{13}\\
-b_{21}e^{-\lambda\tau} & 0 & p_{23}(\lambda) & 0 & -a_{24}\\
0 & -b_{21}e^{-\lambda\tau} & 0 & p_{24}(\lambda) & 0\\
-a_{41} & -a_{31} & -a_{42} & -a_{32} & p_{34}(\lambda)
\end{pmatrix},
$$

where

$$
p_{ij}(\lambda) = 2\lambda - a_{ii} - a_{jj} - k_ik_j\sigma_i\sigma_j, \tag{28}
$$

$i,j = 1,2,3,4$.

Proposition 9 *The characteristic function of (26) is given by:*

$$
h(\lambda,\tau) = detA_{11}(\lambda)detA_{22}(\lambda) \tag{29}
$$

Proof. Consider $R_{ij}(t,s) = e^{\lambda(t+s)}K_{ij}$, $i,j = 1,2,3,4$ with K_{ij} constants. Replacing in (26) we obtain a new system. From the condition that it admits the nontrivial solution we get $h(\lambda,\tau) = 0$, where $h(\lambda,\tau) = 0$ is given by:

$$
h(\lambda,\tau) = det\begin{pmatrix} A_{11}(\lambda) & A_{12}\\ A_{21} & A_{22}(\lambda)\end{pmatrix}.
$$

We obtain (29), because $detA_{12} = 0$.

Proposition 10 *If $k_3 = k_4$, $\sigma_3 = \sigma_4$, $k_1\sigma_1 = k_2\sigma_2$, then:*

$$
\begin{aligned}
&detA_{11}(\lambda) = p_{33}(\lambda)p_{44}(\lambda)p_{12}(\lambda)(p_{11}(\lambda)p_{22}(\lambda) - \\
&- 4a_{12}b_{21}e^{-\lambda\tau}),
\end{aligned}
\tag{30}
$$

$$detA_{22}(\lambda) = (p_{13}(\lambda)p_{23}(\lambda) - a_{12}b_{21}e^{-\lambda\tau})$$
$$(p_{13}(\lambda)p_{23}(\lambda)p_{34}(\lambda) - a_{24}a_{12}p_{13}(\lambda) - a_{12}a_{24}a_{41} -$$
$$-\frac{c_1^2}{c_2^2}a_{24}a_{42} - (\frac{x_{10}^*c_1}{x_{20}^*c_2}a_{41}a_{24}b_{21} + a_{12}b_{21}p_{34}(\lambda))e^{-\lambda\tau}). \quad (31)$$

Proof. From (11) and (28) we get:

$$p_{14}(\lambda) = p_{13}(\lambda), p_{24}(\lambda) = p_{23}(\lambda),$$
$$a_{33} = a_{44}, a_{31} = \frac{c_1}{c_2}a_{42}, a_{32} = \frac{x_{10}^*}{x_{20}^*}a_{41},$$
$$a_{13} = \frac{c_1}{c_2}a_{24}.$$

Replacing the above relation in $detA_{11}(\lambda)$ and $detA_{22}(\lambda)$, then we obtain (30) and (31).

The analysis of the second moments are done studying the roots of the characteristic equation $h(\lambda, \tau) = 0$.

It is well known that the trivial fixed point of (26) is locally asymptotically stable if all the roots λ of the characteristic equation $h(\lambda, \tau) = 0$ satisfy $Re(\lambda) < 0$. In this paper we analyze the roots of the characteristic equation for $\tau = 0$, using the numerical simulation.

For $\tau = 0$ and for given values of the parameters we notice that the solutions of equation (26) is locally asymptotically stable. Therefore, the mean values, the mean square values and the variance of the model's variables are locally asymptotically stable.

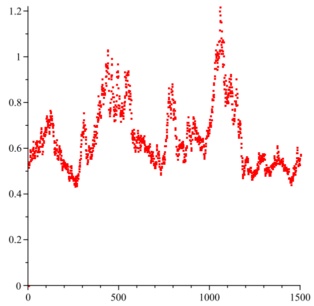
5 Numerical simulations

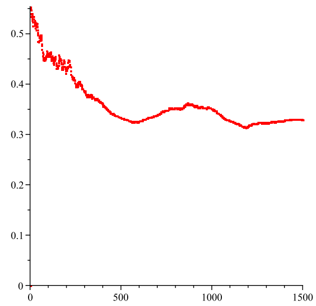
The numerical simulation was made using a program in Maple 12.

In our numerical simulations we consider the following parameters: $c_1 = 0.2$, $c_2 = 2$, $k_1 = 0.1$, $k_2 = 0.2$, $k_3 = 0.1$, $k_4 = 0.1$, $t_1 = 0.19$, $q = 0.8$, $s = 40$, $\sigma_1 = 2$, $\sigma_2 = 1$, $\sigma_3 = 0.8$, $\sigma_4 = 0.8$.

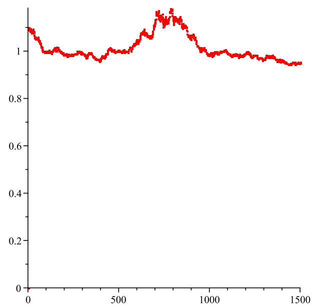
If the price function is $p(x) = \frac{1}{x}$, we obtained the steady state: $x_{10}^* = 0.334710$, $x_{20}^* = 0.334710$, $z_{10}^* = 0.90284090$, $z_{20}^* = 0.08446590$. The real parts of the roots for the equation $h(\lambda, 0) = 0$ are negative. The mean values $E(y_i(t))$, $i = 1, 2, 3, 4$, the mean square values $E(y_i(t)^2)$, $i = 1, 2, 3, 4$ and the variances $D(y_i(t)) = E(y_i(t)^2) - (E(y_i(t)))^2$, $i = 1, 2, 3, 4$ are asymptotically stable. In what follows we consider $\tau = 0$.

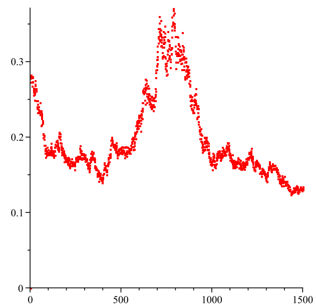
The orbits $(n, x_1(n, \omega))$, $(n, x_2(n, \omega))$ are presented in Fig1 and in Fig2 respectively:



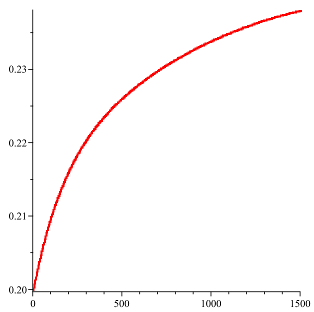


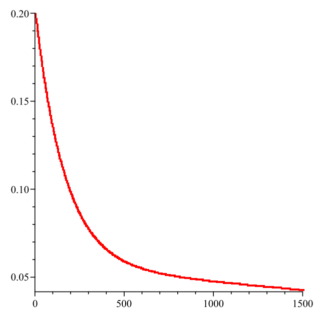
The orbits $(n, z_1(n, \omega))$, $(n, z_2(n, \omega))$ are presented in Fig3 and in Fig4 respectively:



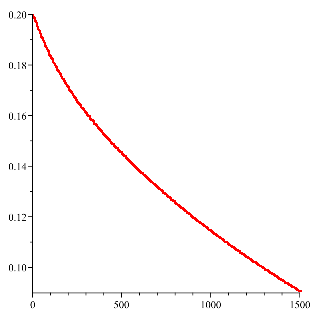


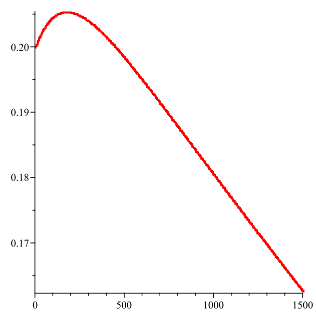
In figures Fig5 and Fig6 the orbits $(n, E(y_1(n, \omega)))$, $(n, E(y_2(n, \omega)))$ are displayed:



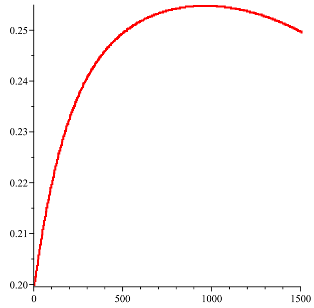


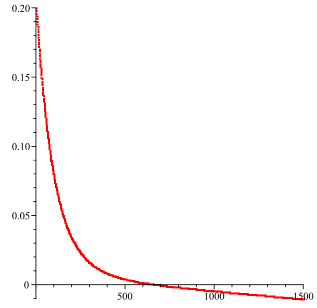
In figures Fig7 and Fig8 the orbits $(n, E(y_3(n, \omega)))$, $(n, E(y_4(n, \omega)))$ are displayed:



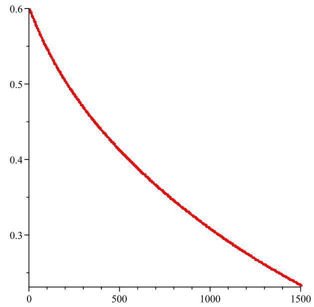


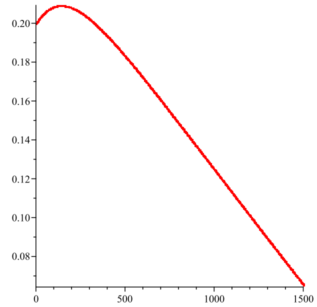
The variances $(n, E(y_1(n, \omega)^2))$, $(n, E(y_2(n, \omega)^2))$ are showed in Fig9 and Fig10:



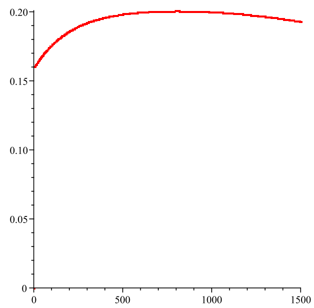


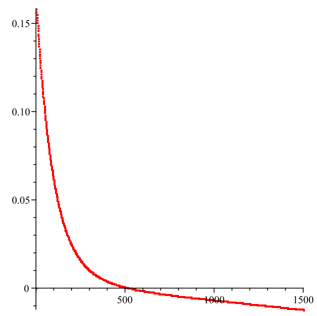
The variances $(n,E(y_3(n,\omega)^2))$, $(n,E(y_4(n,\omega)^2))$ are showed in Fig11 and Fig12:



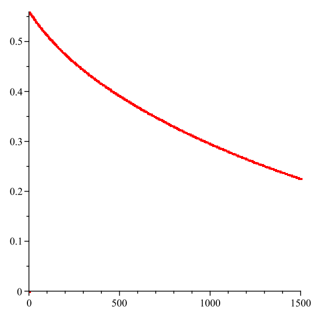


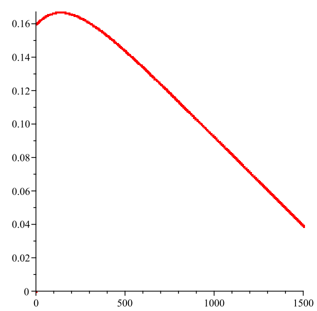
The variances $(n,D(y_1(n,\omega)))$, $(n,D(y_2(n,\omega)))$ can be visualized in Fig13 and Fig14:





The variances $(n,D(y_3(n,\omega)))$, $(n,D(y_4(n,\omega)))$ can be visualized in Fig15 and Fig16:

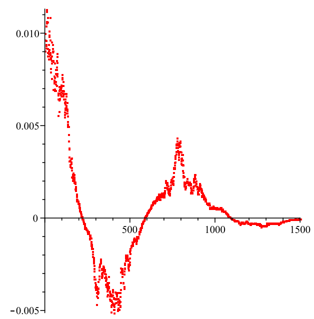


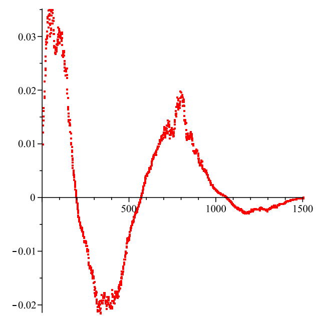


From the above simulations we can notice that the mean values and the variances of the state variables are asymptotically stable.

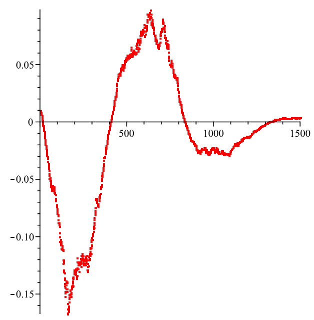
If the price function is $p(x) = 50 - 100x$, we obtained the steady state: $x^*_{10} = 0.6798353910$, $x^*_{20} = 0.6353909467$, $z^*_{10} = 18.84785151$, $z^*_{20} = 17.61525891$. The real parts of the roots for the equation $h(\lambda, 0) = 0$ are negative. The mean values $E(y_i(t))$, $i = 1, 2, 3, 4$, the mean square values $E(y_i(t)^2)$, $i = 1, 2, 3, 4$ and the variances $D(y_i(t)) = E(y_i(t)^2) - (E(y_i(t)))^2$, $i = 1, 2, 3, 4$ are asymptotically stable. In what follows we consider $\tau = 0$.

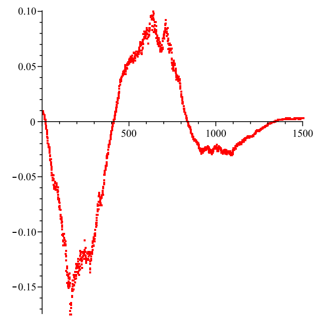
The orbits $(n, x_1(n, \omega))$, $(n, x_2(n, \omega))$ are presented in Fig17 and in Fig18 respectively:



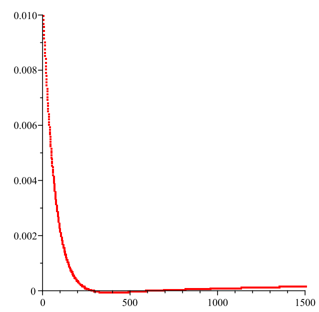


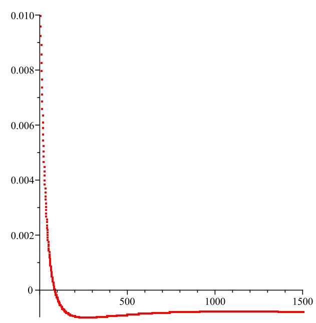
The orbits $(n, z_1(n, \omega))$, $(n, z_2(n, \omega))$ are presented in Fig19 and in Fig20 respectively:



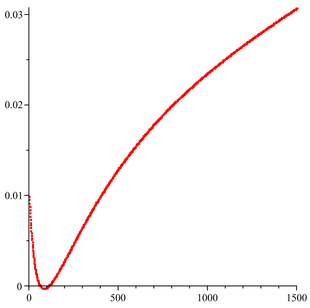


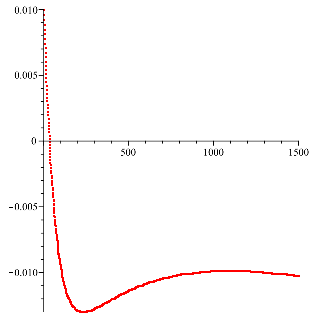
In figures Fig21 and Fig22 the orbits $(n, E(y_1(n, \omega)))$, $(n, E(y_2(n, \omega)))$ are displayed:



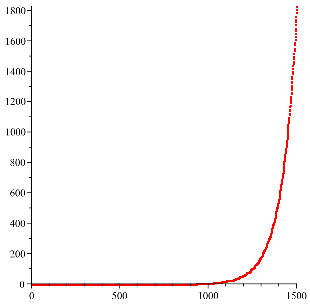


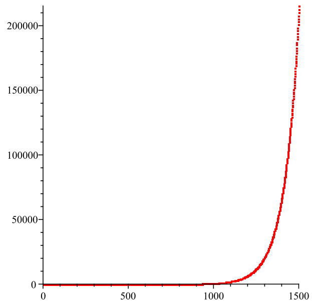
In figures Fig23 and Fig24 the orbits $(n, E(y_3(n,\omega)))$, $(n, E(y_4(n,\omega)))$ are displayed:



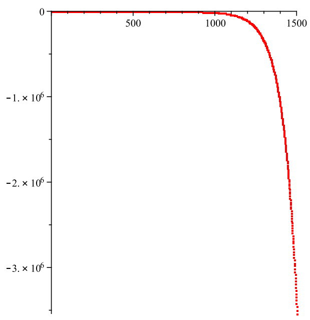


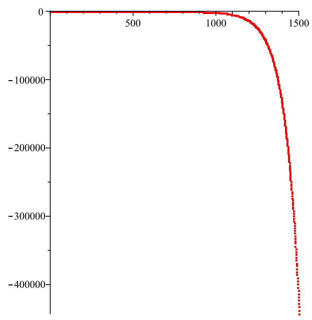
The variances $(n,E(y_1(n,\omega)^2))$, $(n,E(y_2(n,\omega)^2))$ are showed in Fig25 and Fig26:



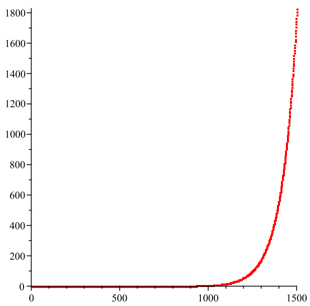


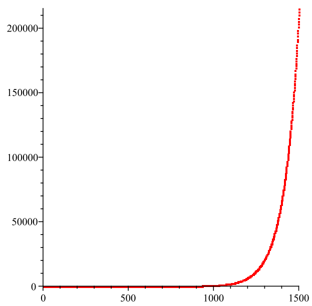
The variances $(n,E(y_3(n,\omega)^2))$, $(n,E(y_4(n,\omega)^2))$ are showed in Fig27 and Fig28:



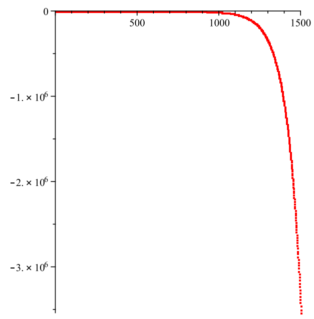


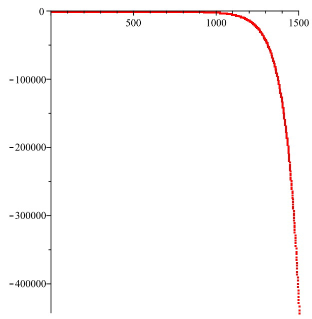
The variances $(n,D(y_1(n,\omega)))$, $(n,D(y_2(n,\omega)))$ can be visualized in Fig29 and Fig30:





The variances $(n,D(y_3(n,\omega)))$, $(n,D(y_4(n,\omega)))$ can be visualized in Fig31 and Fig32:





From the above simulations we can notice that the mean values and the variances of the state variables are asymptotically stable.

6 Conclusions

In the static model with tax evasion two firms enter the market with a homogeneous good. In order to increase their income the firms evade taxes by underreporting their true income. The government combats tax evasion by auditing the taxpayers randomly. The evasion is detected with the probability q. Also, parameter s characterizes the behavior of the firms with respect to the evasion.

For the dynamic model with tax evasion and time delay, using the delay τ as a bifurcation parameter we have shown that a Hopf bifurcation occurs when τ passes through a critical value τ_0.

The direction of the Hopf bifurcation, the stability and the period of the bifurcating periodic solutions will be analyzed in a future paper.

Also, a stochastic approach is considered. We use numerical simulations in order to observe the locally asymptotic stability of the solution.

The findings of the present paper can be extended in an oligopoly case.

Also, the models from this paper can be extended considering the fractional integral [16].

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