On a Degenerate Parabolic Equation from Finance

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Abstract: Consider the degenerate parabolic equation $\partial_{xx}u+u\partial_y u-\partial_t u=f(\cdot,u)$, which comes from mathematics finance, and in which $u(t,x)$ is the utility function of a agent's decision under risk. By Oleinik's line method, the existence and the uniqueness of the local classical solution for the initial boundary problem of the equation are got. Also, the global entropy solution of the Cauchy problem is discussed.

Key–Words: Degenerate parabolic equation, existence, uniqueness, initial boundary problem, global entropy solution

1 Introduction

In this paper, we consider the following initial boundary problem:

$$\partial_{xx}u+u\partial_y u-\partial_t u=f(\cdot,u),(t,x,y)\in(0,T]\times\Omega \tag{1}$$

$$u(\cdot,0)=u_0,(x,y)\in\Omega \tag{2}$$

$$u\mid_{\{x=0\}\times[0,T]}=u_1(0,y,t),u\mid_{\{x=R\}\times[0,T]}=0 \tag{3}$$

where $\Omega=(0,R)\times(0,N)\subset R^2$, T is a suitably small positive constant. The equation (1) arises in mathematics finance, arises when studying nonlinear physical phenomena such as the combined effects of diffusion and convection of matter (cf.[5]), many mathematicians have been interested in it. In [1], Antonelli, Barucci and Mancino introduce a new model for agent's decision under risk, in which the utility function is the solution to (1)-(2). In the sense of the User's guide, i.e.

$$\mid u(x,y,t)-u(\eta,\xi,t)\mid\le C_T(\mid x-\eta\mid+\mid y-\xi\mid) \tag{4}$$

for every $(x,y),(\xi,\eta)\in R^2,t\in[0,T)$, under the assumption that f is uniformly Lipschitz continuous function, Crandall, Ishii and Lions [2] proved the existence of a continuous viscosity solution by means of probability methods. In [3], Citti, Pascucci and Polidoro studied the interior regularity, they proved that the viscosity solutions are indeed in classical sense. In [4], Antonelli and Pascucci showed that u is the limit, uniformly on compacts of $[0,T]\times R^2$, of the family of solutions to the regularized Cauchy Problem: for $(x,y,t)\in R^2\times(0,T]$,

$$\varepsilon^2\partial_{yy}u+\partial_{xx}u+u\partial_y u-\partial_t u=f(\cdot,u), \tag{5}$$

$$u(0,x,y)=u_0(x,y). \tag{6}$$

Other related work, one can refer to [7] etc. However, all of the published papers study the Cauchy problem and get the local classical solutions. As for the existence and uniqueness of the global weak solution for the cauchy problem of (1), there are some differential ways to deal with them, for example, (1) is the special case of the Cauchy problem discussed in [8],[9] etc., we will simply narrate this aspect in the last section of the paper. The main aim of the paper is to study the initial boundary problem (1)-(3).

Clearly, (1) is a degenerate parabolic equation on account of that it lacks the two order partial derivative term $\partial_{yy}u$. It is well-known that there are some rules in how to quote an initial boundary problem of a degenerate parabolic equation, one can refer to Oleinik's books [6],[10] etc. According to these rules, we quote the problem as the form of (1)-(3). We will discuss this problem in a complete different way comparing to [1]-[4].

In order to describe our method, we have to quote the well-known Prandtl system for a non-stationary boundary layer arising in an axially symmetric incompressible flow past a solid body. As well known, Prandtl proposed the conception of the boundary layer in 1904[11]. From then on, the interest in the theory of boundary layer has been steadily growing, due to the mathematical questions it poses, and its important practical applications. According to Prandtl boundary layer theory, the flow about a solid body can be divided into two regions: a very thin layer in the neighborhood of the body (the boundary layer) where viscous friction plays an essential part, and the remaining region outside this layer where friction may be neglected (the outer flow). Thus, for fluids whose viscosity is small, its influence is perceptible only in a very thin region adjacent to the walls of a body in the flow; the said region, according to Prandtl, is called *the boundary layer*. This phenomenon is explained by the fact that the fluid sticks to the surface of a solid body and, this adhesion inhibits the motion of a thin layer of fluid adjacent to the surface. In this thin region the velocity of the flow past a body at rest undergoes a sharp increase: from zero at the surface to the values of the velocity in the outer flow, where the fluid may be regarded as frictionless. Prandtl derived the system of equations for the first approximation of the flow velocity in the boundary layer. This system served as a basis for the development of the boundary layer theory, which has now become one of the fundamental parts of fluid dynamics. Assume that the motion of a fluid occupying a two-dimensional region is characterized by the velocity vector $V = (u, v)$, where u, v are the projections of V onto the coordinate axes x, y, respectively, the Prandtl system for a non-stationary boundary layer arising in an axially symmetric incompressible flow past a solid body has the form as

$$\partial_t u + u\partial_x u + v\partial_y u = \partial_t U + U\partial_x U + \partial_y^2 u,$$

$$\partial_x(ru) + \partial_y(rv) = 0,$$

in a domain $D = \{0 < t < T, 0 < x < X, 0 < y < \infty\}$, where $\nu = const > 0$ is the coefficient of kinematic viscosity; $U(t, x)$ is called the velocity at the outer edge of the boundary layer, $U(t, 0) = 0$, $U(t, x) > 0$ for $x > 0$; $r(x)$ is the distance from that point to the axis of a rotating body, $r(0) = 0$, $r(x) > 0$ for $x > 0$. In recent decades, many scholars have been carrying out research in this field, achievements are abundant in literature on theoretical, numerical experimental aspects of the theory, see [12],[13] etc. If we introduce the Crocco variables

$$\tau = t, \xi = x, \eta = \frac{u(t, x)}{U(t, x)},$$

we obtain the following equation for $w(\tau, \xi, \eta) = \frac{\partial_y u}{U}$:

$$w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w = 0 \quad (7)$$

where A, B are the two known functions derived from the Prandtl system, one can refer to [6] for details. By the above Crocco transform, the Prandtl system is succeeded to be changed to a degenerate parabolic equation (7). On this basic point of view, Oleinik [6] had done excellent works in the boundary theory by the line method. Comparing (1) with (7), we find these two equations are similar as each other, both of them are lacks the two order partial derivative term $\partial_{yy} u$. In view of that Oleinik's line method had been used widely to study a variety of problems, for examples [19][20] et al., it is nature to conjecture that we are able to solve the problem (1)-(3) by Qleinik's line method.

The main result of the paper is the following

Theorem 1 *Assume that*

$$| u_0(x, y) | \leq c(R - x), \ (x, y) \in \Omega, \quad (8)$$

the first order and second order derivatives of u_0 are bounded, u_{0xxx} is bounded too. Assume that u_1 is continuous and smooth near $\partial\Omega$, its first order, second order derivatives at $x = 0$ are all bounded. Suppose f satisfies (11) below and is an uniformly Lipschitz continuous function. Then the initial boundary problem (1)-(3) has a unique solution in classical sense provided that $t \leq T$, T is suitable small (or $y \leq N$, N is suitable small), and moreover, the first order and second order derivatives of u are bounded.

By the way, for the best knowledge of the author, this is the first paper on the initial boundary problem of (1).

2 Line method

For the comparability of signs with the Prandtl system, we rewrite (1)–(3) as following: for $(\xi, \eta, t) \in \Omega \times (0, T)$,

$$w_{\eta\eta} - w_\tau + w w_\xi = f(\eta, \xi, \tau, u), \quad (9)$$

$$w(\eta, \xi, 0) = w_0(\eta, \xi), \ (\xi, \eta) \in \Omega \quad (10)$$

$$w \mid_{\{\eta=0\}\times[0,T]} = \phi(0, \xi, t), \ w \mid_{\{\eta=R\}\times[0,T]} = 0 \quad (11)$$

where $\Omega = (0, N) \times (0, R)$, $w_0 \in C^2(\Omega)$, its first order derivatives and $w_{0\eta\eta}$ are all bounded, $\phi(\eta, \xi, \tau)$ is a smooth function on $\overline{\Omega \times (0, T)}$, $w_0(0, \xi) = \phi(0, \xi, 0)$. and f is a Lipschitz continuous function which satisfies that: when $w_1 - w_2 \geq 0$,

$$c_2(w_1 - w_2) \geq f(\cdot, w_1) - f(\cdot, w_2) \geq c_1(w_1 - w_2) \quad (12)$$

and

$$| f(\cdot, w) | \leq c \mid w \mid^p \quad (13)$$

for some nonnegative number p. Moreover, we assume that $K_1 \leq \frac{1}{2R}$,

$$| w_0(\eta, \xi) | \leq K_1(R - \eta), \ (\eta, \xi) \in \Omega, \phi \leq \frac{1}{2}. \quad (14)$$

It is regret that the author does not know, to get the results of theorem 1.1, whether the condition (14) is necessary or not, it seems that (14) is only a technique request in the proof. If one is able to refine the proof, the (14) may be abandoned or be weaker.

For any functions, we use the following notation

$$f^{m,k}(\eta) = f(\eta, kh, mh), h = const > 0.$$

Instead of equation (9)-(11), let us consider the following system of ordinary differential equations:

$$w^{m,k}_{\eta\eta} - \frac{w^{m,k} - w^{m-1,k}}{h} + w^{m-1,k}\frac{w^{m,k} - w^{m,k-1}}{h} - f(\cdot, w^{m,k}) = 0, \quad (15)$$

$$w^{m,k} \mid_{\eta=R}= 0,\ w^{m,k} \mid_{\eta=0}= \phi(0, kh, mh), \quad (16)$$

where

$$w^{0,k} = w_0(kh, \eta),\ m = 1, \cdots, [Th];\ k = 0, 1, \cdots, [Nh]$$

If $k = 0$, (15) should be

$$w^{m,0}_{\eta\eta} - \frac{w^{m,0} - w^{m-1,0}}{h} - f(\cdot, w^{m,0}) = 0 \quad (17)$$

The solutions of (15)-(16) are defined in the classical sense and we will prove that

$$w^{m,k}_{\eta\eta}, w^{m,k}_{\eta}, \frac{w^{m,k} - w^{m-1,k}}{h}, \frac{w^{m,k} - w^{m,k-1}}{h}$$

are uniformly bounded for any m, k.

Lemma 2 *Under the conditions of (11)-(13), the problem (15)-(16) admits a unique solution for $mh \leq T_0$ and small enough h, where T_0 is a suitable small positive number. The solution satisfies the following estimate*

$$V_0(\eta, mh) \leq w^{m,k} \leq V_1(\eta, mh) \quad (18)$$

where V_0, V_1 are continuous function, positive in $(0, R), V_1 \leq \frac{1}{2}$ and such that

$$V_0 \equiv K_0(R - \eta), V_1 \equiv K_1(R - \eta) \quad (19)$$

in a neighborhood of $\eta = R$, where $K_1 \leq \frac{1}{2R}$ as before.

Proof: By (15) and (17), the existence of $w^{m,k}$ is clearly. Let $Q^{m,k}$ be the difference of two solution $w^{m,k}_1$, $w^{m,k}_2$. Then $Q^{m,k}$ can attain neither a positive maximum nor a negative minimum at $\eta = 0, R$. By the inductive assumption, $\mid w^{m,k} \mid\leq \frac{1}{2}$, so

$$\begin{aligned} 0 &= L_{m,k}(Q^{m,k}) \\ &= Q^{m,k}_{\eta\eta} - \frac{1}{h}Q^{m,k} + w^{m-1,k}\frac{1}{h}Q^{m,k} \\ &\quad + f(\cdot, w^{m,k}_1) - f(\cdot, w^{m,k}_2) \end{aligned} \quad (20)$$

$Q^{m,k}$ can attain neither a positive maximum nor a negative minimum in interior of $(0, R)$ by (12)-(13), provided that $h \leq h_0$ small enough. Consequently, under our assumption, problem (15) cannot have more than one solution. Therefore, we shall prove (18) for m and k under the assumption of that the solutions $w^{m-1,k}$ of (15) admit the following a priori estimate

$$V_1(\eta, (m-1)h) \geq w^{m-1,k} \geq V_0(\eta, (m-1)h). \quad (21)$$

Denote that

$$\begin{aligned} L_{m,k}(u) &= u^{m,k}_{\eta\eta} - \frac{1}{h}(u^{m,k} - u^{m-1,k}) \\ &+ w^{m-1,k}\frac{1}{h}(u^{m,k} - u^{m,k-1}) - f(\cdot, w^{m,k}). \end{aligned}$$

In order to prove the priori estimate (18) for $\tau = mh$, it suffices to show that there exist function V_1 with the properties specified in Lemma 2 and such that

$$\begin{aligned} 0 \geq L_{m,k}(V_1) &= V^{m,k}_{1\eta\eta} - \frac{1}{h}(V^{m,k}_1 - V^{m-1,k}_1) \\ &+ w^{m-1,k}\frac{1}{h}(V^{m,k}_1 - V^{m,k-1}_1) - f(\cdot, V^{m,k}_1) \end{aligned}$$

and

$$V_1(0, mh) \geq \phi(0, kh, mh)$$

under assumption (21). Then the inequality (20) can be proved by induction with respect to m. Indeed, let $q^{m,k} = V_1 - w^{m,k}$. Then $q^{m,k}(0) \geq 0$ and

$$\begin{aligned} 0 &\geq L_{m,k}(V_1) - L_{m,k}(w) \\ &= q^{m,k}_{\eta\eta} - \frac{1}{h}(q^{m,k} - q^{m-1,k}) \\ &\quad + w^{m-1,k}\frac{1}{h}(q^{m,k} - q^{m,k-1}) \\ &\quad + f(\cdot, V^{m,k}_1) - f(\cdot, w^{m,k}) \\ &\geq q^{m,k}_{\eta\eta} - \frac{1}{h}(q^{m,k} - q^{m-1,k}) \\ &\quad + w^{m-1,k}\frac{1}{h}(q^{m,k} - q^{m,k-1}) \\ &\quad + c_2 q^{m,k} \end{aligned} \quad (22)$$

Let $q^{m,k}_1 = e^{\alpha mh} q^{m,k}$. Then

$$\begin{aligned} 0 &\geq q^{m,k}_{1\eta\eta} - \frac{1}{h}(q^{m,k}_1 - q^{m-1,k}_1) \\ &\quad + w^{m-1,k}\frac{1}{h}(q^{m,k}_1 - q^{m,k-1}_1) \\ &\quad + \alpha e^{\alpha h'} w^{m-1,k} + c_2 q^{m,k}_1 \\ &\geq q^{m,k}_{1\eta\eta} - [\frac{1}{h}(1 - w^{m-1,k}) - c_2] q^{m,k}_1 + \frac{1}{h} q^{m-1,k}_1 \end{aligned}$$

$$+w^{m-1,k}(\alpha-\frac{1}{h}q_1^{m,k-1}) \quad (23)$$

where $0<h'<h$. By (23), if we choose $\alpha=\alpha(h)$ large enough, then it is easily to know that $q_1^{m,k}$ can not attain negative minimum in interior of $(0,R)$ by maximal principle.

$$q_1^{m,k}\mid_{\eta=0}=e^{\alpha mh}q^{m,k}\mid_{\eta=0}\geq 0$$

$$q_1^{m,k}\mid_{\eta=R}=e^{\alpha mh}q^{m,k}\mid_{\eta=R}=0,$$

so (18) is true .

Under the condition (21), let us show that there is a positive T_0 such that for $mh\leq T_0$ there exist function V_1 satisfying the desired inequality. Let $\varphi_1(s)$ be a smooth function such that for $\eta>\frac{R}{2}$,

$$\varphi_1(s)=R-\eta,$$

for $\frac{1}{4}R\leq s\leq\frac{1}{2}R$,

$$\frac{R}{2}\leq\varphi_1\leq R,$$

for $\eta<\frac{1}{4}R$,

$$\varphi_1(s)=R.$$

Set

$$V_1=M\varphi_1(\eta)\varphi_2(\beta_1\eta)e^{\beta_2 mh}, \quad (24)$$

where φ_2 is a smooth function such that for $0\leq s\leq R$,

$$\varphi_2(s)=4-e^{\frac{1}{R}s},$$

for $s\geq 2R$,

$$\varphi_2(s)=1,$$

for $R\leq s\leq 2R$

$$1\leq\varphi_2(s)\leq 3.$$

The constant M is chosen from the condition $V_1(\eta,\xi,0)\geq\frac{1}{2}$. The positive constants β_1,β_2 will be specified shortly.

Clearly

$$\begin{aligned}L(V_1) &= Me^{\beta_2 mh}(\varphi_1(\eta)\varphi_2(\beta_1\eta))_{\eta\eta}\\ &\quad -\frac{1}{h}M\varphi_1(\eta)\varphi_2(\beta_1\eta)(e^{\beta_2 mh}-e^{\beta_2(m-1)h})\\ &\quad -f(\cdot,w^{m,k}).\end{aligned}$$

For a given small positive number δ, if $R-\eta<\delta$, we can choose β_1 such that $\beta_1\eta\geq 2R$ then $\varphi_1(\eta)\varphi_2(\beta_1\eta)=R-\eta$, so choosing β_2 large enough, $mh\leq T_0$ small enough

$$\begin{aligned}L(V_1) &= -\frac{1}{h}M(R-\eta)(e^{\beta_2 mh}-e^{\beta_2(m-1)h})+c_1(w^{m,k})^p\\ &\leq -\frac{1}{h}M(R-\eta)(e^{\beta_2 mh}-e^{\beta_2(m-1)h})\\ &\quad +c_1(M(R-\eta)e^{\beta_2 mh})^p\\ &\leq M(R-\eta)[-\beta_2 e^{\beta_2 h'}+c_2]\leq 0.\end{aligned}$$

If $R-\eta>\delta$, notice that $\mid(\varphi_1(\eta)\varphi_2(\beta_1\eta))_{\eta\eta}\mid\leq c_3$, then

$$\begin{aligned}L(V_1) &\leq -\frac{1}{h}M\varphi_1(\eta)\varphi_2(\beta_1\eta)(e^{\beta_2 mh}-e^{\beta_2(m-1)h})\\ &\quad +c_3Me^{\beta_2 mh}+c_1\mid w^{m,k}\mid^p\\ &\leq -\frac{1}{h}M\varphi_1\varphi_2(e^{\beta_2 mh}-e^{\beta_2(m-1)h})\\ &\quad +c_3Me^{\beta_2 mh}+c_1(M\varphi_1\varphi_2 e^{\beta_2 mh})^p\\ &\leq M(\varphi_1\varphi_2)[-\beta_2 e^{\beta_2 h'}+c_4]\leq 0.\end{aligned}$$

At the same time, set

$$V=\mu\varphi(\alpha_1\eta)\varphi_1(\eta)e^{-\alpha_2 mh},$$

where μ is small enough such that $V_0(0,mh)\leq\phi(0,kh,mh)$ and $\varphi(s)$ is a smooth function such that for $0\leq s\leq R$,

$$\varphi(s)=e^{\frac{1}{R}s},$$

for $R\leq s\leq\frac{3}{2}R$,

$$1\leq k\leq 3,$$

for $s>\frac{3}{2}R$,

$$\varphi(s)=1,$$

also by α_1,α_2 large enough, $mh\leq T_0$ small enough, we have

$$L(V)\geq 0.$$

Thus we can get (18) easily.

Lemma 3 *Assume that the conditions of Lemma 2 are fulfilled, w_0 has bounded first order derivatives, $w_{0\eta\eta}$ is bounded, then*

$$w_\eta^{m,k},\frac{1}{h}(w^{m,k}-w^{m-1,k}),\frac{1}{h}(w^{m,k}-w^{m,k-1}),w_{\eta\eta}^{m,k}$$

are bounded for $mh\leq T_1$ and $h\leq h_0$, uniformly with respect to h, where the positive constants $T_1\leq T_0$.

Proof: Let $\Phi^{m,k}(\eta)$ be the functions defined as follows: for $k\geq 1,m\geq 1$,

$$\Phi^{m,k}(\eta)=(\frac{w^{m,k}-w^{m-1,k}}{h})^2+(\frac{w^{m,k}-w^{m,k-1}}{h})^2 \quad (25)$$

and for $k = 0, m \geq 1$

$$\Phi^{m,k}(\eta) = (\frac{w^{m,k} - w^{m-1,k}}{h})^2. \quad (26)$$

Also, we need to define $\Phi^{0,k}(\eta) = (\frac{w^{0,k}-w^{-1,k}}{h})^2$. So let $w^{-1,k} = w^{-1,k}(\eta, kh)$ be a bounded function such that

$$\frac{w^{0,k} - w^{-1,k}}{h} = w^{0,k}_{\eta\eta} + w^{-1,k}\frac{w^{0,k} - w^{0,k-1}}{h} - f(\cdot, w^{0,k}). \quad (27)$$

$w^{-1,k}$ is well defined because $\frac{1}{h} + \frac{w^{0,k}}{h} > 0$. Clearly, on account of w_0 has bounded first order derivatives and $w_{0\eta\eta}$ is bounded, $\frac{w^{0,k}-w^{-1,k}}{h}$ is uniformly bounded with respect to h, so $\mid \Phi^{0,k} \mid \leq c$.

Let

$$r^{m,k} = \frac{w^{m,k} - w^{m,k-1}}{h}, \rho^{m,k} = \frac{w^{m,k} - w^{m-1,k}}{h}.$$

Now we will deduce the equation for $\Phi^{m,k}(\eta)$. To this end, we subtract from equation (15) for $w^{m,k}$ equation (13) for $w^{m-1,k}$ and multiply the result by $\frac{2\rho^{m,k}}{h}$ to get the first equation; from (15) for $w^{m,k}$ we subtract (13) for $w^{m,k-1}$ and multiply the result by $\frac{2r^{m,k}}{h}$ to get the second equation. We find the equations for $\Phi^{m,k}(\eta)$ with $k = 0, m \geq 1$ by taking only the first equation. In order to derive the equation for $\Phi^{m,k}(\eta)$ with $m = 1$, we utilize the relation (27) which defines the values of $w^{-1,k}$. Taking the sum of the three equations just obtained we get the equation for $\Phi^{m,k}(\eta), k \geq 1$. Say, we have

$$\begin{aligned} &((15)^{m,k} - (15)^{m-1,k})\frac{2\rho^{m,k}}{h} \\ = &\ 2\rho^{m,k}\rho^{m,k}_{\eta\eta} - \frac{2\rho^{m,k}}{h}(\rho^{m,k} - \rho^{m-1,k}) \\ &+\frac{2\rho^{m,k}}{h}(w^{m-1,k}r^{m,k} - w^{m-2,k}r^{m-1,k}) \\ &-\frac{2\rho^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m-1,k}) \\ = &\ 2\rho^{m,k}\rho^{m,k}_{\eta\eta} - \frac{2\rho^{m,k}}{h}(\rho^{m,k} - \rho^{m-1,k}) \\ &+\frac{2\rho^{m,k}}{h}[w^{m-1,k}(\rho^{m,k} - w^{m-1,k}\rho^{m,k-1})] \\ &+\frac{2\rho^{m,k}}{h}(w^{m-1,k} - w^{m-1,k-1})\rho^{m-1,k} \\ &-\frac{2\rho^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m-1,k}); \end{aligned}$$

$$\begin{aligned} &((18)^{m,k} - (18)^{m,k-1})\frac{2r^{m,k}}{h} \\ = &\ 2r^{m,k}r^{m,k}_{\eta\eta} - \frac{2r^{m,k}}{h}(\rho^{m,k} - \rho^{m,k-1}) \\ &+\frac{2r^{m,k}}{h}(w^{m-1,k}r^{m,k} - w^{m-1,k-1}r^{m,k-1}) \\ &-\frac{2r^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m,k-1})) \\ = &\ 2r^{m,k}r^{m,k}_{\eta\eta} - \frac{2r^{m,k}}{h}(r^{m,k} - r^{m-1,k}) \\ &+\frac{2r^{m,k}}{h}(w^{m-1,k}r^{m,k} - w^{m-1,k-1}r^{m,k-1}) \\ &-\frac{2r^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m,k-1}); \end{aligned}$$

$$\Phi^{m,k}_\eta = 2\rho^{m,k}\rho^{m,k}_\eta + 2r^{m,k}r^{m,k}_\eta;$$

$$\Phi^{m,k}_{\eta\eta} = 2\rho^{m,k}\rho^{m,k}_{\eta\eta} + 2r^{m,k}r^{m,k}_{\eta\eta} + 2(\rho^{m,k}_\eta)^2 + 2(r^{m,k}_\eta)^2;$$

$$-\frac{1}{h}(\Phi^{m,k} - \Phi^{m-1,k}) = -\frac{1}{h}[(\rho^{m,k})^2 - (\rho^{m-1,k})^2 + (r^{m,k})^2 - (r^{m-1,k})^2];$$

$$\begin{aligned} &\frac{w^{m-1,k}}{h}(\Phi^{m,k} - \Phi^{m,k-1}) \\ = &\ \frac{w^{m-1,k}}{h}[(\rho^{m,k})^2 - (\rho^{m,k-1})^2] \\ &+\frac{w^{m-1,k}}{h}[(r^{m,k})^2 - (r^{m,k-1})^2]; \end{aligned}$$

$$\begin{aligned} &\Phi^{m,k}_{\eta\eta} - \frac{1}{h}(\Phi^{m,k} - \Phi^{m-1,k}) \\ &+\frac{w^{m-1,k}}{h}(\Phi^{m,k} - \Phi^{m,k-1}) \\ = &\ 2(\rho^{m,k}_\eta)^2 + 2(r^{m,k}_\eta)^2 \\ - &\ \frac{1}{h}[(\rho^{m,k})^2+(r^{m,k})^2]+\frac{1}{h}[(\rho^{m-1,k})^2+(r^{m-1,k})^2] \\ + &\ \frac{w^{m-1,k}}{h}[(\rho^{m,k})^2-(\rho^{m,k-1})^2+(r^{m,k})^2-(r^{m,k-1})^2] \\ + &\ \frac{2\rho^{m,k}}{h}(\rho^{m,k} - \rho^{m-1,k}) - \frac{2\rho^{m,k}}{h}[w^{m-1,k}\rho^{m,k} \\ + &\ (w^{m-1,k} - w^{m-1,k-1})\rho^{m-1,k} - w^{m-1,k}\rho^{m,k-1}] \\ + &\ \frac{2\rho^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m-1,k}) \\ + &\ \frac{2r^{m,k}}{h}(r^{m,k} - r^{m,k-1}) \\ - &\ \frac{2r^{m,k}}{h}(w^{m-1,k}r^{m,k} - w^{m-1,k-1}r^{m,k-1}) \\ + &\ \frac{2r^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m,k-1}). \end{aligned}$$

By (12)-(13), we have

$$\Phi^{m,k}_{\eta\eta} - \frac{1}{h}(\Phi^{m,k} - \Phi^{m-1,k}) + \frac{w^{m-1,k}}{h}(\Phi^{m,k} - \Phi^{m,k-1})$$

$$\begin{aligned}
\geq\ & 2(\rho_\eta^{m,k})^2+2(r_\eta^{m,k})^2-\frac{1}{h}[(\rho^{m,k})^2+(r^{m,k})^2]\\
&+\frac{1}{h}[(\rho^{m-1,k})^2+(r^{m-1,k})^2]\\
&+\frac{w^{m-1,k}}{h}[(\rho^{m,k})^2-(\rho^{m,k-1})^2]\\
&+\frac{w^{m-1,k}}{h}[(r^{m,k})^2-(r^{m,k-1})^2]\\
&+\frac{2\rho^{m,k}}{h}(\rho^{m,k}-\rho^{m-1,k})\\
&-\frac{2\rho^{m,k}}{h}[w^{m-1,k}(\rho^{m,k}-\rho^{m,k-1})]\\
&-\frac{2\rho^{m,k}}{h}[(w^{m-1,k}-w^{m-1,k-1})\rho^{m-1,k}]\\
&+\frac{2c_1}{h}(\rho^{m,k})^2+\frac{2r^{m,k}}{h}(r^{m,k}-r^{m,k-1})\\
&-\frac{2r^{m,k}}{h}(w^{m-1,k}r^{m,k}-w^{m-1,k-1}r^{m,k-1})\\
&+\frac{2c_1}{h}(r^{m,k})^2;
\end{aligned}$$

(i). If at the maximal value point of $\Phi^{m,k}$, suppose $\Phi^{m,k}-\Phi^{m,k-1}\geq 0$, then

$$\begin{aligned}
&\Phi_{\eta\eta}^{m,k}-\frac{1}{h}(\Phi^{m,k}-\Phi^{m-1,k})\\
+\ &\frac{w^{m-1,k}}{h}(\Phi^{m,k}-\Phi^{m,k-1})-\alpha\Phi^{m,k}+\beta\Phi^{m,k-1}\\
\geq\ & 2(\rho_\eta^{m,k})^2+2(r_\eta^{m,k})^2\\
+\ & (\frac{2c_1+1-w^{m-1,k}}{h}-\alpha)[(\rho^{m,k})^2+(r^{m,k})^2]\\
+\ & (\beta-\frac{w^{m-1,k}}{h})[(\rho^{m,k-1})^2+(r^{m,k-1})^2]\\
+\ & \frac{1}{h}[(\rho^{m-1,k})^2+(r^{m-1,k})^2]\\
-\ & \frac{2\rho^{m,k}}{h}[(w^{m-1,k}-w^{m-1,k-1}+1)\rho^{m-1,k}-w^{m-1,k}\rho^{m,k-1}]\\
-\ & \frac{2r^{m,k}}{h}r^{m,k-1}+\frac{2r^{m,k}}{h}w^{m-1,k-1}r^{m,k-1};
\end{aligned}$$

If we choose$\alpha=\alpha(h)\geq$, $\beta=\beta(h)$ large enough, such that $\beta-\frac{w^{m-1,k}}{h}>0$, $\frac{2c_1+1-w^{m-1,k}}{h}-\alpha>0$, this is possible because that (18). Then by Cauchy inequality, we have

$$\begin{aligned}
&\Phi_{\eta\eta}^{m,k}-\frac{1}{h}(\Phi^{m,k}-\Phi^{m-1,k})\\
+\ &\frac{w^{m-1,k}}{h}(\Phi^{m,k}-\Phi^{m,k-1})-\alpha\Phi^{m,k}+\beta\Phi^{m,k-1}\\
=\ &\Phi_{\eta\eta}^{m,k}-\frac{1}{h}(\Phi^{m,k}-\Phi^{m-1,k})\\
-\ &(\beta-\frac{w^{m-1,k}}{h})(\Phi^{m,k}-\Phi^{m,k-1})\\
-\ &(\alpha-\beta)\Phi^{m,k}>0. \qquad (28)
\end{aligned}$$

Now, we have two cases. The first cases is that at the maximal value point of $\Phi^{m,k}$, $\Phi^{m,k}-\Phi^{m-1,k}\geq 0$, then by the maximal principle, $\Phi^{m,k}$ can not attain its maximum in the interior of $(0,R)$. The second case is that at the maximal point of $\Phi^{m,k}$, $\Phi^{m,k}-\Phi^{m-1,k}\leq 0$, let $\tilde\Phi=e^{-\gamma mh}\Phi$. Then by (28)

$$\begin{aligned}
&\tilde\Phi_{\eta\eta}^{m,k}-\tfrac{1}{h}(\tilde\Phi^{m,k}-\tilde\Phi^{m-1,k})-\gamma e^{-\gamma h_1}\tilde\Phi^{m-1,k}\\
&-e^{-\gamma mh}(\beta-\tfrac{w^{m-1,k}}{h})(\tilde\Phi^{m,k}-\tilde\Phi^{m,k-1})-(\alpha-\beta)\tilde\Phi^{m,k}\\
&>0
\end{aligned}$$

and

$$\begin{aligned}
&\tilde\Phi_{\eta\eta}^{m,k}-\tfrac{1}{h}\tilde\Phi^{m,k}+(\tfrac{1}{h}-\gamma e^{-\gamma h_1})\tilde\Phi^{m-1,k}\\
&-e^{-\gamma mh}(\beta-\tfrac{w^{m-1,k}}{h})(\tilde\Phi^{m,k}-\tilde\Phi^{m,k-1})-(\alpha-\beta)\tilde\Phi^{m,k}\\
&>0
\end{aligned}$$

where $h_1<h$. If we choose $\gamma=\gamma(h)>\frac{1}{h}$ large enough, then $\tilde\Phi^{m,k}$can not attain its maximum in the interior of $(0,R)$. $\Phi(\eta)=e^{\gamma mh}\tilde\Phi(\eta)$ also can not attain its maximum in the interior of $(0,R)$.

(ii). If at the maximal value point of $\Phi^{m,k}$, $\Phi^{m,k}-\Phi^{m,k-1}\leq 0$, let $\Phi_1=\Phi+1$. Then

$$\begin{aligned}
&\Phi_{1\eta\eta}^{m,k}-\frac{1}{h}\Phi_1^{m,k}+\frac{w^{m-1,k}}{h}(\Phi_1^{m,k}-\Phi_1^{m,k-1})\\
&-\alpha\Phi_1^{m,k}+\beta\Phi_1^{m,k-1}\\
\geq\ & 2(\rho_\eta^{m,k})^2+2(r_\eta^{m,k})^2+(-\alpha+\beta-\frac{1}{h})\\
&+(\frac{2c_1+1-w^{m-1,k}}{h}-\alpha)[(\rho^{m,k})^2+(r^{m,k})^2]\\
&+(\beta-\frac{w^{m-1,k}}{h})[(\rho^{m,k-1})^2+(r^{m,k-1})^2]\\
&-\frac{2\rho^{m,k}}{h}\rho^{m-1,k}+\frac{2\rho^{m,k}}{h}w^{m-2,k}\rho^{m-1,k}\\
&-\frac{2r^{m,k}}{h}r^{m,k-1}+\frac{2r^{m,k}}{h}w^{m-1,k-1}r^{m,k-1};
\end{aligned}$$

If we choose $\alpha=\alpha(h)$, $\beta=\beta(h)$ large enough, such that$\frac{2c_1+1-w^{m-1,k}}{h}-\alpha>0$, $\frac{1}{h}+\alpha\leq\beta$, which implies that $\beta-\frac{w^{m-1,k}}{h}>0$, then by Cauchy inequality, we have

$$\begin{aligned}
&\Phi_{1\eta\eta}^{m,k}-\frac{1}{h}\Phi_1^{m,k}+\frac{w^{m-1,k}}{h}(\Phi_1^{m,k}-\Phi_1^{m,k-1})\\
&-\alpha\Phi_1^{m,k}+\beta\Phi_1^{m,k-1}\\
=\ &\Phi_{1\eta\eta}^{m,k}-\frac{1}{h}\Phi_1^{m,k}+\frac{w^{m-1,k}}{h}(\Phi_1^{m,k}-\Phi_1^{m,k-1})\\
&-(\alpha-\beta)(\Phi_1^{m,k}-\Phi_1^{m,k-1})-\beta\Phi_1^{m,k}>0.
\end{aligned}$$

By the maximal principle, $\Phi_1^{m,k}$ can not attain its maximum in the interior of $(0,R)$. Thus $\Phi^{m,k}$ can not attain its maximum in the interior of $(0,R)$ too.

When $\eta=R,\Phi^{m,k}=0$. When $\eta=0$, because $\phi(\eta,\xi,\tau)$ is a smooth function on $\overline{\Omega\times(0,T)}$, clearly we have $\Phi^{m,k}(0)\leq c$, so

$$\Phi^{m,k}(\eta)\leq c,\forall\eta\in[0,R]. \qquad (29)$$

Now by (15), $\mid w_{\eta\eta}^{m,k} \mid\leq c$. This implies that $\mid w_{\eta}^{m,k} \mid\leq c$. So Lemma 2 is proved.

Theorem 4 *Under the assumption of Lemma 2 and Lemma 3, the problem (9)-(11) admits a solution w with the following properties: w is continuous,*

$$\mid w \mid\leq K_1(R-\eta) \tag{30}$$

in the domain $(0,T)\times\Omega$, w has bounded weak derivatives w_η, w_ξ, w_τ,

$$\mid w_\xi \mid\leq c(R-\eta), \mid w_\tau \mid\leq c(R-\eta); \tag{31}$$

the weak derivative w_η, $w_{\eta\eta}$ exists and bounded , equation (15) holds almost everywhere in the same domain.

Proof : First, let us prove the uniqueness of the solution. Assume the contrary, namely, that w_1 and w_2 are two solutions of problem (9)-(11). Then, the function $z = w_1 - w_2$ satisfies the following equation

$$z_{\eta\eta}-z_\tau+zz_\xi+(w_1w_{2\xi}+w_2w_{1\xi}) = f(\cdot,w_1)-f(\cdot,w_2),$$

$$z(\eta,\xi,0) = 0, z\mid_{[0,T]\times\Omega}= 0.$$

Let $z_1 = e^{-\alpha\tau}z$. If we choose α large enough then it is easily to prove that $z\equiv 0$ by the maximal principle.

Now, we will prove the existence of the solution of (9)-(11). The solutions $w^{m,k}$ of problem (15)-(16) should be linearly extended to the domain $(0,T)\times\Omega$. First, when $(k-1)h < \xi \leq kh$, $k = 1,2,\cdots,k(h)$, $k(h) = [Nh]$, let

$$\begin{aligned} & w_h^m(\eta,\xi) = w_h^m(\eta,(k-1)h\lambda+(1-\lambda)kh) \\ = \; & (1-\lambda)w^{m,k}(\eta)+\lambda w^{m,k-1}(\eta). \end{aligned} \tag{32}$$

Secondly, when $(m-1)h < \tau < mh$, $m = 1,2,\cdots,m(h)$, $m(h) = [Th]$, let

$$\begin{aligned} & w_h(\tau,\xi,\eta) = w_h(\eta,\xi,mh(1-\sigma)+(m-1)h\sigma) \\ & = (1-\sigma)w_h^m(\eta,\xi)+\sigma w_h^{m-1}(\eta,\xi). \end{aligned} \tag{33}$$

According to Lemma 2, Lemma 3, the functions $w_h(\eta,\xi,\tau)$ from this family satisfy the Lipschitz condition with respect to ξ, τ, and have uniformly (in h) bounded first derivative in η for $0\leq\xi\leq N, 0\leq\eta\leq R$. By the Arzela Theorem, there is a sequence $h_i\to 0$ such that w_h uniformly converge to some $w(\eta,\xi,\tau)$. It follows from Lemma 2, Lemma 3 that $w(\eta,\xi,\tau)$ has bounded weak derivatives w_τ, $w_\xi, w_\eta, w_{\eta\eta}$ in $(0,T)\times\Omega$. Moreover,

$$\mid w_\xi \mid\leq c(R-\eta), \mid w_\tau \mid\leq c(R-\eta).$$

The sequence w_{h_i} may be assumed such that the derivatives $w_\tau, w_\xi, w_\eta, w_{\eta\eta}$ in the domain $(0,T)\times\Omega$ coincide with weak limits in $L^2((0,T)\times\Omega)$ of the respective functions

$$\frac{w_{h_i}(\eta,\xi,\tau+h_i)-w_{h_i}(\eta,\xi,\tau)}{h_i},$$

$$\frac{w_{h_i}(\eta,\xi+h_i,\tau)-w_{h_i}(\eta,\xi,\tau)}{h_i}, \quad w_{h_i\eta}, w_{h_i\eta\eta}.$$

Denoting $w_h^{m,k} = w_h(\eta,\xi,\tau) = w(\eta,kh,mh)$, by (15),

$$\begin{aligned} & w_{h\eta\eta}^{m,k} - \frac{w_h^{m,k}-w_h^{m-1,k}}{h} + w_h^{m-1,k}\frac{w_h^{m,k}-w_h^{m,k-1}}{h} \\ & -f(\cdot,w_h^{m,k}) = 0. \end{aligned} \tag{34}$$

Now, suppose that $\varphi(\eta,\xi,\tau)$ be a smooth function, which support set is compact in $(0,T)\times\Omega$. Let

$$\varphi^{m,k}(\eta) = \varphi(\eta,kh,mh).$$

Let us multiply with $h\varphi^{m,k}(\eta)$ at the two side of (34), integrating the resulting equation in η from 0 to R, and taking the sum over k,m from 1 to $k(h), m(h)$ respectively, we obtain

$$\begin{aligned} & \sum_{m,k} h\int_{-R}^{R}\varphi^{m,k}[w_{h\eta\eta}^{m,k} - \frac{w_h^{m,k}-w_h^{m-1,k}}{h} \\ & + w_h^{m-1,k}\frac{w_h^{m,k}-w_h^{m,k-1}}{h} - f(\cdot,w^{m,k})]d\eta = 0. \end{aligned} \tag{35}$$

Denote the function $\bar{f}\ (\tau,\xi,\eta)$ on $[0,T]\times\Omega$ as: for $(m-1)h < \tau < mh, (k-1)h < \xi\leq kh$,

$$\bar{f}(\eta,\xi,\tau) = f(\eta,kh,mh), \tag{36}$$

and denote

$$(\frac{\Delta w_h}{h})_1^m = \frac{w_h^{m,k}-w^{m-1,k}}{h},$$

$$(\frac{\Delta w_h}{h})_2^k = \frac{w_h^{m,k}-w^{m,k-1}}{h}.$$

Then we can rewrite (35) to

$$\begin{aligned} & \int_0^T\int(\bar{w}_{h\eta\eta}\bar{\varphi} - (\overline{\frac{\Delta w_h}{h}})_1^m\bar{\varphi} + (\overline{\frac{\Delta w_h}{h}})_2^k\bar{\varphi}\bar{w}_h \\ & -\overline{f(\cdot,w^{m-1,k})}\bar{\varphi})d\tau d\xi d\eta = 0. \end{aligned} \tag{37}$$

Since

$$\mid \bar{w}-w \mid\leq\mid \bar{w}-w_h \mid + \mid w_h-w \mid\leq Mh+\mid w_h-w \mid,$$

when $h\to 0$, $\bar{w}\to w$. Just likely, $\bar{\varphi}\to\varphi$, $\overline{f(\cdot,w^{m-1,k})\varphi}\to g(\cdot,w)\varphi$. At the same time,

$$(\overline{\frac{\Delta w_h}{h}})_1 \rightharpoonup w_\tau, (\overline{\frac{\Delta w_h}{h}})_2 \rightharpoonup w_\xi, \bar{w}_{h\eta\eta} \rightharpoonup w_{\eta\eta},$$

in $L^2((0,T)\times(0,R)\times(0,N))$, so, if let $h\to 0$ in (37), then

$$\int_0^T\int_0^R\int_0^N (w_{\eta\eta}-w_\tau+ww_\xi-g(z,w))\varphi d\tau d\xi d\eta = 0.$$

By the arbitrary of φ, we get ours result.

3 Parallel results

Consider the following initial boundary problem

$$w_{\eta\eta}-w_\tau+ww_\xi = f(\eta,\xi,\tau,u),(\xi,\eta,t)\in\Omega\times(0,T) \quad (38)$$

$$w(\eta,\xi,0) = w_0(\eta,\xi),\ (\xi,\eta)\in\Omega \quad (39)$$

$$w\mid_{\{\eta=0\}\times[0,T]} = \phi(0,\xi,t),\ w\mid_{\{\eta=R\}\times[0,T]} = 0, \quad (40)$$

$$w\mid_{\xi=0} = \varphi(\eta,0,\tau), \quad (41)$$

where $\Omega = (0,N)\times(0,R), w_0(0,\xi) = \phi(0,\xi,0)$ as before, φ is compatible with the functions w_0,ϕ, which satisfy the condition (14), and f satisfies the condition (12)-(13). In addition, we must assume that $K_2\le\frac{1}{2R}$,

$$\mid\varphi(\eta,\tau)\mid\le K_2(R-\eta),\varphi\le\frac{1}{2}. \quad (42)$$

Instead of equation (38)-(41), let us consider the following system of ordinary differential equations:

$$w_{\eta\eta}^{m,k}-\frac{w^{m,k}-w^{m-1,k}}{h}+w^{m-1,k}\frac{w^{m,k}-w^{m,k-1}}{h} -f(\cdot,w^{m,k}) = 0, \quad (43)$$

$$w^{m,k}\mid_{\eta=R} = 0,\ w^{m,k}\mid_{\eta=0} = \phi(0,kh,mh), \quad (44)$$

where

$$w^{0,k} = w_0(kh,\eta),\ w^{m,0}(\eta) = \varphi(\eta,mh),$$
$$m = 1,\cdots,[Th];\ k = 0,1,\cdots,[Nh].$$

Similarly, we have

Lemma 5 *Under the conditions of (11), (12) and (42), the problem (43)-(44) admits a unique solution for $kh\le N_0$ and small enough h, where N_0 is a suitable small positive number. The solution satisfies the following estimate*

$$V_0(\eta,kh)\le w^{m,k}\le V_1(\eta,kh), \quad (45)$$

where V_0,V_1 are continuous functions, positive in $(0,R)$, $V_1\le\frac{1}{2}$ and such that

$$V_0\equiv K_0(R-\eta),V_1\equiv K_1(R-\eta),K_1\le\frac{1}{2R}, \quad (46)$$

in a neighborhood of $\eta = R$.

Lemma 6 *Assume that the conditions of Lemma 5 are fulfilled, w_0 has bounded first order derivatives, $w_{0\eta\eta}$ is bounded, then*

$$w_\eta^{m,k},\frac{1}{h}(w^{m,k}-w^{m-1,k}),\frac{1}{h}(w^{m,k}-w^{m,k-1}),w_{\eta\eta}^{m,k}$$

are bounded for $kh\le N_1$ and $h\le h_0$, uniformly with respect to h, where the positive constants $N_1\le N_0$.

Theorem 7 *Under the assumption of Lemma 5 and Lemma 6, problem (38)-(41) admits a solution w with the following properties: w is continuous,*

$$\mid w\mid\le K_1(R-\eta) \quad (47)$$

in the domain $(0,T)\times\Omega$, w has bounded weak derivatives w_η,w_ξ,w_τ,

$$\mid w_\xi\mid\le c(R-\eta),\mid w_\tau\mid\le c(R-\eta); \quad (48)$$

the weak derivative w_η, $w_{\eta\eta}$ exists and bounded , equation (38) holds almost everywhere in the same domain.

4 The proof of Theorem 1

Let $p = \partial_x u$ and differential (1) with respect to x. Then we get

$$\partial_{xx}p+p\partial_y u+u\partial_y p-\partial_t p = \frac{\partial}{\partial u}f(\cdot,u)p.$$

Consider the following problem: for $z = (t,x,y)\in(0,T)\times\Omega$,

$$\partial_{xx}p+u\partial_y p-\partial_t p = \frac{\partial}{\partial u}f(\cdot,u)p-p\partial_y u = g(\cdot,p), \quad (49)$$

$$p(0,\cdot) = p_0(x,y) = u_{0y}(x,y), \quad (50)$$

$$p\mid_{\{x=0\}\times[0,T]} = u_{1x}(0,y,t),\ p\mid_{\{x=R\}\times[0,T]} = 0. \quad (51)$$

where $g(\cdot,p) = p(\frac{\partial}{\partial u}f(\cdot,u)-\partial_y u)$. By Theorem 4, we can assume that $g(\cdot,p)$ is a Lipschitz function and satisfies with (11). Just like the discussion of section 2, on account of (27)-(28), we are able to get the boundedness of the weak first order derivatives of p. Then $\partial_{xy}u = \partial_x p,\partial_{yt}u = \partial_t p,\partial_{xx}p = \partial_{xxx}u$ are bounded. Which means that $\partial_x u,\partial_{xx}u,\partial_y u,\partial_t u$ are actually continuous functions. So (1)-(3) has the solution in classical sense.

Similarly, one is able to prove the problem (38)-(41) has the solution in classical sense, and so the theorem is got.

5 The global solution of Cauchy problem

However, the known results in the equation (1) are all in local solutions, say, when T is suitably small positive constant. The methods used in these papers seem difficult to be generalized to the cases when T is an any given positive constant. The equation (1) is a degenerate parabolic equation, so it only has the weak global solution in general. To consider the weak global solution of the cauchy problem of (1)-(2) (certainly, the Ω in (1) should be changed to R^2), let us firstly consider the following nonlinear degenerate parabolic equation of the form

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x^i}(a^{ij}(u)\frac{\partial u}{\partial x^j})-\sum_{i=1}^{n}\frac{\partial b_i(u)}{\partial x_i} = c(t,x,u),\ (t,x)\in Q_T=(0,T)\times R^n, \quad (52)$$

$$u(0,\cdot)=u_0(x,y),\ \text{ in } R^N \quad (53)$$

where

$$a^{ij}\xi_i\xi_j\geq 0, \forall\xi=(\xi_1,\xi_2,\cdots,\xi_N)\in R^N, \quad (54)$$

and pairs of equal indices imply the summation from 1 up to N. Equation (52) arises in many applications, including two phase flow in porous media (cf. [14] and references cited therein), sedimentation-consolidation processes (cf. [15] and references cited therein).

We notice that (1) is a very special case of (52), say, a^{ij} in (1) has the form

$$(a^{ij})=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

We had got the posedness of (52) in [16] and [17], so we can get the existence and uniqueness of the global weak solution for the Cauchy problem (1)-(2) from [17] etc. In details, we can discuss the global weak solution as follows.

Following reference [18], $u\in BV(Q_T), Q_T=R^N\times(0,T)$ if and only if $u\in L^1_{loc}(Q_T)$ and

$$\int_0^T\int_{B_\rho}|\, u(x_1+h_1,\cdots,x_N+h_N,t+h_{N+1}) -u(x,t)\,|\,dxdt\leq K\,|\,h\,|, \quad (55)$$

where

$$B_\rho=\{x\in R^N;|\,X\,|<\rho\},\ h=(h_1,h_2,\cdots h_N,h_{N+1})$$

and K is a positive constant. This is equivalent to that the generalized derivatives of every function in $BV(Q_T)$ are regular Radon measures on Q_T.

Let $S_\eta(s)=\int_0^s h_\eta(\tau)d\tau$ for small $\eta>0$, where $h_\eta(s)=\frac{2}{\eta}(1-\frac{|s|}{\eta})_+$. Obviously $h_\eta(s)\in C(R)$ and

$$h_\eta(s)\geq 0,\ |\,sh_\eta(s)\,|\leq 1,\ |\,S_\eta(s)\,|\leq 1; \quad (56)$$

$$\lim_{\eta\to 0}S_\eta(s)=\text{sgn}(s),\ \lim_{\eta\to 0}sS'_\eta(s)=0, \quad (57)$$

where sgn represents the sign function.

According to the idea of [16] and [17], we can introduce the following notion.

Definition 8 *A function $u\in BV(Q_T)\cap L^\infty(Q_T)$ is said to be a weak solution of the problem (52)-(53) if*

1. *there exist the functions $g^i\in L^2_{loc}(Q_T), i=1,2,\cdots,N$ such that*

$$\int\int_{Q_T}\phi(t,x)\widehat{r^{ij}}(u)\frac{\partial u}{\partial x_i}dxdt = \int\int_{Q_T}\phi(x,t)g^i(t,x)dxdt. \quad (58)$$

for any $\phi\in C^2_0(Q_T)$, where $(r^{ij}(u))$ is the square root of the matrix $(a^{ij}(u))$, and $\widehat{r^{ij}}(u)$ is the composite mean value of r^{ij} and u as usual.

2. *u satisfies*

$$\int\int_{Q_T}\{I_\eta(u-k)\varphi_t-B^i_\eta(u,k)\varphi_{x_i} + A^{ij}_\eta(u,k)\varphi_{x_ix_j}+c(t,x,u)S_\eta(u-k)\varphi - \sum_{i=1}^{N}S'_\eta(u-k)(g^i)^2\varphi\}dxdydt\geq 0, \quad (59)$$

for any $\varphi\in C^2_0(Q_T)$, $\varphi\geq 0$, any $k\in R$, $\eta>0$.

3.

$$\lim_{\eta\to 0}\int_\Omega|\,u(t,x,y)-u_0(x,y)\,|\,dxdy=0, \quad (60)$$

where

$$I_\eta(u-k)=\int_0^{u-k}S_\eta(s-k)ds,$$

$$B^i_\eta(u,k)=\int_k^u b_{is}S_\eta(s-k)ds, \quad (61)$$

$$A^{ij}_\eta(u,k)=\int_k^u a^{ij}(s)S_\eta(s-k)ds. \quad (62)$$

Review the following general definition of the entropy solution for Cauchy problem of (52) (cf. [18] etc.),

$$\int\int_{Q_T}[|u-k|\varphi_t-\text{sgn}(u-k)(b_i(u)-b_i(k))\varphi_{x_i} +\text{sgn}(u-k)(A^{ij}(u)-A^{ij}(k))\varphi_{x_ix_j} +c(t,x,u)\varphi]dxdt\geq 0, \quad (63)$$

where

$$A^{ij}(u)=\int_0^u a^{ij}(s)ds,$$

on account of the arbitraries of the constant k, from (5.13), it is well known that (1.1) is true in the sense of distribution, i.e. for any $\phi(x,t)\in C_0^\infty(Q_T)$,

$$\begin{aligned}&\int\int_{Q_T}\frac{\partial u}{\partial t}\phi(t,x)\\&=\int\int_{Q_T}A^{ij}\phi_{x_ix_j}dxdt+\int\int_{Q_T}b_i\phi_{x_i}dxdt\\&+\int\int_{Q_T}c(\cdot,u)\phi dxdt.\end{aligned}\tag{64}$$

This fact means that the entropy solution is stronger than the general weak solution.

Clearly, (60) implies (64), the entropy solution defined in Definition 8 is stronger than the general one defined as (63). At the same time, (60) implies (64) certainly.

Considering the problem (1)-(2), the dimension of the space variables is $N=2$, we can simply denote $x=(x_1,x_2)=(x,y)$. Then we quote the following

Definition 9 *A function $u\in BV(Q_T)\cap L^\infty(Q_T)$ is said to be a weak solution of the problem (1)-(2) if u satisfies (58), (60) and*

$$\begin{aligned}&\int\int_{Q_T}\{I_\eta(u-k)\varphi_t-B_\eta(u,k)\varphi_y\\&+I_\eta(u-k)\varphi_{xx}-f(t,x,y,u)S_\eta(u-k)\varphi\\&-S'_\eta(u-k)(\partial_x u)^2\varphi\}dxdydt\geq 0,\end{aligned}\tag{65}$$

for any $\varphi\in C_0^2(Q_T)$, $\varphi\geq 0$, any $k\in R$, $\eta>0$. Here

$$B_\eta(u,k)=-\int_k^u sS_\eta(s-k)ds.$$

Immediately, by [16], [17], we have

Theorem 10 *Suppose that*

$$u_0(x,y)\in L^\infty(R^2),$$

$f_r(t,x,y,r)$ is bounded, and $u_0(x,y),f(t,x,y,r)$ are suitably smooth. Then problem (1)-(2) has a generalized solution in the sense of Definition 9.

The outline of the proof of Theorem 10: Consider the following regularized equation

$$\partial_{xx}u+u\partial_y u-\partial_t u-\epsilon\triangle u=f(\cdot,u),(t,x,y)\in(0,T]\times\Omega,$$

with the initial value (2). As in [18] we can prove

$$|u_\epsilon|\leq M,\tag{66}$$

$$\int_{Q_T}(a^{ij}(u_\epsilon)u_{\epsilon x_i}u_{\epsilon x_j}+\epsilon\mid\nabla u_\epsilon\mid^2)\omega_\lambda(x)dxdt\leq C\tag{67}$$

$$\int_{Q_T}(\mid\frac{\partial u_\epsilon}{\partial t}\mid+\mid\nabla u_\epsilon\mid)\omega_\lambda(x)dxdt\leq C\tag{68}$$

where C,M are constants independent of ϵ, and

$$\omega_\lambda(x,y)=\exp\{-\lambda\sqrt{1+x^2+y^2}\},$$

for a given positive constant λ.

Thus there exists a subsequence $\{u_{\epsilon_n}\}$ of $\{u_\epsilon\}$ and a function $u\in BV(Q_T)\cap L^\infty(Q_T)$ such that

$$u_{\epsilon_n}\to u,\quad \text{a.e. in } Q_T.$$

We now prove that u is a generalized solution of (1)-(2). From (67), we have

$$\int_{Q_T}\mid r^{ij}\frac{\partial u_\epsilon}{\partial xj}\mid^2 w_\lambda(x)dxdt\leq C\ \ i=1,\cdot,N,$$

where $\{r^{ij}\}$ is the square root of the matrix $\{a^{ij}\}$ generally, while for our special case, $r^{ij}=a^{ij}$. This means that $r^{ij}\frac{\partial u_\epsilon}{\partial xj}$ is weakly compact in $L^2_{loc}(Q_T)$. Without loss of generality, we may assume that $r^{ij}\frac{\partial u_\epsilon}{\partial xj}$ itself converges weakly in $L^2_{loc}(Q_T)$ to a function $g^i\in L^2_{loc}(Q_T)$. Thus for any $\phi\in C_0^1(Q_T)$

$$\begin{aligned}&\int_{Q_T}\phi g^i dxdt=\lim_{\epsilon\to 0}\int_{Q_T}\phi r^{ij}\frac{\partial u_\epsilon}{\partial xj}dxdt\\=&\lim_{\epsilon\to 0}\int_{Q_T}\phi(\int_0^{u_\epsilon}r^{ij}(s)ds)_{x_j}dxdt\\&-\int_{Q_T}\phi\int_0^u r^{ij}_{x_j}(s)dsdxdt\\=&-\int_{Q_T}\phi_{x_j}\int_0^u r^{ij}(s)dsdxdt\\&-\int_{Q_T}\phi\int_0^u r^{ij}_{x_j}(s)dsdxdt\\=&\int_{Q_T}\phi\widehat{r^{ij}(u)}\frac{\partial u}{\partial x_j}dxdt.\end{aligned}$$

This implies u satisfies (58) in Definition 8.

Let $\phi\in C_0^2(Q_T)$, $\phi\geq 0$, $k\in R$, $\eta>0$. Multiply (9) by $\phi S_\eta(u_\epsilon-k)$ and integrate over Q_T, we obtain

$$\begin{aligned}-&\int_{Q_T}[I_\eta(u_\epsilon-k)\varphi_t-B_\eta(u_\epsilon,k)\varphi_y+I_\eta(u_\epsilon-k)\varphi_{xx}\\&-S'_\eta(u_\epsilon-k)\varphi a^{ij}\frac{\partial u_\epsilon}{\partial x_i}\frac{\partial u_\epsilon}{\partial x_j}\\&+f(t,x,u_\epsilon)\varphi S_\eta(u_\epsilon-k)]dxdt\\-&\ \varepsilon\int_{Q_T}S'_\eta(u_\epsilon-k)(u_\epsilon-k)\varphi_{x_i}\frac{\partial u}{\partial x_i}dxdt\\-&\ \varepsilon\int_{Q_T}(u_\epsilon-k)S_\eta(u_\epsilon-k)\varphi_{x_ix_i}dxdt\end{aligned}$$

$$- \quad \varepsilon \int_{Q_T} S'_\eta(u_\epsilon - k)(\frac{\partial u_\epsilon}{\partial x_i})^2 \varphi dxdt = 0. \qquad (69)$$

Notice that, on the left hand side, the seventh term and the eighth term tend to zero as $\varepsilon \to 0$, the last term is nonnegative, and by (69), on account of that,

$$\begin{aligned} &\liminf_{\epsilon\to 0} \int_{Q_T} S'_\eta(u_\epsilon - k) a^{ij}(u_\epsilon) \frac{\partial u_\epsilon}{\partial x_i}\frac{\partial u_\epsilon}{\partial x_j} \phi dxdt \\ \geq & \int_{Q_T} S'_\eta(u-k) g^i g^i \phi dxdt \end{aligned}$$

Letting $\epsilon \to 0$ in (69), we get (65).

Theorem 11 *Let u, v be solutions of (1.1)-(1.2) with initial values $u_0(x), v_0(x) \in L^\infty(R^2)$ respectively. Then*

$$\begin{aligned} &\int_\Omega | u(x,y,t) - v(x,y,t) | \omega_\lambda(x,y) dxdy \\ \leq & C \int_\Omega | u_0 - v_0 | \omega_\lambda(x,y) dxdy. \qquad (70) \end{aligned}$$

The outline of the proof of Theorem 11: Let Γ_u be the set of all jump points of $u \in BV(Q_T), \nu$ the unit normal vector of Γ_u at $X = (x,y,t)$, $u^+(X)$ and $u^-(X)$ the approximate limits of u at $X \in \Gamma_u$ with respect to $(\nu, Y - X) > 0$ and $(\nu, Y - X) < 0$ respectively.

Let u be a solution of (52)-(53). Then we can prove that (see [16][17])

$$\int_{u^-}^{u^+} \gamma^{ij}(s,t,x) ds \nu_i = 0, \quad \text{a.e.}(t,x) \in \Gamma_u, j = 1,2. \qquad (71)$$

Let u, v be two generalized solutions of (1) with initial values

$$u(x_1,x_2,0) = u_0(x_1,x_2),\ v(y_1,y_2,0) = v_0(y_1,y_2).$$

By Definition 9, we have for any $\varphi \in C_0^2(Q_T),\ \varphi \geq 0,\ k,l \in R$,

$$\begin{aligned} &\int_{Q_T} \{I_\eta(u-k)\varphi_t - B_\eta(u,k)\varphi_{x_2} + I_\eta(u-k)\varphi_{x_1x_1} \\ &- \sum_{n=1}^N S'_\eta(u-k) g_1^j g_1^j \varphi \\ &- f(t,x_1,x_2,u)\varphi S_\eta(u-k)\} dxdt \geq 0, \qquad (72) \end{aligned}$$

$$\begin{aligned} &\int_{Q_T} \{I_\eta(v-l)\varphi_\tau - B_\eta(v,l)\varphi_{y_2} + I_\eta(v-l)\varphi_{y_1y_1} \\ &- \sum_{n=1}^N S'_\eta(v-l) g_2^j g_2^j \varphi \\ &- f(\tau,y_1,y_2,v)\varphi S_\eta(v-l)\} dyd\tau \geq 0. \qquad (73) \end{aligned}$$

For simplicity, we denote $x = (x_1,x_2), y = (y_1,y_2)$. Let $\psi(t,x,\tau,y) \geq 0$, $\psi \in C^2(Q_T \times Q_T)$, $\text{supp}\psi(\cdot,\cdot,\tau,y) \subset Q_T$ if $(\tau,y) \in Q_T$, $\text{supp}\psi(t,x,\cdot,\cdot) \subset Q_T$. We choose $k = v(\tau,y),\ l = u(t,x),\ \varphi = \psi(t,x,\tau,y)$ in (72) (73) and integrate over Q_T, to get

$$\begin{aligned} &\int_{Q_T}\int_{Q_T} \{I_\eta(u-v)(\psi_t+\psi_\tau) - (B_\eta(u,v)\psi_{x_2} + B_\eta(v,u)\psi_{y_2}) \\ &+ I_\eta(u-v)\psi_{x_1x_1} + I_\eta(v-u)\psi_{y_1y_1} \\ &- S'_\eta(u-v) \sum_{n=1}^N (g_1^j g_1^j + g_2^j g_2^j)\psi \\ &+ \psi(f(t,x,u)S_\eta(u-v) + f(\tau,y,v)S_\eta(v-u))\} \\ &\cdot dxdtdyd\tau \geq 0. \end{aligned} \qquad (74)$$

Choose $\psi(t,x,\tau,y) = \phi(t,x) j_h(t-\tau, x-y)$, where $\phi(t,x) \geq 0$, $\phi(t,x) \in C_0^\infty(Q_T)$, and

$$j_h(t-\tau, x-y) = \omega_h(t-\tau)\Pi_{i=1}^N \omega_h(x_i - y_i),$$

$$\omega_h(s) = \frac{1}{h}\omega(\frac{s}{h}),$$

$$\omega(s) \in C_0^\infty(R),\ \omega(s) \geq 0,\ \omega(s) = 0\ if\ \ | s |> 1,$$

$$\int_{-\infty}^{\infty} \omega(s) ds = 1.$$

Clearly

$$\frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = 0, \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0,\ i = 1,\cdots,N;$$

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \phi}{\partial t} j_h,\ \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i} j_h.$$

Then (74) becomes

$$\begin{aligned} &\int_{Q_T}\int_{Q_T} \{I_\eta(u-v)(\psi_t + \psi_\tau) \\ &- (B_\eta(u,v)\psi_{x_2} + B_\eta(v,u)\psi_{y_2}) \\ &+ I_\eta(u-v)\psi_{x_1x_1} + I_\eta(v,u)\psi_{y_1y_1} \\ &- S'_\eta(u-v) \sum_{n=1}^N (g_1^j g_1^j + g_2^j g_2^j)\psi \\ &+ \psi(f(t,x,u)S_\eta(u-v) + f(\tau,y,v)S_\eta(v-u))\phi j_h\} \\ &\cdot dxdtdyd\tau \geq 0. \qquad (75) \end{aligned}$$

Letting $\eta \to 0,\ h \to 0$ in (75), by (71), we are able to get

$$\begin{aligned} &\int_{Q_T} \{|u(x,t) - v(x,t)|\phi_t - \frac{1}{2}\text{sgn}(u-v)(u^2-v^2)\phi_{x_2} \\ &+ |u(x,t) - v(x,t)|\phi_{x_1x_1} \\ &+ \text{sgn}(u-v)(f(t,x,u) - f(t,x,v))\} \geq 0. \qquad (76) \end{aligned}$$

Let

$$\eta(t) = \int_{\tau-t}^{s-t} \alpha_\epsilon(\sigma) d\sigma,\ \ \epsilon < \min\{\tau, T-s\},$$

where $\alpha_\epsilon(t)$ is the kernel of mollifier with $\alpha_\epsilon(t) = 0$ for $t \notin (-\epsilon, \epsilon)$. By approximation, we can choose ϕ with $\phi(x,t) = \omega_\lambda(x)\eta(t)$ in (76), where $\omega_\lambda(x)$ is the function as before. Using the estimates

$$| \nabla\omega_\lambda | \leq C_\lambda\omega_\lambda(x), \quad | \Delta\omega_\lambda(x) | \leq C_\lambda\omega_\lambda(x),$$

and letting $\epsilon \to 0$ in (76), we obtain

$$\int_{R^N} |u(s,x) - v(s,x)|\omega_\lambda(x)dx$$

$$\leq \int_{R^N} |u(\tau,x) - v(\tau,x)|\omega_\lambda(x)dx$$

$$+C\int_\tau^s \int_{R^N} |u(t,x) - v(t,x)|\omega_\lambda(x)dxdt.$$

Hence by Gronwall lemma, we obtain

$$\int_{R^N} |u(s,x) - v(s,x)|\omega_\lambda(x)dx$$

$$\leq C\int_{R^N} |u(\tau,x) - v(\tau,x)|\omega_\lambda(x)dx.$$

Letting $\tau \to 0$, the proof of Theorem 11 is completed.

At the last of the paper, we would like to point that the uniqueness of the initial boundary problem of (52) is still an open problem. So, it also seem very difficult to solve the posedness of the global weak solution of (1)-(3) for the time being.

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