Improvements of preconditioned SOR iterative method for L-matrices

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Abstract: In this paper, we present some comparison theorems on preconditioned iterative method for solving L-matrices linear systems. Comparison results and numerical examples show that the rate of convergence of the preconditioned Gauss-Seidel iterative method is faster than the rate of convergence of the preconditioned SOR iterative method.

Key-Words: Preconditioner; L-matrix; SOR method; Gauss-Seidel method; Spectral radius; Iteration

1 Introduction

The solutions of many problems in scientific computing are eventually turned into the solutions of the large linear systems, that is,

$$Ax = b, \qquad (1)$$

where $A \in R^{n\times n}$ is a known nonsingular matrix, $b \in R^{n\times 1}$ is given and $x \in R^{n\times 1}$ is unknown. To solve (1) iteratively, the efficient splitting of the coefficient matrix A is usually required, one can see [11-13] for details. For any splitting, $A = M - N$ with $\det(M) \neq 0$, the basic iterative method solving (1) is

$$x^{i+1} = M^{-1}Nx^i + M^{-1}b, i = 1, 2, \cdots$$

Under the assumption that $a_{ii} \neq 0$, $i = 1, 2, \cdots, n$. Without loss of generality, we assume that

$$A = I - L - U, \qquad (2)$$

where I is an identity matrix, L and U are, respectively, strictly lower and upper triangular matrices obtained from A. By the above splitting (2) of A, the classical SOR iterative method is defined by:

$$x^{i+1} = (I - wL)^{-1}[(1-w)I + wU]x^i + (I - wL)^{-1}wb,$$

where $i = 1, 2, \cdots, n.$ Its iterative matrix is

$$L_w = (I - wL)^{-1}[(1-w)I + wU], \qquad (3)$$

where $w \neq 0$ is a parameter, called the relaxation parameter. It is known that for $w = 1$, the SOR method is reduced to the Gauss-Seidel method.

The spectral radius of the iterative matrix is decisive for the convergence of the corresponding iterative method, and the smaller it is, the faster the iterative method converges when the spectral radius is smaller than 1. In order to accelerate the convergence of iterative method for solving the linear systems (1), the preconditioned methods are often used. That is,

$$PAx = Pb,$$

where the preconditioner P is a non-singular matrix.

Let

$$PA = D^* - L^* - U^*.$$

Based on the SOR iterative method, it is easy to get the corresponding preconditioned SOR iterative methods, whose iterative matrices are

$$L_w^* = (D^* - wL^*)^{-1}[(1-w)D^* + wU^*], w \neq 0.$$

To improve the convergence rate of the iterative method, many preconditioners have been proposed [1-2,7-8,10,14-17]. Recently, the preconditioner $P = I + \tilde{S}$ was considered in [1-2,9] with

$$\tilde{S} = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

whose effect on A is to eliminate the elements of the first upper diagonal to improve the convergence of the iterative method where the matrix A has to L-matrix with $0 < a_{i+1,i}a_{i,i+1} < 1, i = 1, 2, \cdots, n.$

In this paper, under assumptions weaker than that [1-2, 9], we consider the preconditioned SOR-type iterative method for solving linear systems. Some comparison theorems on preconditioned iterative methods are provided. Also the optimal parameter is presented. The comparison results and numerical examples show that the rate of convergence of the preconditioned Gauss-Seidel method is faster than the rate of convergence of the preconditioned SOR iterative method with $0 < w \le 1$.

2 Preliminaries

For convenience, we shall now briefly explain some of the terminology and lemmas. Let $C = (c_{ij}) \in R^{n\times n}$ be an $n\times n$ real matrix. By $diag(C)$, it denotes the $n\times n$ diagonal matrix coinciding in its diagonal with c_{ii}. For $A = (a_{ij})$, $B = (b_{ij}) \in R^{n\times n}$, $A \ge B$ if $a_{ij} \ge b_{ij}$ holds for all $i, j = 1, 2, \cdots, n$. Calling A is nonnegative if $A \ge 0$ $a_{ij} \ge 0; i, j = 1, 2, \cdots, n$. It says that $A - B \ge 0$ if and only if $A \ge B$. These definitions carry immediately over to vectors by identifying them with $n\times 1$ matrices. $\rho(\cdot)$ denotes the spectral radius of a matrix.

Definition 1[3] A matrix A is an L-matrix if $a_{ii} \ge 0;\ i = 1, 2, \cdots, n$ and $a_{ij} \le 0$, for all $i, j = 1, 2, \cdots, n; i \ne j$.

Definition 2[4] A matrix A is irreducible if the directed graph associated to A is strongly connected.

Lemma 1[4] Let $A \in R^{n\times n}$ be a nonnegative and irreducible $n\times n$ matrix. Then

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;

(ii) for $\rho(A)$, there corresponds an eigenvector $x > 0$;

(iii) $\rho(A)$ is a simple eigenvalue of A.

Lemma 2[5] Let A be a nonnegative matrix. Then

(1) If $\alpha x \le Ax$ for some nonnegative vector x, $x \ne 0$, then $\alpha \le \rho(A)$.

(2) If $Ax \le \beta x$ for some positive vector x, then $\rho(A) \le \beta$. Moreover, if A is irreducible and if $0 \ne \alpha x \le Ax \le \beta x$ for some nonnegative vector x, then

$$\alpha \le \rho(A) \le \beta$$

and x is a positive vector.

Lemma 3[4] Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \ge 0$. If $N_2 \ge N_1 \ge 0$, then

$$0 \le \rho(M_1^{-1}N_1) \le \rho(M_2^{-1}N_2) \le 1.$$

If, moreover, $A^{-1} \ge 0$ and if $N_2 \ge N_1 \ge 0$ equality excluded, then

$$0 < \rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2) < 1.$$

3 The preconditoined SOR iterative method

Consider the preconditioned linear systems,

$$\tilde{A}x = \tilde{b}, \tag{4}$$

where $\tilde{A} = (I + \tilde{S})A$ and $\tilde{b} = (I + \tilde{S})b$ with

$$\tilde{S} = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

And the preconditioned linear systems

$$\bar{A}x = \bar{b}, \tag{5}$$

where $\bar{A} = (I + \bar{S})A$ and $\bar{b} = (I + \bar{S})b$ with

$$\bar{S} = \begin{bmatrix} 0 & -\alpha_1^{-1}a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2^{-1}a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1}^{-1}a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

We express the coefficient matrix of (4) as

$$\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U},$$

where $\tilde{D} = diag(\tilde{A})$, $\tilde{L}$ and $\tilde{U}$ are strictly lower and upper triangular matrices obtained from $\tilde{A}$, respectively. By calculation, it obtains that

$$\tilde{D} = \begin{bmatrix} 1 - a_{12}a_{21} & & & \\ & 1 - a_{23}a_{32} & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

$$\tilde{L} = \begin{bmatrix} 0 & & & & \\ -a_{12} + a_{23}a_{31} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ -a_{n-1,1} + a_{n-1,n}a_{n1} & -a_{n-1,2} + a_{n-1,n}a_{n2} & \cdots & 0 & \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & 0 \end{bmatrix},$$

$$\tilde{U} = \begin{bmatrix} 0 & 0 & -a_{13} + a_{12}a_{23} & \cdots & -a_{1n} + a_{12}a_{2n} \\ & 0 & 0 & \cdots & -a_{2n} + a_{23}a_{3n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}.$$

The coefficient matrix of (5) can be expressed as

$$\bar{A} = \bar{D} - \bar{L} - \bar{U}, \tag{6}$$

where $\bar{D} = diag(\bar{A})$, $\bar{L}$ and $\bar{U}$ are strictly lower and upper triangular matrices obtained from $\bar{A}$, respectively. By calculation, we also get that

$$\bar{D} = \begin{bmatrix} 1-\frac{a_{12}a_{21}}{\alpha_1} & & & \\ & 1-\frac{a_{23}a_{32}}{\alpha_2} & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

$$\bar{L} = \begin{bmatrix} 0 & & & & \\ -a_{12}+\alpha_2^{-1}a_{23}a_{31} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ -a_{n-1,1}+\alpha_{n-1}^{-1}a_{n-1,n}a_{n1} & -a_{n-1,2}+\alpha_{n-1}^{-1}a_{n-1,n}a_{n2} & \cdots & 0 & \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & 0 \end{bmatrix},$$

$$\bar{U} = \begin{bmatrix} 0 & -a_{12}+\alpha_1^{-1}a_{12} & -a_{13}+\alpha_1^{-1}a_{12}a_{23} & \cdots & -a_{1n}+\alpha_1^{-1}a_{12}a_{2n} \\ & 0 & -a_{23}+\alpha_2^{-1}a_{23} & \cdots & -a_{2n}+\alpha_2^{-1}a_{23}a_{3n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & -a_{n-1,n}+\alpha_{n-1}^{-1}a_{n-1,n} \\ & & & & 0 \end{bmatrix}.$$

Applying the SOR method to the preconditioned linear systems (4) and (5), respectively, we have the corresponding preconditioned SOR iterative method whose iterative matrices are

$$\tilde{L}_w = (\tilde{D} - w\tilde{L})^{-1}[(1-w)\tilde{D} + w\tilde{U}], \qquad (7)$$

and

$$\bar{L}_w = (\bar{D} - w\bar{L})^{-1}[(1-w)\bar{D} + w\bar{U}]. \qquad (8)$$

First, we need the following lemmas for our proof.

Lemma 4 Let A and $\tilde{A}$ be the coefficient matrices of the linear systems (1) and (4), respectively. If $0 < w \le 1$, A is an L-matrix such that $a_{i+1,i} \ne 0, i = 1, 2, \cdots, n-1$ and $a_{1n} \ne 0$, and there exists a nonempty set of $\alpha \in N = \{1, 2, \cdots, n-1\}$ such that

$$\begin{cases} 0 < a_{i,i+1}a_{i+1,i} < 1, i \in \alpha, \\ a_{i,i+1}a_{i+1,i} = 0, i \in N \setminus \alpha. \end{cases}$$

Then the iterative matrices L_w and $\tilde{L}_w$ associated to the SOR method applied to the linear systems (1) and (4), respectively, are nonnegative irreducible.

Proof: From that A is an L-matrix, then $L \ge 0$ is a strictly lower triangular matrix. So

$$(I - wL)^{-1} = I + wL + w^2L^2 + \cdots + w^{n-1}L^{n-1} \ge 0.$$

By (3), we have

$$\begin{aligned} L_w &= (I - wL)^{-1}[(1-w)I + wU] \\ &= [I + wL + w^2L^2 + \cdots + w^{n-1}L^{n-1}] \\ &\quad \times [(1-w)I + wU] \\ &= (1-w)I + wU + w(1-w)L + w^2LU \\ &\quad + (w^2L^2 + \cdots + w^{n-1}L^{n-1})[(1-w)I + wU] \\ &= (1-w)I + wU + w(1-w)L + T, \end{aligned}$$

where

$$T = w^2LU + (w^2L^2 + \cdots + w^{n-1}L^{n-1}) \times [(1-w)I + wU] \ge 0.$$

So L_w is nonnegative. Then, from Lemma 1 of [6], we have that L_w is irreducible.

By (7), we have

$$\begin{aligned} \tilde{L}_w &= (\tilde{D} - w\tilde{L})^{-1}[(1-w)\tilde{D} + w\tilde{U}] \\ &= (I - w\tilde{D}^{-1}\tilde{L})^{-1}[(1-w)I + w\tilde{D}^{-1}\tilde{U}] \\ &= (1-w)I + w(1-w)\tilde{D}^{-1}\tilde{L} + w\tilde{D}^{-1}\tilde{U} + \tilde{T}, \end{aligned}$$

where

$$\tilde{T} = w^2(\tilde{D}^{-1}\tilde{L})(\tilde{D}^{-1}\tilde{U}) + [w^2(\tilde{D}^{-1}\tilde{L})^2 + \cdots + w^{n-1}(\tilde{D}^{-1}\tilde{L})^{n-1}][(1-w)I + w\tilde{D}^{-1}\tilde{U}] \ge 0.$$

So we have $\tilde{T} \ge 0$ and $\tilde{L}_w \ge 0$ from $\tilde{D} \ge 0$, $\tilde{L} \ge 0$ and $\tilde{U} \ge 0$. As $\tilde{L}_w$, we have $\tilde{L}_w$ is nonnegative and irreducible too. □

Analogously, we have the following lemma.

Lemma 5 Let A and $\bar{A}$ be the coefficient matrices of the linear systems (1) and (5), respectively. If $0 < w \le 1$, $\alpha_i \ge 1$ $i = 1, 2, \cdots, n-1$, A is an L-matrix such that $a_{i+1,i} \ne 0, i = 1, 2, \cdots, n-1$ and $a_{1n} \ne 0$, and there exists a nonempty set of $\beta \in N = \{1, 2, \cdots, n-1\}$ such that

$$\begin{cases} 0 < a_{i,i+1}a_{i+1,i} < \alpha_i, i \in \beta, \\ a_{i,i+1}a_{i+1,i} = 0, i \in N \setminus \beta. \end{cases}$$

Then the iterative matrices L_w and $\bar{L}_w$ associated to the SOR method applied to the linear systems (1) and (5), respectively, are nonnegative irreducible.

We need the following equalities to prove Theorem 1, which are easily proved.

$(E1)$ $\tilde{D} - \tilde{L} = I - L - \tilde{S}L;$

$(E2)$ $\tilde{L} = \tilde{D} - I + L + \tilde{S}L;$

$(E3)$ $\tilde{U} = \tilde{S}U - \tilde{S} + U.$

Theorem 1 Let L_w and $\tilde{L}_w$ be the iterative matrices of the SOR method given (3) and (7), respectively. If $0 < w \le 1$, A is an L-matrix such that $a_{i+1,i} \ne 0, i = 1,2,\cdots,n-1$ and $a_{1n} \ne 0$, and there exists a nonempty set of $\alpha \in N = \{1,2,\cdots,n-1\}$ such that

$$\begin{cases} 0 < a_{i,i+1}a_{i+1,i} < 1, i \in \alpha, \\ a_{i,i+1}a_{i+1,i} = 0, i \in N \setminus \alpha. \end{cases}$$

Then

(1) $\rho(\tilde{L}_w) < \rho(L_w)$, if $\rho(L_w) < 1$;

(2) $\rho(\tilde{L}_w) = \rho(L_w)$, if $\rho(L_w) = 1$;

(3) $\rho(\tilde{L}_w) > \rho(L_w)$, if $\rho(L_w) > 1$.

Proof: From Lemma 4, it is clear that L_w and $\tilde{L}_w$ are nonnegative irreducible matrices. Thus, from Lemma 1 there exists a positive vector $x = [x_1, x_2, \cdots, x_n]^T$, $x_i > 0, i = 1,2,\cdots,n,$ such that

$$L_w x = \lambda x, \tag{9}$$

where $\lambda = \rho(L_w)$, or, equivalently,

$$[(1-w)I + wU]x = \lambda(I - wL)x. \tag{10}$$

Therefore, for this $x > 0$,

$$\tilde{L}_w x - \lambda x = (\tilde{D} - w\tilde{L})^{-1}[(1-w)\tilde{D} + w\tilde{U} - \lambda(\tilde{D} - w\tilde{L})]x. \tag{11}$$

Since $\lambda(\tilde{D} - w\tilde{L})x = \lambda(1-w)\tilde{D} + w\lambda(\tilde{D} - \tilde{L})x$, we get

$$\tilde{L}_w x - \lambda x = (\tilde{D} - w\tilde{L})^{-1}[(1-w)\tilde{D} + w\tilde{U} - \lambda(1-w)\tilde{D} - w\lambda(\tilde{D} - \tilde{L})]x. \tag{12}$$

Since $\tilde{U} = \tilde{S}U - \tilde{S} + U$, from (12) we obtain

$$\tilde{L}_w x - \lambda x = (\tilde{D} - w\tilde{L})^{-1}[(1-w)\tilde{D} + w(\tilde{S}U - \tilde{S} + U) - \lambda(1-w)\tilde{D} - w\lambda(\tilde{D} - \tilde{L})]x.$$

By $\tilde{L} = \tilde{D} - I + L + \tilde{S}L$, from the above equation we have

$$\tilde{L}_w x - \lambda x = (\tilde{D} - w\tilde{L})^{-1}[(1-w-\lambda+w\lambda)\tilde{D} + w(\tilde{S}U - \tilde{S} + U) - w\lambda(I - L - \tilde{S}L)]x. \tag{13}$$

By simple computations, from (13) we get

$$\begin{aligned}\tilde{L}_w x - \lambda x &= (\tilde{D} - w\tilde{L})^{-1}[(1-w)(1-\lambda)\tilde{D} - w\lambda(I - L - \tilde{S}L) + w(I + \tilde{S})U - w\tilde{S}]x \\ &= (\tilde{D} - w\tilde{L})^{-1}\{(1-w)(1-\lambda)\tilde{D} - w\lambda(I - L - \tilde{S}L) - w\tilde{S} + (I + \tilde{S})[(\lambda - 1 + w)I - w\lambda L]\}x \\ &= (\tilde{D} - w\tilde{L})^{-1}[(1-w)(1-\lambda)(\tilde{D} - I) - (1-\lambda)\tilde{S}]x \\ &= (\lambda - 1)(\tilde{D} - w\tilde{L})^{-1}[(w-1)(\tilde{D} - I) + \tilde{S}]x.\end{aligned}$$

Let

$$\tilde{E} = (w-1)(\tilde{D} - I) + \tilde{S} = \begin{bmatrix} -(w-1)a_{12}a_{21} & -a_{12} & 0 & \cdots & 0 \\ 0 & -(w-1)a_{23}a_{32} & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

We have

$$\begin{aligned}\tilde{L}_w x - \lambda x &= (\lambda - 1)(\tilde{D} - w\tilde{L})^{-1}\tilde{E}x \\ &= (\lambda - 1)(\tilde{D} - w\tilde{L})^{-1}(\mu_1, \mu_2, \cdots, \mu_{n-1}, 0)^T,\end{aligned}$$

where

$$\mu_i = -(w-1)a_{i,i+1}a_{i+1,i}x_i - a_{i,i+1}x_{i+1} \ge 0, \quad (i = 1,2,\cdots,n-1).$$

(1) If $0 < \lambda < 1$, then $\tilde{L}_w x - \lambda x \le 0$ but not equal to 0. By Lemma 2, we get $\rho(\tilde{L}_w) < \lambda = \rho(L_w)$.

(2) If $\lambda = 1$, then $\tilde{L}_w x - \lambda x = 0$. By Lemma 2, we get $\rho(\tilde{L}_w) = \lambda = \rho(L_w)$.

(3) If $\lambda > 1$, then $\tilde{L}_w x - \lambda x \ge 0$ but not equal to 0. By Lemma 2, we get $\rho(\tilde{L}_w) > \lambda = \rho(L_w)$.

Remark 1 It is easy to get that if $\alpha = N$, our preconditioner is reduced to the preconditioner in [1].

We need the following equalities to prove Theorem 2, which are easily proved.

$(E1')$ $\bar{D} - \bar{L} = I - L - \bar{S}L;$

$(E2')$ $\bar{L} = \bar{D} - I + L + \bar{S}L;$

$(E3')$ $\bar{U} = \bar{S}U - \bar{S} + U.$

Theorem 2 Let L_w and $\overline{L}_w$ be the iterative matrices of the SOR method given (3) and (8), respectively. If $0 < w \le 1$, $\alpha_i \ge 1$ $i = 1,2,\cdots,n-1$, A is an L-matrix such that $a_{i+1,i} \ne 0, i = 1,2,\cdots,n-1$ and $a_{1n} \ne 0$, and there exists a nonempty set of $\beta \in N = \{1,2,\cdots,n-1\}$ such that

$$\begin{cases} 0 < a_{i,i+1}a_{i+1,i} < \alpha_i, i \in \beta, \\ a_{i,i+1}a_{i+1,i} = 0, i \in N \setminus \beta. \end{cases}$$

Then

(1) $\rho(\overline{L}_w) < \rho(L_w)$, if $\rho(L_w) < 1$;

(2) $\rho(\overline{L}_w) = \rho(L_w)$, if $\rho(L_w) = 1$;

(3) $\rho(\overline{L}_w) > \rho(L_w)$, if $\rho(L_w) > 1$.

Proof: From Lemma 5, it is clear that L_w and $\overline{L}_w$ are nonnegative irreducible matrices. Thus, from Lemma 1 there exists a positive vector $x = [x_1, x_2, \cdots, x_n]^T$, such that

$$[(1-w)I + wU]x = \lambda(I - wL)x. \qquad (14)$$

Therefore, for this $x > 0$,

$$\overline{L}_w x - \lambda x = (\overline{D} - w\overline{L})^{-1}[(1-w)\overline{D} + w\overline{U} - \lambda(\overline{D} - w\overline{L})]x. \qquad (15)$$

Since $\lambda(\overline{D} - w\overline{L})x = \lambda(1-w)\overline{D} + w\lambda(\overline{D} - \overline{L})x$, then we get

$$\overline{L}_w x - \lambda x = (\overline{D} - w\overline{L})^{-1}[(1-w)\overline{D} + w\overline{U} - \lambda(1-w)\overline{D} - w\lambda(\overline{D} - \overline{L})]x. \qquad (16)$$

From $\overline{U} = \overline{S}U - \overline{S} + U$, from (16) we obtain

$$\overline{L}_w x - \lambda x = (\overline{D} - w\overline{L})^{-1}[(1-w)\overline{D} + w(\overline{S}U - \overline{S} + U) - \lambda(1-w)\overline{D} - w\lambda(\overline{D} - \overline{L})]x.$$

By $\overline{L} = \overline{D} - I + L + \overline{S}L$, from the above equation we have

$$\overline{L}_w x - \lambda x = (\overline{D} - w\overline{L})^{-1}[(1 - w - \lambda + w\lambda)\overline{D} + w(\overline{S}U - \overline{S} + U) - w\lambda(I - L - \overline{S}L)]x. \qquad (17)$$

By simple computations, from (17) we get

$$\overline{L}_w x - \lambda x = (\overline{D} - w\overline{L})^{-1}[(1-w)(1-\lambda)\overline{D} + w(\overline{S} + I)U - w\overline{S} - w\lambda(I - L - \overline{S}L)]x.$$

$$\begin{aligned}\overline{L}_w x - \lambda x &= (\overline{D} - w\overline{L})^{-1}\Big\{(1-w)(1-\lambda)\overline{D} - w\lambda(I - L - \overline{S}L) - w\overline{S} \\ &\quad + (I + \overline{S})[(\lambda - 1 + w)I - w\lambda L]\Big\} x \\ &= (\overline{D} - w\overline{L})^{-1}[(1-w)(1-\lambda)(\overline{D} - I) - (1-\lambda)\overline{S}]x \\ &= (1-\lambda)(\overline{D} - w\overline{L})^{-1}[(1-w)(\overline{D} - I) - \overline{S}]x.\end{aligned}$$

Let

$$\overline{E} = (1-w)(\overline{D} - I) - \overline{S} = \begin{bmatrix} (w-1)\frac{a_{12}a_{21}}{\alpha_1} & \frac{a_{12}}{\alpha_1} & 0 & \cdots & 0 \\ 0 & (w-1)\frac{a_{23}a_{32}}{\alpha_2} & \frac{a_{23}}{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \frac{a_{n-1,n}}{\alpha_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

We have

$$\begin{aligned}\overline{L}_w x - \lambda x &= (1-\lambda)(\overline{D} - w\overline{L})^{-1}\overline{E}x \\ &= (1-\lambda)(\overline{D} - w\overline{L})^{-1}(\overline{\mu}_1, \overline{\mu}_2, \cdots, \overline{\mu}_{n-1}, 0)^T,\end{aligned}$$

where

$$\overline{\mu}_i = \frac{1}{\alpha_i}[(w-1)a_{i,i+1}a_{i+1,i}x_i + a_{i,i+1}x_{i+1}] \le 0, \quad (i = 1,2,\cdots,n-1).$$

The following proof is similar to Theorem 1. Here is omitted. □

It is well known that, when $w = 1$, SOR iterative method is reduced to Gauss-Seidel iterative method. So we can easily get the following corollaries.

Corollary 1 Let L_w and $\tilde{L}_w$ be the iterative matrices of the Gauss-Seidel iterative method associated to the linear systems (1) and (4), respectively. If A is an L-matrix such that $a_{i+1,i} \ne 0, i = 1,2,\cdots,n-1$ and $a_{1n} \ne 0$, and there exists a nonempty set of $\alpha \in N = \{1,2,\cdots,n-1\}$ such that

$$\begin{cases} 0 < a_{i,i+1}a_{i+1,i} < 1, i \in \alpha, \\ a_{i,i+1}a_{i+1,i} = 0, i \in N \setminus \alpha. \end{cases}$$

Then

(1) $\rho(\tilde{L}_w) < \rho(L_w)$, if $\rho(L_w) < 1$;

(2) $\rho(\widetilde{L}_w) = \rho(L_w)$, if $\rho(L_w) = 1$;

(3) $\rho(\widetilde{L}_w) > \rho(L_w)$, if $\rho(L_w) > 1$.

Corollary 2 Let L_w and $\overline{L}_w$ be the iterative matrices of the Gauss-Seidel iterative method associated to the linear systems (1) and (5), respectively. If $\alpha_i \geq 1$ $i=1,2,\cdots,n-1$, A is an L-matrix such that $a_{1n} \neq 0$ and $a_{i+1,i} \neq 0$ $i=1,2,\cdots,n-1$, and there exists a nonempty set of $\beta \in N = \{1,2,\cdots,n-1\}$ such that

$$\begin{cases} 0 < a_{i,i+1}a_{i+1,i} < \alpha_i, i \in \beta, \\ a_{i,i+1}a_{i+1,i} = 0, i \in N \setminus \beta. \end{cases}$$

Then

(1) $\rho(\overline{L}_w) < \rho(L_w)$, if $\rho(L_w) < 1$;

(2) $\rho(\overline{L}_w) = \rho(L_w)$, if $\rho(L_w) = 1$;

(3) $\rho(\overline{L}_w) > \rho(L_w)$, if $\rho(L_w) > 1$.

Theorem 3 Let $0 < w_1 < w_2 \leq 1$ and $A^{-1} \geq 0$. Under the hypothesis of Theorem 1, then $0 < \rho(\widetilde{L}_{w2}) < \rho(\widetilde{L}_{w1}) < 1$, if $0 < \lambda < 1$.

Proof: Let

$$\widetilde{A} = \widetilde{M}_w - \widetilde{N}_w$$

where

$$\widetilde{M}_w = \frac{1}{w}\widetilde{D} - \widetilde{L}, \ \widetilde{N}_w = \frac{1-w}{w}\widetilde{D} + \widetilde{U}.$$

Since $0 < w_1 < w_2 \leq 1$, then $0 \leq \widetilde{N}_{w2} \leq \widetilde{N}_{w1}$. By Lemma 3, this completes the proof. □

Analogously, we have the following Theorem.

Theorem 4 Let $0 < w_1 < w_2 \leq 1$ and $A^{-1} \geq 0$. Under the hypothesis of Theorem 2, then $0 < \rho(\overline{L}_{w2}) < \rho(\overline{L}_{w1}) < 1$, if $0 < \lambda < 1$.

Remark 2 From the above discussing, it is easy to get that $w = 1$ is the optimal value. That is, the rate of convergence of the preconditioned Gauss-Seidel iterative method is faster than that the preconditioned SOR iterative method with $0 < w \leq 1$.

4 Numerical example

Now let us consider the following example to illustrate the results obtained.

The matrix A of the coefficient matrix of the linear system (1) is the following form:

$$A = \begin{bmatrix} 1 & q & r & s & q & \cdots \\ s & 1 & q & r & \ddots & q \\ q & s & 1 & q & \ddots & s \\ r & q & s & 1 & \ddots & r \\ s & \ddots & \ddots & \ddots & \ddots & q \\ \cdots & s & r & q & s & 1 \end{bmatrix}_{n \times n}$$

where $q = -\frac{2}{n}$, $r = 0$ and $s = -\frac{1}{n+2}$. For convenience, we set up the tested problem so that the right hand side is equal to $b = (1,1,\cdots,1)^T$. All tests are started from the zero vector, performed in Matlab 7.0. The error is chosen as $ERR = \|x^{k+1} - x^k\|$.

The stopping criterion is chosen as

$$\frac{\|x^{k+1} - x^k\|}{\|x^k\|} \leq 10^{-6}.$$

Let 'sor' denote the non-preconditioned SOR method, 'psor' denote the preconditioned SOR method of the present paper and 'pesor' denote the preconditioned SOR method in [7] with $P = I + S$, where

$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & 0 & \cdots & 0 \end{bmatrix}.$$

In Tables 1-4, we list the value of the spectral radius of iterative matrix $(\rho(\cdot))$, the iteration number (IT), the CPU time (CPU(s)), the error (ERR) with the different value of w and n when the SOR iterative method are used to solve the linear systems (1) with the preconditioner $I + \widetilde{S}$ and $I + S$, respectively.

The purpose of these experiments is just to investigate the influence of the spectral radius of iterative matrix and the convergence behavior of SOR iterative method with the preconditioner $I + \widetilde{S}$ and $I + S$, respectively.

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9583	251	0.1410	6.9581×10^{-6}
pesor	0.9582	251	0.1250	6.9129×10^{-6}
psor	0.9555	237	0.1090	6.8481×10^{-6}

Table 1. Numerical illustration of Theorem 1 with $w = 0.8$ and $n = 50$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9735	380	0.7030	9.9395×10^{-6}
pesor	0.9734	380	0.7030	9.9063×10^{-6}
psor	0.9725	368	0.6720	9.9031×10^{-6}

Table 2. Numerical illustration of Theorem 1 with $w=0.9$ and $n=100$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9882	793	3.8590	1.2150×10^{-5}
pesor	0.9882	793	3.8570	1.2150×10^{-5}
psor	0.9880	779	3.7650	1.2109×10^{-5}

Table 3. Numerical illustration of Theorem 1 with $w=0.7$ and $n=150$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9902	931	9.6250	1.4049×10^{-5}
pesor	0.9901	931	9.6250	1.4044×10^{-5}
psor	0.9900	918	9.4220	1.4035×10^{-5}

Table 4. Numerical illustration of Theorem 1 with $w=0.75$ and $n=200$

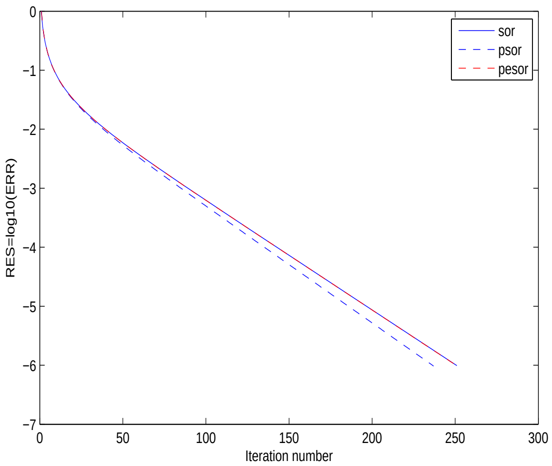

Fig.1 Iteration number with $w=0.8$ and $n=50$

Remark 3 Fig. 1 corresponds to Table 1. Tables 2-4 corresponding to figures are similar to Table 1, which are omitted here. From Tables 1-4 and Fig. 1, it is easy to get that Theorem 1 holds.

Next , we study the Gauss-Seidel iterative method to illustrate Corollary 1.

Similarity, let 'gs', 'pgs' and 'pegs' , respectively, denote the non-preconditioned Gauss-Seidel method, the preconditioned Gauss-Seidel method of the present paper and the preconditioned Gauss-Seidel method in [7]. The spectral radius of the iterative matrix $(\rho(\cdot))$, the iteration number (IT), the CPU time (CPU(s)) and the error (ERR) are listed in Tables 5-8 with the different value of w and n.

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
gs	0.9379	174	0.094	6.8179×10^{-6}
pegs	0.9379	174	0.094	6.7626×10^{-6}
pgs	0.9330	162	0.078	6.8110×10^{-6}

Table 5. Numerical illustration of Corollary 1 with $n=50$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
gs	0.9676	317	0.5780	9.9348×10^{-6}
pegs	0.9676	317	0.5620	9.8974×10^{-6}
pgs	0.9663	306	0.5470	9.8781×10^{-6}

Table 6. Numerical illustration of Corollary 1 with $n=100$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9782	455	2.2190	1.2117×10^{-5}
pesor	0.9782	455	2.2190	1.2116×10^{-5}
psor	0.9776	445	2.1410	1.2016×10^{-5}

Table 7. Numerical illustration of Corollary 1 with $n=150$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9836	590	6.0470	1.3969×10^{-5}
pesor	0.9836	590	6.0630	1.3962×10^{-5}
psor	0.9833	579	5.8910	1.3928×10^{-5}

Table 8. Numerical illustration of Corollary 1 with $n=200$

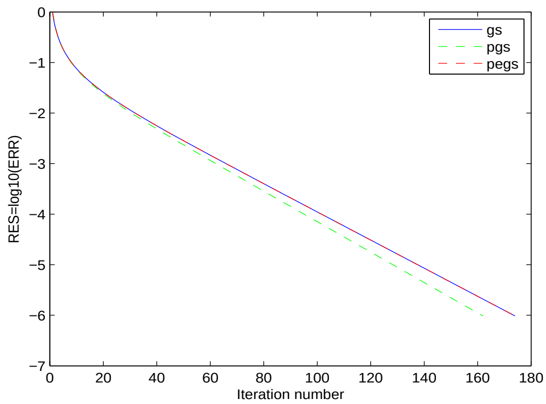

Fig.2 Iteration number with $n=50$

Remark 4 The following Fig. 2 corresponds to Table 5.

From Tables 5-8 and Fig. 2, it is not difficult to find that Corollary 1 holds.

To illustrate Remark 2 obtained, here we give the following Figs 3-4. Fig. 3 is to show that the non-preconditioned Gauss-Seidel method is faster than the non-preconditioned SOR method. Subsequently, Fig. 4 shows that the preconditioned Gauss-Seidel method is faster than the preconditioned SOR method.

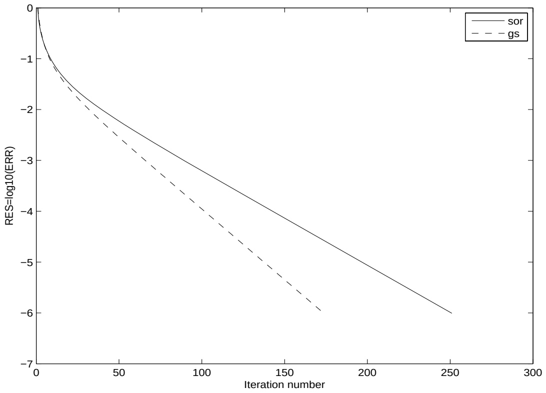

Fig.3 Unpreconditioned comparison results with $n = 50$

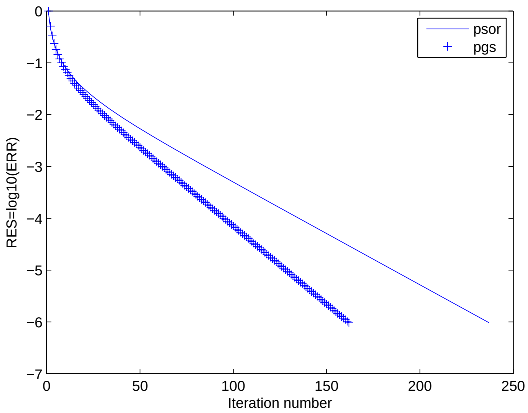

Fig.4 Preconditioned comparison results with $n = 50$

To demonstrate Theorem 2, for simplicity, here $\alpha_i = 2, (i = 1, 2, \cdots, n-1)$. As before, we set up the tested problem so that the right hand side is equal to $b=(1,1,\cdots,1)^T$. All tests are started from the zero vector. The error is chosen as $ERR = \|x^{k+1} - x^k\|$. The stopping criterion is chosen as

$$\frac{\|x^{k+1} - x^k\|}{\|x^k\|} \le 10^{-6}.$$

Some results are presented to illustrate the behavior of the convergence of the SOR method with the preconditioner $I + \overline{S}$, which are listed in Tables 9-12. The purpose of these experiments is just to investigate the influence of the spectral radius of iterative matrix and the convergence behavior of SOR iterative method with the preconditioner $I + \overline{S}$.

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9509	216	0.0781	4.3557×10^{-6}
psor	0.9477	204	0.0469	4.3642×10^{-6}

Table 9. Numerical illustration of Theorem 2 with $w = 0.5$ and $n = 20$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9661	304	0.1250	6.1536×10^{-6}
psor	0.9649	294	0.0781	6.2858×10^{-6}

Table 10. Numerical illustration of Theorem 2 with $w = 0.6$ and $n = 40$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9645	291	0.250	7.7326×10^{-6}
psor	0.9635	284	0.1875	7.7071×10^{-6}

Table 11. Numerical illustration of Theorem 2 with $w = 0.8$ and $n = 60$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9733	378	0.6094	8.8828×10^{-6}
psor	0.9728	371	0.500	8.8977×10^{-5}

Table 12. Numerical illustration of Theorem 2 with $w = 0.8$ and $n = 80$

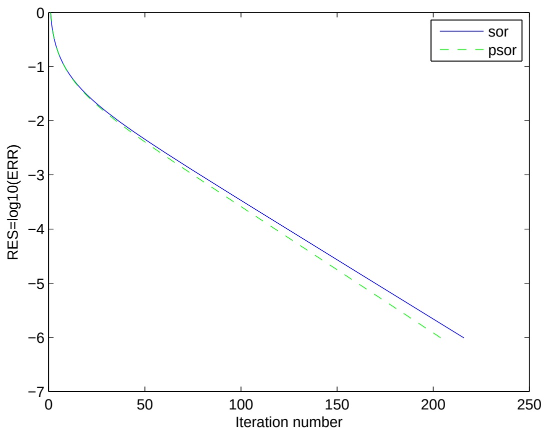

Fig.5 Iteration number with $w = 0.5$, $n = 20$ and $\alpha = 2$

Similarly, the above Fig. 5 corresponds to Table 9. From Tables 9-12 and Fig. 5, we get that Theorem 2 holds.

In the sequel, we investigate the Gauss-Seidel method with preconditioner $I+\overline{S}$. In other words, we consider the spectral radius of of the iterative matrix $(\rho(\cdot))$, the iteration number (IT), the CPU time (CPU(s)) and the error (ERR) with the different value of w and n when the Gauss-Seidel method is used to solve the linear systems (1) with preconditioner $I+\overline{S}$.

From our numerical experiments we get Tables 13-16.

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
gs	0.8595	80	0.0469	4.1360×10^{-6}
pgs	0.8465	73	0.0156	4.3739×10^{-6}

Table 13. Numerical illustration of Corollary 2 with $n=20$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
gs	0.9225	141	0.0938	6.1743×10^{-6}
pgs	0.9187	135	0.0625	6.0958×10^{-6}

Table 14. Numerical illustration of Corollary 2 with $n=40$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9471	202	0.1875	7.5164×10^{-5}
psor	0.9454	196	0.1406	7.5418×10^{-6}

Table 15. Numerical illustration of Corollary 2 with $n=60$

Iterative method	$\rho(\cdot)$	IT	CUP(s)	ERR
sor	0.9602	262	0.4129	8.8849×10^{-6}
psor	0.9592	257	0.3125	8.5927×10^{-6}

Table 16. Numerical illustration of Corollary 2 with $n=80$

To illustrate Remark 2 obtained further, here we give the above Figs 6-7. Fig. 6 illustrates that the non-preconditioned Gauss-Seidel method is faster than the non-preconditioned SOR method, too. Subsequently, Fig. 7 shows that the preconditioned Gauss-Seidel method is faster than the preconditioned SOR method as well as.

From the above numerical experiments, it is easy to get that Theorems 1-2 and Corollaris 1-2 hold. By observing a mass of experiments, we also get that Theorems 3-4 hold and our preconditioner is superior to the preconditioner in [7]. What is more,

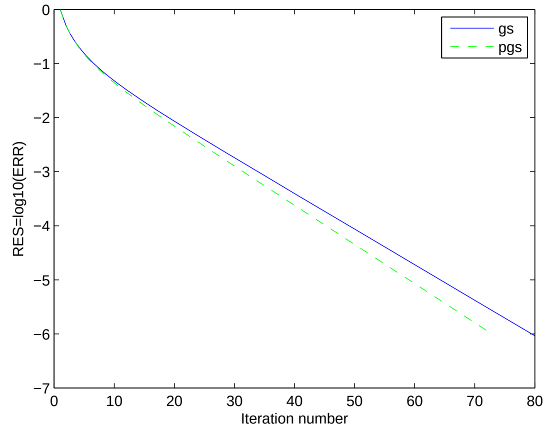

Fig.6 Iteration number with $n=20$ and $\alpha=2$

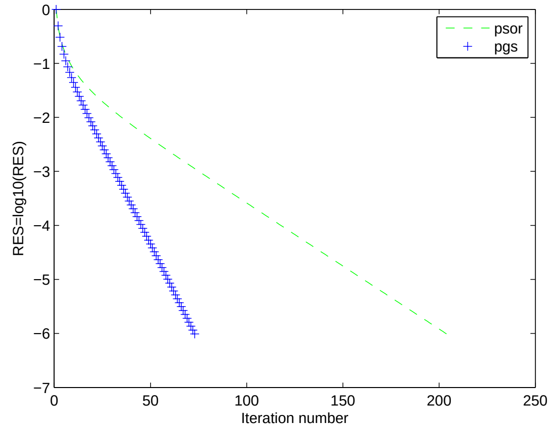

Fig.7 Preconditioned comparison results with $n=20$ and $\alpha=2$

the rate of convergence of the preconditioned Gauss-Seidel iterative method is faster than that the preconditioned SOR iterative method with $0<w\le1$.

Recently, Darvishi and Azimbeigi [17] proposed the preconditioner $P'=I+S'$ with

$$S'=\begin{bmatrix} 0 & -\alpha_1^{-1}a_{12} & 0 & 0 & \cdots & 0 \\ -\beta_1^{-1}a_{21} & 0 & -\alpha_2^{-1}a_{23} & 0 & \cdots & 0 \\ 0 & -\beta_2^{-1}a_{12} & 0 & -\alpha_3^{-1}a_{34} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -\beta_{n-2}^{-1}a_{n-1,n-2} & 0 & -\alpha_{n-1}^{-1}a_{n-1,n} \\ 0 & 0 & 0 & 0 & -\beta_{n-1}^{-1}a_{n,n-1} & 0 \end{bmatrix}.$$

To inspect the efficiency of the preconditioner P and P' for Gauss-Seidel method by the above discussion, we mainly discuss two cases:

(I) $\alpha_i=\beta_i=1$ $(i=1,2,\cdots,n)$;

(II) $\alpha_i\ne\beta_i\ne1$ $(i=1,2,\cdots,n)$.

n	20	30	40	50
$\rho(P)$	0.8321	0.8858	0.9148	0.9330
$\rho(P')$	0.8321	0.8858	0.9148	0.9330

Table 17. Spectral radius of iterative matrix with the different values of n in Case (I)

n	20	30	40	50
$\rho(P)$	0.8461	0.8925	0.9187	0.9355
$\rho(P')$	0.8461	0.8925	0.9187	0.9355

Table 18. Spectral radius of iterative matrix with the different values of n and $\alpha_i = \beta_i = 2$ for Case (II)

n	20	30	40(0.6)	50(0.8)
$\rho(P)$	0.8458	0.8924	0.9186	0.9355
$\rho(P')$	0.8458	0.8924	0.9186	0.9355

Table 19. Spectral radius of iterative matrix with the different values of n, $\beta_i = 1$ and $\alpha_i = 2$ for Case (II)

n	20	30	40	50
$\rho(P)$	0.8322	0.8858	0.9148	0.9330
$\rho(P')$	0.8322	0.8858	0.9148	0.9330

Table 20. Spectral radius of iterative matrix with the different values of n, $\beta_i = 2$ and $\alpha_i = 1$ for Case (II)

In Tables 17-20, we list the value of the spectral radius of iterative matrix $\rho(P)$ and $\rho(P')$ for Case (I) and (II).

From Tables 17-20, under certain conditions, we are interested in finding that the spectral radius $\rho(P)$ of iterative matrix is the same as the spectral radius $\rho(P')$ of iterative matrix when Gauss-Seidel method is applied to solve the linear systems (1) with *L*-matrices. In other words, the convergence rate of Gauss-Seidel method with the preconditioner P is the same as the convergence rate of Gauss-Seidel method with the preconditioner P'. Whereas, based on the structure of preconditioner and the memory requirement, the preconditioner P is less than the preconditioner P'. In this case, the preconditioner P is superior to the preconditioner P'.

4 Conclusion

In this paper, we have studied the preconditioned SOR iterative method for solving L-matrices linear systems (1). Some comparison theorems on the preconditioned SOR iterative method are presented. The optimal parameter is presented as well as. The comparison results and the numerical example show that the rate of convergence of the preconditioned Gauss-Seidel method is faster than the rate of convergence of the preconditioned SOR iterative method with $0 < w \leq 1$.

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