Locating Zeros of Polynomials Associated With Daubechies Orthogonal Wavelets

JALAL KARAM
Alfaisal University
Faculty of Science and General Studies
Riyadh
KINGDOM OF SAUDI ARABIA
email: jkaram@alfaisal.edu Website: www.alfaisal.edu/faculty/cv/jalal

Abstract: In the last decade, Daubechies orthogonal wavelets have been successfully used and proved their practicality in many signal processing paradigms. The construction of these wavelets via two channel perfect reconstruction filter bank requires the identification of necessary conditions that the coefficients of the filters and the roots of binomial polynomials associated with them should exhibit. In this paper, the low pass and high pass filters that generate a Daubechies mother wavelet are considered. From these filters, a new set of high pass and low pass filters are derived by using the "Alternating Flip" techniques. The new set of filters maintain perfect reconstruction status of an input signal, thus allowing the construction of a new mother wavelet and a new scaling function that are reflective to those of the originals. Illustration are given and new reflective wavelets are derived. Also, a subclass of polynomials is derived from this construction process by considering the ratios of consecutive binomial polynomials' coefficients. A mathematical proof of the residency of the roots of this class of polynomials inside the unit circle is presented along with an illustration for $db6$, a member of the Daubechies orthogonal wavelets family. The Kakeya-Enestrom theorem is discussed along with some of its generalizations. Finally, a λ-dependent difference among the coefficients of the new set of polynomials is examined and optimized locations of the roots are derived.

Key–Words: Zeros, Polynomials, Daubechies Orthogonal Wavelets, Reflective Wavelets, Kakeya-Enestrom Theorem.

1 Introduction

Coding data with the minimum amount of coefficients is one of the essential applications of real orthogonal wavelets [9] [10]. The normalization of these wavelets basis functions ensures that the data is covered in the time domain and in the frequency domain and that a perfect reconstruction is then achievable [2] [15]. There are many different real orthogonal wavelets. They usually take the name of their inventors like the Meyer wavelets [13], the Daubechies wavelets [2], and the Haar wavelets [13]. To construct a Daubechies baby wavelet we need to find its father (the scaling function) and its mother (the wavelet function). This construction is best described via a perfect reconstruction filter banks, [2] [15] which depends on the distribution of the zeros of some polynomials in the plane. The literature provides many theorems describing geometric locations of the roots of certain polynomials [1] [3] [4] [5] [14]. Identifying necessary conditions for the coefficients of the filters associated with the construction is vital for orthogonality and alias cancellation. The distribution of the zeros of the binomial polynomial related to the construction of these wavelets were proved to reside inside the unit circle [6].

In Section 2, the construction of Daubechies filters is detailed and a new set of high pass and low pass filters are derived while maintaining perfect reconstruction status of input signals, thus allowing the construction of a new mother wavelet and a new scaling function that are reflective of the originals. In Section 3, conditions on the filters coefficients involved in the construction to maintain perfect reconstruction filter are derived.

Namely, the coefficients of the low pass synthesis filter is derived from those of their analysis counterpart by an order flip. In Section 4, the Kakeya-Enestrom theorem is discussed along with some of its generalizations and its analytical proof. The new class of polynomials is then formulated and the distribution of their zeros is examined and mathematically shown to reside inside the unit circle. An example is then presented illustrating the location for the zeros of the derived polynomial of the orthogonal mother wavelet $db6$. Also the conditions of λ - difference among the new coefficients are optimized. Section 5 contains the conclusion.

2 Construction of Daubechies Orthogonal Wavelets

A two-channel filter bank has a low-pass and a high-pass filter in the decomposition (analysis) phase depicted in Fig. 1 and two more in the reconstruction (Synthesis) phase displayed in Fig. 2.

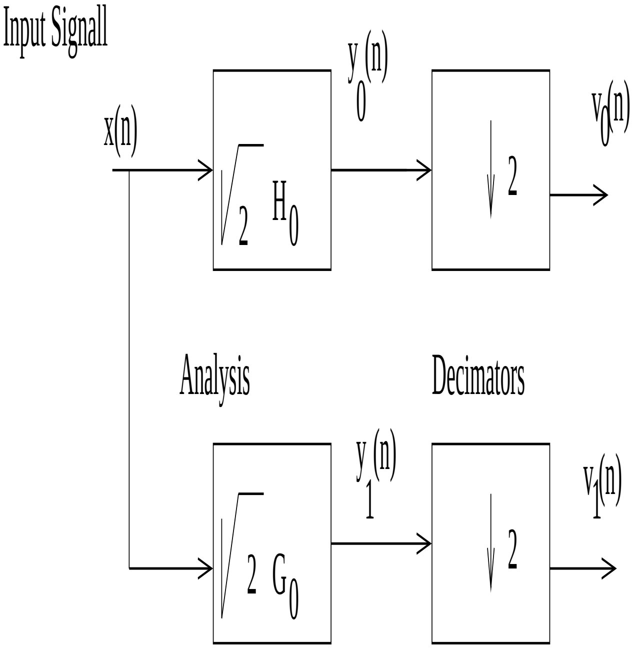

Fig. 1: Analysis phase filters of a two-channels filter bank.

Let $h_0(n)$ and $g_0(n)$ denote the low-pass filter coefficients and the high-pass filter coefficients respectively in the analysis phase. To obtain perfect reconstruction, these two filters must satisfy the following conditions [2] [15]

1. For the low-pass filter $h_0(n)$:

$$\sum_n h_0(n) = 1 \quad \sum_n h_0^2(n) = \frac{1}{2} \quad (1)$$
$$2\sum_n h_0(n)h_0(n-2k) = \delta(k)$$

2. For the high-pass filter $g_0(n)$:

$$\sum_n g_0(n) = 0 \quad \sum_n g_0^2(n) = \frac{1}{2} \quad (2)$$
$$2\sum_n g_0(n)g_0(n-2k) = \delta(k)$$

where $\delta(k)$ is the Dirac delta function defined by

$$\delta(k) = \begin{cases} 1 & \text{if k=0} \\ or & \\ 0 & \text{otherwise.} \end{cases}$$

Given the coefficients of h_0, it is shown in [15] and [16] that the coefficients of the filters $h_1(n)$, $g_0(n)$ and $g_1(n)$ that lead to orthogonality can easily be derived from the coefficients of h_0. Therefore, to construct a Daubechies orthogonal wavelet, all we need to do, is to find the coefficients of the filter h_0 associated with it.

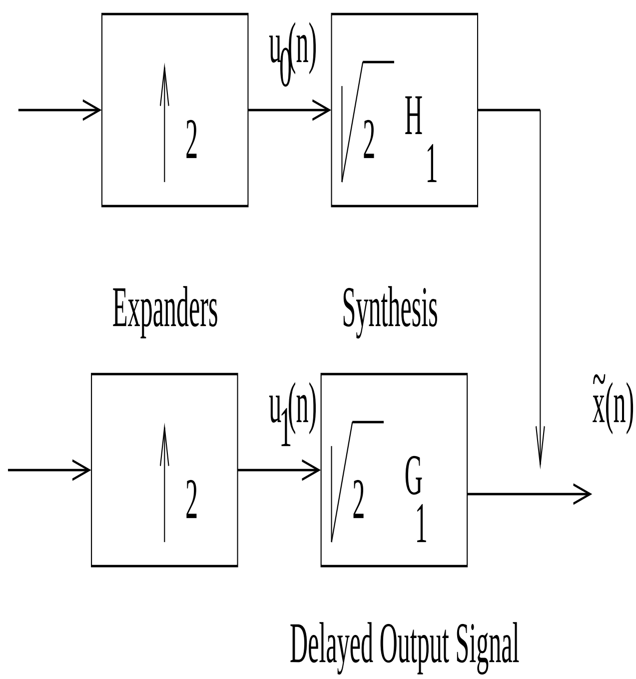

Fig. 2: Synthesis phase filters of a two-channels filter bank.

The construction of the filter bank amounts to [15]

1. Design a product low-pass filter P_0 satisfying
$$P_0(z) - P_0(-z) = 2z^{-l} \tag{3}$$
2. Factor P_0 in H_1H_0, then find G_1 and G_0.

and can be reduced even further by defining $P(z) = z^l P_0(z)$ and substituting $P(-z)$ by $-z^l P_0(-z)$. Hence, the perfect reconstruction condition becomes [18]:
$$P(z) + P(-z) = 2 \tag{4}$$
which implies that $P(z)$ is a half band filter [15] with all of its the coefficients zeros except the constant term 1. Furthermore, the odd powers cancel when we add $P(z)$ to $P(-z)$. The design of the low-pass and the high-pass filters of the synthesis and analysis filter banks of a Daubechies orthogonal wavelet, considers the following two properties [15]

1. These wavelets filters must be orthogonal;
2. And must have maximum flatness at $w = 0$ and $w = \pi$ in their frequency responses.

The low-pass filters will have p zeros at π, and have a total of $2p$ coefficients, (length of the filters). This filter bank is orthogonal and the product filters $P_0(z)$ and $P_1(z)$ have a length of $4p - 2$. The construction of Daubechies orthogonal wavelets begins by choosing the number of zeros p at π. The zeros the filters associated with the $db6$ are depicted in Fig. 4. Here, we also need to choose the binomial polynomial $B_p(y)$ associated with it which has a degree of $p - 1$. The coefficients of these polynomials can be found recursively for p by:
$$b(i) = b(i+1) * \frac{(2p - i - 1)}{4 * (p - i)} \tag{5}$$
For a given value p, the coefficients of $B_p(y)$ are in an ascending order [6]. To get the roots of $B_p(y)$, one scales b by 4 and to facilitate the numerical calculations, one uses the variable $4y$ instead of y. The ratio of any two consecutive coefficients is:
$$r_k = \frac{b_k}{b_{k+1}} = \frac{(p+k-2)!}{(p-1)! * (k-1)!} * \frac{(p-1)! * k!}{(p+k-1)!} \tag{6}$$
Which in its simplest form can be expressed as:
$$r_i = \frac{i-1}{i+p}, \quad i = 0, 1, 2, ..., p-2 \tag{7}$$

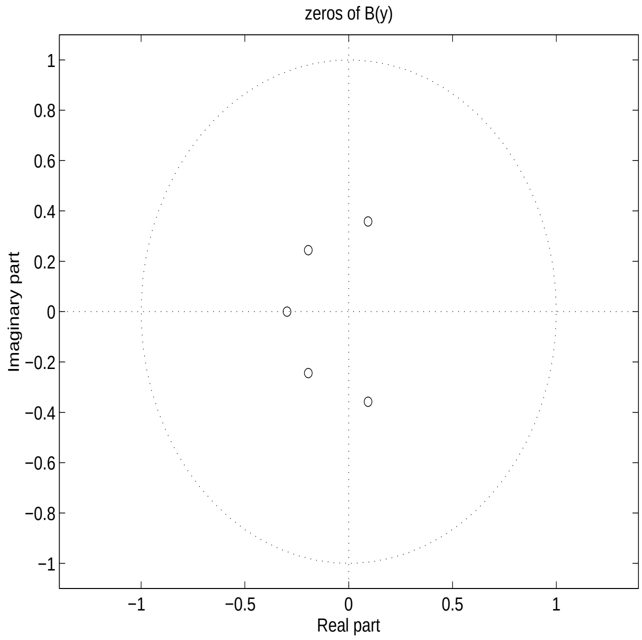

Fig. 3: Zeros of $B(y)$ for db6.

To compute the $2p - 2$ zeros of $P(z)$ other than -1, we note that according to [13] [15], the frequency response of the half-band filter $P(w)$ is given by:
$$P(w) = 2(1 - y)^p B(y)$$
where $\cos(w) = 1 - 2y$ or $y = (1 - \cos(w))/2$. On the unit circle we have:
$$(z + 1/z)/2 = \cos(w) = 1 - 2y.$$
Also, off the unit circle we use the same relation between z and y. Rearranging these terms leads to
$$(1/z)(z^2 - 2(1 - 2y)z + 1) = 0. \tag{8}$$
Now let $x = 1 - 2y$ and $u = \sqrt{(x^2 - 1)}$ then,
$$(1/z)(z^2 - 2xz + 1) = 0$$
with $z \neq 0$, this implies that
$$z = x + u$$
and
$$z = x - u$$
are the two roots of $P(z)$ for each root y of B(y). Note that $x - u = \frac{1}{x+u}$. That is, we have $p - 1$ roots and their inverses, namely
$$x = 1 - 2 * y$$

and

$$u = \sqrt{x^2 - 1}$$

and

$$z = [x + u, x - u].$$

The distribution of these zeros in the plane is shown in Fig. 4. From $P(z)$, $P_0(z)$ is then derived and all is left is to factorize $P_0(z)$. Daubechies did the following factorization found in [15]:

$$P_0(z) = (\frac{1 + z^{-1}}{2})^{2p} Q_{2p-2}(z) \tag{9}$$

where $Q_{2p-2}(z)$ is a polynomial of degree $2p-2$.

2.1 The Case of $db6$

For $p = 6$, the $db6$ wavelet is obtained. Fig. 3 shows the location in the complex plane for the zeros of $B_6(y)$ associated with the Daubechies $db6$ orthogonal wavelet.

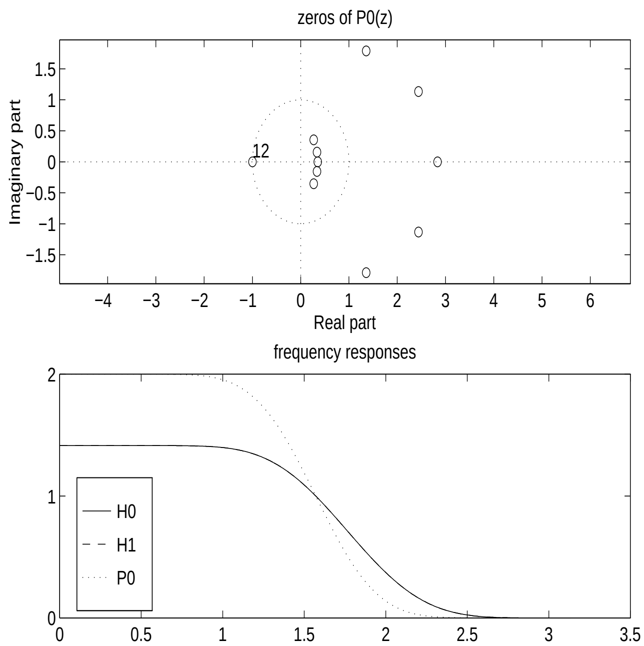

Fig. 4: Zeros of P(z) for db6 and the frequency responses of the filters P, H_0 and H_1.

The frequency responses of the analysis low-pass filter $H_0(z)$ and synthesis low-pass filter $H_1(z)$ of this wavelet are depicted in Fig. 4. Therefore completing the construction of the scaling function along with the mother wavelet. The decomposition and reconstruction functions for the mother wavelet $db6$ are plotted in Fig. 5, and Fig. 6. shows the impulse response of the four filters associated with it.

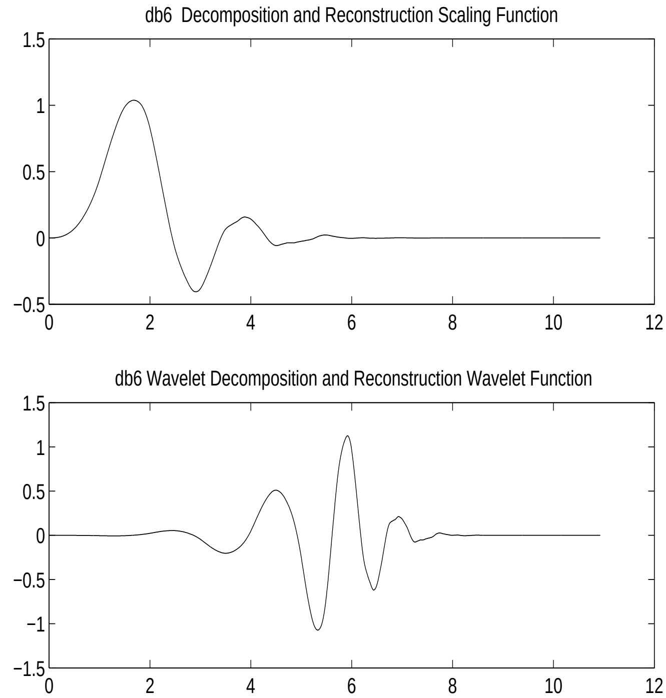

Fig. 5: The $db6$ analysis and synthesis scaling and wavelet functions.

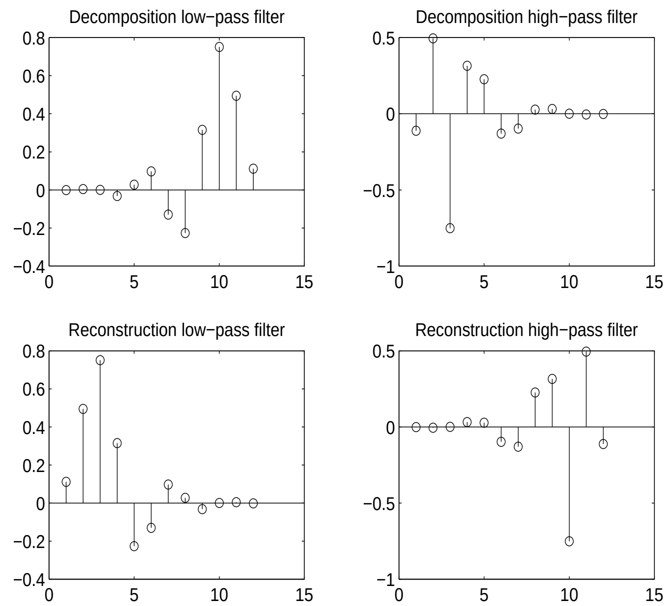

Fig. 6: Impulse response for the reconstruction and decomposition filters of db6.

Now, given the coefficients $h(0)$, $h(1)$, ..., $h(n-1)$ of the low-pass filter h_0, it is shown in [15] and [16] that the coefficients of the filters $h_1(n)$, $g_0(n)$ and $g_1(n)$ that lead to orthogonality can be derived from the coefficients of h_0 as follows

Firstly, the coefficients of the high-pass filter $g_0(n)$ of the analysis bank are obtained from those of the low-pass filter $h_0(n)$ by the " alternating flip".

This can be represented by three operations on the coefficients of $h_0(n)$.

1. Reverse the order;
2. Alternate the signs;
3. Shift by an odd number l.

This takes the low-pass filter coefficients into an orthogonal high-pass filter [15] which is represented in the following equation:

$$g_0(n) = (-1)^n h_0(l-n) \tag{10}$$

Then, the coefficients of the high-pass filter $g_1(n)$ of the synthesis bank are obtained by the reverse of the coefficients of the high-pass filter $g_0(n)$ of the analysis bank. They can be generated by the following equation

$$g_1(n) = h_0(l-n) \tag{11}$$

The coefficients of the low-pass filter $g_1(n)$ of the synthesis bank are the alternating flip of the coefficients of $g_0(n)$. They can be generated by the following equation

$$h_1(n) = (-1)^n g_1(l-n) \tag{12}$$

where l is the length of the low-pass filter $h_0(n)$ [15].
The scaling and wavelet functions are then derived from the coefficients of these filters. The scaling function satisfies the equation [15]

$$\phi(t) = 2\sum_{k=0}^{l-1} h_1(k)\phi(2t-k) \tag{13}$$

where $h_1(k)$ is the reverse of $h_0(k)$, and the wavelet function is then derived from the scaling function by the equation

$$W(t) = 2\sum_{k=0}^{l-1} g_1(k)\phi(2t-k) \tag{14}$$

3 Constructing the New Reflective Orthogonal Wavelets

In a perfect reconstruction filter bank, the coefficients of F_0 are obtained from those of H_0 by an order flip which in the Z-domain, translates into the equation [10]

$$F_0(z) = H_1(-z) = z^N H_0(z^{-1}) \tag{15}$$

The coefficients of the filter F_1 are obtained from those of H_0 by the alternating sign property which in the Z-domain, translates into the equation

$$F_1(z) = -H_0(-z) \tag{16}$$

The coefficients of the filter H_1 are obtained from those of F_0 by the alternating sign property which is the order flip of F_1 and in the Z-domain, translates into the equation

$$H_1(z) = -F_1(-z) \tag{17}$$

An example of obtaining the coefficients of the other three filters from those of H_0 in the time domain is the Haar filters depicted in Fig. 7.

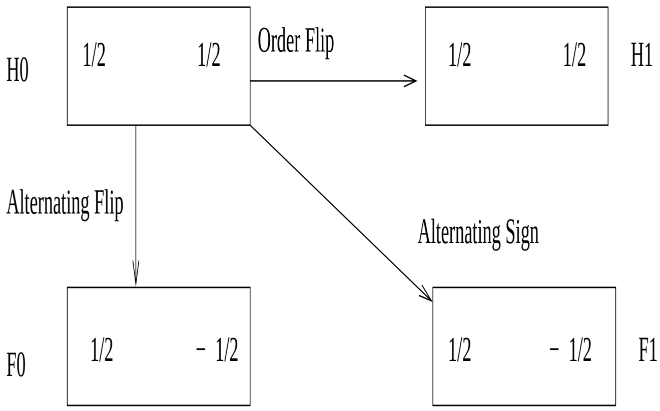

Fig. 7: The case of the Haar Filter Bank.

Theorem 1 *Reversing the coefficients of the filters in the time-domain preserves perfect reconstruction of hypermedia elements.*

Proof If $H_0 = [h_0, h_1, ..., h_N]$, then reversing its coefficients produces the filter

$$Rev(H_0) = H_{0r} = [h_N, h_{N-1}, \dots, h_0].$$

The alternating flip filter of $Rev(H_0)$ is the filter

$$H_{1r} = [h(0), -h(1), +h(2), ..., +h(n)].$$

H_0	H_1
-0.0011	0.1115
0.0048	0.4946
0.0006	0.7511
-0.0316	0.3153
0.0275	-0.2263
0.0975	-0.1298
-0.1298	0.0975
-0.2263	0.0275
0.3153	-0.0316
0.7511	0.0006
0.4946	0.0048
0.1115	-0.0011

Table 1: The coefficients of the low-pass and high-pass filters of $db6$.

H_0	H_1
0.1115	-0.0011
0.4946	0.0048
0.7511	0.0006
0.3153	-0.0316
-0.2263	0.0275
-0.1298	0.0975
0.0975	-0.1298
0.0275	-0.2263
-0.0316	0.3153
0.0006	0.7511
0.0048	0.4946
-0.0011	0.1115

Table 2: The coefficients of the new low-pass and new high-pass filters.

The order flip of H_{0r} is the filter

$$F_{0r} = [h_0, h_1, ..., h_N].$$

Finally, the alternating sign filter of $Rev(H_0)$ is the filter

$$F_{1r} = [-h_N, +h_{N-1}, ..., +h_0].$$

The filters H_{0r}, H_{1r}, F_{0r} and F_{1r} satisfy all the three time domain conditions of perfect reconstruction.

This result [8], allows the construction of new scaling and wavelet functions from the coefficients of the wavelet $db6$ in Table 1. These new coefficients of the new orthogonal reflective wavelet [10], are listed in Table 2. Fig. 8, shows the scaling and wavelet functions of the derived wavelet and Fig. 9, represents the plot of the frequency responses of the filters for the new wavelet. One notes the reflective similarities among the scaling and wavelet functions of $db6$ with those of the derived wavelet.

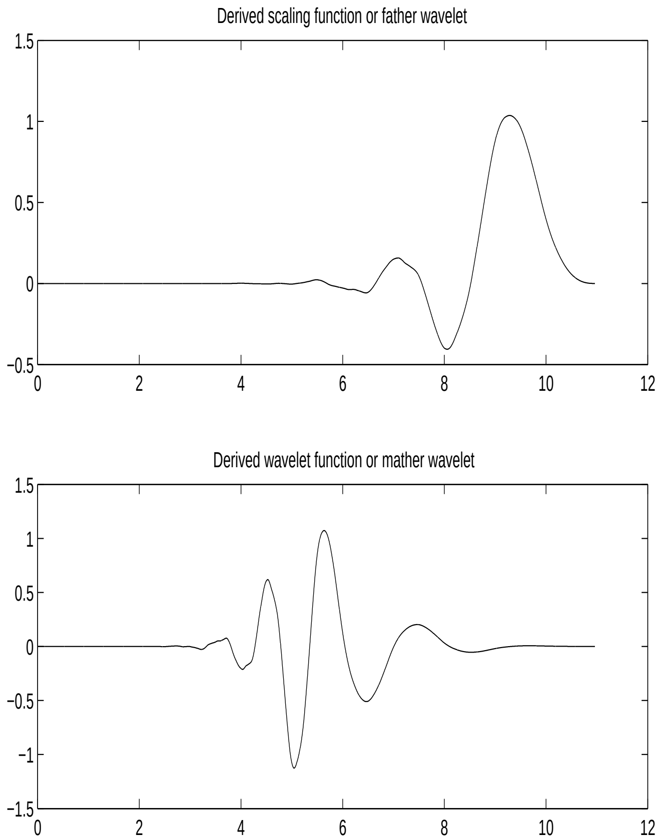

Fig. 8: The scaling and wavelet functions of the derived wavelet.

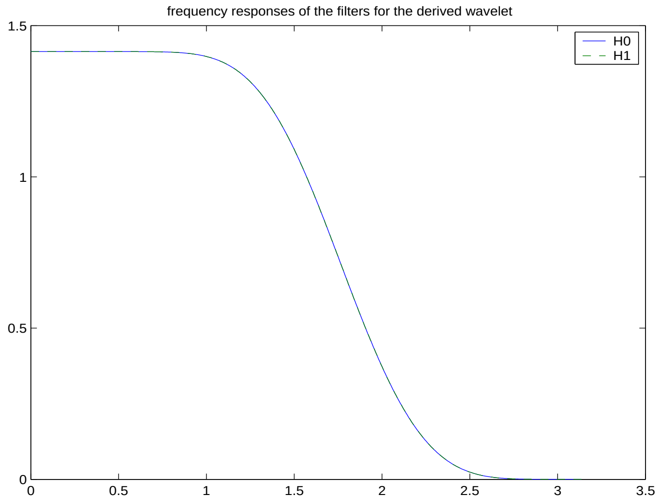

Fig. 9: The frequency responses of the filters.

4 Zeros of Associated Polynomials

In this section, we show that if y is a root of a binomial polynomial of degree $p-1$, then, $\frac{1}{2p} \leq |y| \leq \frac{1}{2}$. And, the class of polynomials with coefficients those of the ratios obtained in Equation 7, also have their zeros inside the unit circle. Let's first introduce the Kakeya-Enestrom theorem, which can be stated in many different ways. Here are 2 of them along with a proof which we will follow to show the result on the locations of the zeros for associated polynomials of Daubechies wavelets:

Theorem 2 *(Statement 1): Given a polynomial* $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ *where* a_i *is real and positive for all* $i = 1, 2, \ldots, n$. *If we denote* α *(resp.* β*) to be the smallest (resp. the largest) of the quotients* $\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{n-1}}{a_n}$, *then if* z *is a root of* $P(z)$ *one should have* $\alpha \leq |z| \leq \beta$.

Proof: Assume that $|z| = \gamma < 1$, then the identity

$$(1-z)P(z) = a_0 - (a_0 - a_1)z - (a_1 - a_2)z^2 - \cdots - (a_{n-1} - a_n)z^n - a_n z^{n+1}$$

implies that

$$|(1-z)P(z)| \geq a_0 - (a_0 - a_1)\gamma - (a_1 - a_2)\gamma^2 - \cdots - (a_{n-1} - a_n)\gamma^n - a_n \gamma^{n+1}$$

and so

$$|(1-z)P(z)| \geq (1-\gamma)P(\gamma) > 0. \quad (18)$$

Applying this result to the polynomial

$$P(\alpha z) = a_0 + \alpha a_1 z + \alpha^2 a_2 z^2 + \ldots + \alpha^n a_n z^n \quad (19)$$

and to the polynomial

$$z^n P(\frac{z}{\beta}) = \beta^n a_n + \beta^{n-1} a_{n-1} z + \ldots + a_0 z^n, \quad (20)$$

one can establish the desired result.

Theorem 3 *(Statement 2): If the coefficients of* $P(z)$ *in (Statement 1) satisfy* $a_n > a_{n-1} > \ldots > a_1 > a_0 > 0$, *then all the roots of* $P(z)$ *fall inside the unit circle.*

The ratios of the coefficients of the binomial polynomial of degree $p-1$ as defined here, have a minimum of $r = \frac{1}{p}$ obtained for $i = 0$, and a maximum of $s = 1/2$ obtained for $i = p-2$. Thus, by Statement 2 of Theorem 1 , the left hand side of the inequality is proved. For the complete proof of this theorem, Please refer to [7].

4.1 Zeros of Ratio Coefficients Polynomials

To show the second result, consider the class of polynomials with coefficients those of the ratios obtained in Equation (7). They can be expressed as follows

$$R_{p-2}(z) = \frac{-1}{p} + 0 \cdot z^1 + \frac{1}{2+p} z^2 + \frac{3}{4+p} z^3 + \cdots + \frac{p-2}{2p-2} z^{p-2}.$$

We note that these coefficients are in an ascending order where $a_k - a_{k-1} > 0$.

Theorem 4 *The roots of the polynomials* R_{p-2} *lie inside the unit disk for all* p.

Proof: For $a_0 = -\frac{1}{p} < 0$, consider

$$(1-z)R(z) = \phi(z) - a_{p-2} z^{p-1}$$

where

$$\phi(z) = a_0 + \sum_{k=1}^{p-2} (a_k - a_{k-1}) z^k.$$

For $z \neq 0$, we have

$$z_{p-2}\phi(\frac{1}{z}) = a_0 z^{p-2} + \sum_{k=1}^{p-2}(a_k - a_{k-1})z^{p-2-k}.$$

Now for $|z| \leq 1$,

$$\begin{aligned} |z^{p-2}\phi(\frac{1}{z})| &\leq |a_0| + \sum_{k=1}^{p-2}(a_k - a_{k-1}) \\ &\leq |a_0| - a_0 + a_{p-2} \end{aligned}$$

and

$$|\phi(\frac{1}{z})| \leq \frac{|a_0| - a_0 + a_{p-2}}{|z^{p-2}|}.$$

Replacing z with $\frac{1}{z}$, we get

$$\phi(z) \leq (|a_0| - a_0 + a_{p-2})|z^{p-2}|$$

for $z \neq 0$. Hence, if

$$|z| > \frac{(|a_0| - a_0 + a_{p-2})}{|a_{p-2}|}(\text{i.e. } |z| \geq 1),$$

then

$$\begin{aligned} & |(1-z)R(z)| \\ = \ & |\phi(z) - a_{p-2}z^{p-1}| \\ \geq \ & |a_{p-2}||z|^{p-1} - |\phi(z)| \\ \geq \ & |a_{p-2}||z^{p-1}| - (|a_0| - a_0 + a_{p-2})|z|^{p-2} \\ > \ & 0. \end{aligned}$$

4.2 An Example

For the Daubechies wavelet ”db6” this polynomial is given by $R_4(z) = -1/6 + 0 + 1/8z + 2/9z^2 + 3/10z^3 + 4/11z^4$ with a maximum module of 0.9325. The roots of this polynomial are depicted below in Fig. 10 and observed to reside all in the unit circle.

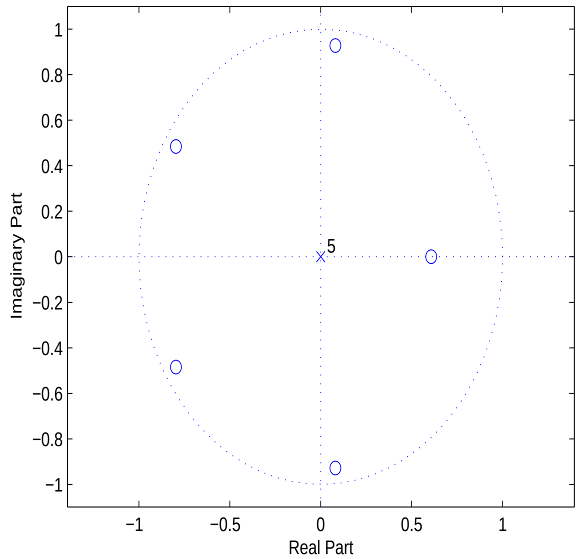

Fig. 10: Zeros of $R_4(z)$ for Daubechies db6 orthogonal wavelet.

4.3 On the Optimal Limits

One of the Kakeya-Enestrom theorem’s numerous generalizations [12] asserts:

Theorem 5 *Let $P(z) = a_n z^n + ... + a_1 z + a_0$ be a polynomial such that $a_k - \lambda a_{k-1} > 0$ for $k = 1, 2, ..., n$, with $\lambda > 0$. Then, $P(z)$ has all of its zeros in the closed disk:*

$$|z| \leq \frac{a_n - a_0\lambda^n + |a_0|\lambda^n}{\lambda|a_n|} \tag{21}$$

As noted before, the coefficients of $R_{p-2}(z)$ are in an ascending order where $a_k - a_{k-1} > 0$. It is worthwhile mentioning here that if a_0 was positive, then all the roots of this class of polynomials lie inside the unit circle as the result of Theorem 4 is then reduced to the Kakeya-Enestrom theorem [11].

While the value of λ has to be positive for the validity of Theorem 4, the discussion that follows considers two ranges of that value. For the first range of $0 < \lambda < 1$, a larger radius is obtained, but the conditions for the coefficients are weaker than those of Theorem 3.

For $\lambda > 1$, we obtain a smaller radius of the disk in which the zeros of $R(z)$ are located, with stronger conditions than the ones of Theorem 3.

Now since $a_0 = \frac{-1}{p} < 0$ and $a_{p-2} > 0$, an optimal (smallest) estimate of the radius of the disk containing all the roots of $R(z)$ can be obtained for the choice of $\lambda = (\frac{B}{A(p-3)})^{1/n}$ with $A = -2a_0$ $= \frac{2}{p}$ and $B = a_{p-2} > 0$. That is:

$$\lambda = (\frac{a_{p-2}}{\frac{2}{p}(p-3)})^{\frac{1}{p-2}} \tag{22}$$

For the case of $db6$ and the polynomial, $R_4(z)$, this value is

$$\lambda = (\frac{\frac{3}{10}}{\frac{2}{6}(6-3)})^{\frac{1}{6-2}} \tag{23}$$

and

$$\lambda = [\frac{3}{10}]^{\frac{1}{4}} \tag{24}$$

5 Conclusion

In this paper we construct the Daubechies orthogonal mother wavelet $db6$ via the two channel perfect reconstruction filter bank. We derived the coefficients of the filters associated with it and the roots of the binomial polynomial that made this construction possible. From the Daubechies family of wavelets, we derived a new class of wavelets called reflective wavelets along with their scaling functions. The case of the mother wavelet $db6$ was fully examined as an example and the coefficients and frequency responses of the derived filters were calculated. The locations of the zeros of the polynomials involved in this construction were found and their locations were discussed. The distribution of the zeros of a family of polynomials having their coefficients as the ratios of those of the binomial polynomials was examined and were proved mathematically to reside inside the unit circle. Finally, the λ based difference of the ratio coefficients is optimized.

Acknowledgements: The author would like to thank Alfaisal University and its Office of Research for securing the time, environment and funds to complete this research project. This work was also supported by the Alfaisal University Start-Up Fund (No. 410111410091).

Dedication: This paper is dedicated to my friend and former dean, Professor Ray Weisenborn.

References:

[1] T.–H. Chang, Verallgemeinerung des Stazes von Kakeya. *Tohoku Math. J.* 43, 1937, pp: 79–83.

[2] I. Daubechies, Ten Lectures on Wavelets, *Philadelphia, SIAM*,1992.

[3] G. Enestrom, Harlednini af en allman formel for antalet pensionarer, som vid en godtyeklig tidpunkt forefinnas inom en sluten pensionskassa. *Ofv. af. kungl. Vetenskaps-Akademiens Forhandlingar* 6, 1893, pp: 405–415.

[4] S. Kakeya, On the limits of the roots of an algebraic equation with positive coefficients, *Tohoku Math. J.* 2,1912-1913, pp: 140–142, .

[5] J. Karam, *On the Kakeya-Enestrom Theorem*, MSc. thesis, Dalhousie University, Halifax, 1995.

[6] J. Karam, Connecting Daubechies wavelets with the Kakeya-Enestrom Theorem, *International Journal of Applied Mathematics*, Vol. (14), 2003, No. 2, pp: 109–124.

[7] J. Karam, On the roots of Daubechies polynomials, *International Journal of Applied Mathematics*, IJAM, Vol. 20, 2007, No. 8, pp: 1069–1076.

[8] J. Karam and M. Karam, Perfect reconstruction of Hypermedia elements via filter banks, *International Journal of Computational and Numerical Analysis and Applications*, Vol 5, 2004, No. 1, pp: 51–57.

[9] J. Karam, A comprehensive approach for speech related multimedia applications, *WSEAS Transactions on Signal Processing*, Issue 1, Volume 6, January 2010, pp: 12–21.

[10] J. Karam, A new approach in wavelet based speech compression, *Proceedings of the 10th WSEAS International Conference on Mathematical Methods, Computational Techniques and Intelligent Systems*, Corfu, Greece, October 26-28, 2008, pp: 228–233.

[11] J. Karam On the Distribution of zeros for Daubechies orthogonal wavelets and associated polynomials, *The 15th WSEAS International Conference on Applied Mathematics*, Vouliagmeni, Athens, Greece, December, 29-31, 2010, pp: 101–104.

[12] M.–A. Kovacevic and I.–Z. Milovaovic, On a generalization of Enestrom-Kakeya Theorem, *Pu. Math. Appl.* Ser. A 3, 1992, pp: 43–47.

[13] M. Misiti, Y. Misiti, G. Oppenheim and J. Poggi, *Matlab wavelet tool box*, 1997.

[14] C.–A. Muresan, Comparative methods for the polynomial isolation, *Proceedings of the 13th WSEAS International Conference on Computers*, 2009, pp: 634–638.

[15] G. Strang, and T. Nguyen, *Wavelets and Filter Banks*, Wellesley MA, Wellesley–Cambridge Press, Wellesley, MA, 1996.

[16] M. Vetterli, Wavelets and filter banks: theory and design, *IEEE Transactions on Signal Processing*, Vol.40, 1992, pp: 2207-2232.

[17] M. Vetterli, and C. Herley, Wavelets and filter banks: relationships and new results, *IEEE Transactions, Signal Processing*, 1990, pp: 1723–1726.

[18] M. Vetterli and J. Kovacevic, *Wavelets and Subband Coding*, Prentice Hall, PTR, Englewood Cliffs, New Jersey, 1995.