On a Degenerate Parabolic Equation from Finance

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Abstract: Consider the degenerate parabolic equation \( \partial_{xx} u + u \partial_y u - \partial_t u = f(\cdot, u) \), which comes from mathematics finance, and in which \( u(t, x) \) is the utility function of a agent’s decision under risk. By Oleinik’s line method, the existence and the uniqueness of the local classical solution for the initial boundary problem of the equation are got. Also, the global entropy solution of the Cauchy problem is discussed.

Key–Words: Degenerate parabolic equation, existence, uniqueness, initial boundary problem, global entropy solution

1 Introduction

In this paper, we consider the following initial boundary problem:

\[
\begin{align*}
\partial_{xx} u + u \partial_y u - \partial_t u &= f(\cdot, u), \quad (t, x, y) \in (0, T] \times \Omega \\
u(\cdot, 0) &= u_0, \quad (x, y) \in \Omega \\
u \mid_{\{x=0\} \times [0,T]} &= u_1(0, y, t), \quad u \mid_{\{x=R\} \times [0,T]} = 0
\end{align*}
\]

(1) (2) (3)

where \( \Omega = (0, R) \times (0, N) \subset R^2 \), \( T \) is a suitably small positive constant. The equation (1) arises in mathematics finance, arises when studying nonlinear physical phenomena such as the combined effects of diffusion and convection of matter (cf.[5]), many mathematicians have been interested in it. In [1], Antonelli, Barucci and Mancino introduce a new model for agent’s decision under risk, in which the utility function is the solution to (1)-(2). In the sense of the User’s guide, i.e.

\[
| u(x, y, t) - u(\eta, \xi, t) | \leq C_T(| x - \eta | + | y - \xi |)
\]

for every \((x, y), (\xi, \eta) \in R^2, t \in [0, T)\), under the assumption that \( f \) is uniformly Lipschitz continuous function, Crandall, Ishii and Lions [2] proved the existence of a continuous viscosity solution by means of probability methods. In [3], Citti, Pascucci and Polidoro studied the interior regularity, they proved that the viscosity solutions are indeed in classical sense. In [4], Antonelli and Pascucci showed that \( u \) is the limit, uniformly on compacts of \([0, T] \times R^2\), of the family of solutions to the regularized Cauchy Problem: for \((x, y, t) \in R^2 \times (0, T] \),

\[
\varepsilon^2 \partial_{yy} u + \partial_{xx} u + u \partial_y u - \partial_t u = f(\cdot, u),
\]

(4)

\[
\varepsilon^2 \partial_{yy} u + \partial_{xx} u + u \partial_y u - \partial_t u = f(\cdot, u),
\]

(5)

1 Other related work, one can refer to [7] etc. However, all of the published papers study the Cauchy problem and get the local classical solutions. As for the existence and uniqueness of the global weak solution for the cauchy problem of (1), there are some differential ways to deal with them, for example, (1) is the special case of the Cauchy problem discussed in [8],[9] etc., we will simply narrate this aspect in the last section of the paper. The main aim of the paper is to study the initial boundary problem (1)-(3).

Clearly, (1) is a degenerate parabolic equation on account of that it lacks the two order partial derivative term \( \partial_{yy} u \). It is well-known that there are some rules in how to quote an initial boundary problem of a degenerate parabolic equation, one can refer to Oleinik’s books [6],[10] etc. According to these rules, we quote the problem as the form of (1)-(3). We will discuss this problem in a complete different way comparing to [1]-[4].

In order to describe our method, we have to quote the well-known Prandtl system for a non-stationary boundary layer arising in an axially symmetric incompressible flow past a solid body. As well known, Prandtl proposed the conception of the boundary layer in 1904[11]. From then on, the interest in the theory of boundary layer has been steadily growing, due to the mathematical questions it poses, and its important practical applications. According to Prandtl boundary layer theory, the flow about a solid body can be divided into two regions: a very thin layer in the neighborhood of the body (the boundary layer) where viscous friction plays an essential part, and the remaining region outside this layer where friction may be ne-
neglected (the outer flow). Thus, for fluids whose viscosity is small, its influence is perceptible only in a very thin region adjacent to the walls of a body in the flow; the said region, according to Prandtl, is called the boundary layer. This phenomenon is explained by the fact that the fluid sticks to the surface of a solid body and, this adhesion inhibits the motion of a thin layer of fluid adjacent to the surface. In this thin region the velocity of the flow past a body at rest undergoes a sharp increase: from zero at the surface to the values of the velocity in the outer flow, where the fluid may be regarded as frictionless. Prandtl derived the system of equations for the first approximation of the flow velocity in the boundary layer. This system served as a basis for the development of the boundary layer theory, which has now become one of the fundamental parts of fluid dynamics. Assume that the flow velocity in the boundary layer. This system, one can refer to [6] for details.

By the way, for the best knowledge of the author, this is the first paper on the initial boundary problem (1)-(3) has a unique solution in classical sense provided that $t \leq T$, $T$ is suitable small (or $y \leq N$, $N$ is suitable small), and moreover, the first order and second order derivatives of $u$ are bounded.

The main result of the paper is the following

**Theorem 1** Assume that

$$|u_0(x, y)| \leq c(R - x), (x, y) \in \Omega,$$  

(8)

the first order and second order derivatives of $u_0$ are bounded, $u_{0xx}$ is bounded too. Assume that $u_1$ is continuous and smooth near $\partial\Omega$, its first order, second order derivatives at $x = 0$ are all bounded. Suppose $f$ satisfies (11) below and is an uniformly Lipschitz continuous function. Then the initial boundary problem (1)-(3) has a unique solution in classical sense provided that $t \leq T$, $T$ is suitable small (or $y \leq N$, $N$ is suitable small), and moreover, the first order and second order derivatives of $u$ are bounded.

2 **Line method**

For the comparability of signs with the Prandtl system, we rewrite (1)–(3) as following: for $(\xi, \eta, t) \in \Omega \times (0, T)$,

$$w_{\eta\eta} - w_{r} + w w_{\xi} = f(\xi, \eta, \tau, u),$$  

(9)

$$w(\eta, \xi, 0) = w_0(\eta, \xi), (\xi, \eta) \in \Omega$$  

(10)

$$w |_{\{\eta=0\} \times [0,T]} = \phi(0, \xi, t), w |_{\{\eta=R\} \times [0,T]} = 0$$  

(11)

where $\Omega = (0, N) \times (0, R)$, $w_0 \in C^2(\Omega)$, its first order derivatives and $w_{0\gamma\eta}$ are all bounded, $\phi(\eta, \xi, t)$ is a smooth function on $\Omega \times (0, T)$, $w_0(0, \xi) = \phi(0, \xi, 0)$. and $f$ is a Lipschitz continuous function which satisfies that: when $w_1 - w_2 \geq 0$,

$$c_2(w_1 - w_2) \geq f(\cdot, w_1) - f(\cdot, w_2) \geq c_1(w_1 - w_2)$$

(12)

and

$$|f(\cdot, w)| \leq c |w|^p$$

(13)

for some nonnegative number $p$. Moreover, we assume that $K_1 \leq \frac{1}{2\pi}$.

$$|w_0(\eta, \xi)| \leq K_1(R - \eta), (\eta, \xi) \in \Omega, \phi \leq \frac{1}{2}$$

(14)
It is regret that the author does not know, to get the results of theorem 1.1, whether the condition (14) is necessary or not, it seems that (14) is only a technique request in the proof. If one is able to refine the proof, the (14) may be abandoned or be weaker.

For any functions, we use the following notation

\[ f^{m,k}(\eta) = f(\eta, kh, mh), h = \text{const} > 0. \]

Instead of equation (9)-(11), let us consider the following system of ordinary differential equations:

\[ w^{m,k}_\eta = \frac{w^{m,k} - w^{m-1,k}}{h} + w^{m-1,k} \frac{w^{m,k} - w^{m,k-1}}{h} - f(\cdot, w^{m,k}) = 0, \]

\[ w^{m,k}|_{\eta=R}= 0, w^{m,k}|_{\eta=0} = \phi(0, kh, mh), \]

where

\[ w^{0,k} = w_0(kh, \eta), m = 1, \ldots, [Th]; k = 0, 1, \ldots, [Nh] \]

If \( k = 0, \) (15) should be

\[ \frac{w^{m,0}_\eta - w^{m-1,0}_\eta}{h} - f(\cdot, w^{m,0}) = 0 \]

The solutions of (15)-(16) are defined in the classical sense and we will prove that

\[ w^{m,k}_\eta, w^{m,k}, \frac{w^{m,k} - w^{m-1,k}}{h}, \frac{w^{m,k} - w^{m,k-1}}{h} \]

are uniformly bounded for any \( m, k. \)

**Lemma 2** Under the conditions of (11)-(13), the problem (15)-(16) admits a unique solution for \( mh \leq T_0 \) and small enough \( h, \) where \( T_0 \) is a suitable small positive number. The solution satisfies the following estimate

\[ V_0(\eta, mh) \leq w^{m,k} \leq V_1(\eta, mh) \]

where \( V_0, V_1 \) are continuous functions, positive in \((0, R), V_1 \leq \frac{1}{2} \) and such that

\[ V_0 \equiv K_0(R-\eta), V_1 \equiv K_1(R-\eta) \]

in a neighborhood of \( \eta = R, \) where \( K_1 \leq \frac{1}{2\pi} \) as before.

**Proof:** By (15) and (17), the existence of \( w^{m,k} \) is clearly. Let \( Q^{m,k} \) be the difference of two solution \( w_1^{m,k}, w_2^{m,k}. \) Then \( Q^{m,k} \) can attain neither a positive maximum nor a negative minimum at \( \eta = 0, R. \) By the inductive assumption, \( |w^{m,k}| \leq \frac{1}{2}, \) so

\[ 0 = L_{m,k}(Q^{m,k}) \]

\[ = Q^{m,k}_{\eta}\eta - \frac{1}{h}Q^{m,k} + w^{m-1,k} \frac{1}{h} Q^{m,k} + f(\cdot, w^{m,k}) - f(\cdot, w^{m,k}) \]

\[ Q^{m,k} \] can attain neither a positive maximum nor a negative minimum in interior of \((0, R)\) by (12)-(13), provided that \( h \leq h_0 \) small enough. Consequently, under our assumption, problem (15) cannot have more than one solution. Therefore, we shall prove (18) for \( m \) and \( k \) under the assumption of that the solutions \( w^{m-1,k} \) of (15) admit the following a priori estimate

\[ V_1(\eta, (m-1)h) \geq w^{m-1,k} \geq V_0(\eta, (m-1)h). \]

Denote that

\[ L_{m,k}(u) = u^{m,k}_\eta - \frac{1}{h} (w^{m,k} - w^{m-1,k}) + w^{m-1,k} \frac{1}{h} (V_1^{m,k} - V_1^{m-1,k}) \]

In order to prove the priori estimate (18) for \( \tau = mh, \) it suffices to show that there exist function \( V_1 \) with the properties specified in Lemma 2 and such that

\[ 0 \geq L_{m,k}(V_1) = V_1^{m,k}_\eta - \frac{1}{h} (V_1^{m,k} - V_1^{m-1,k}) \]

\[ + w^{m-1,k} \frac{1}{h} (V_1^{m,k} - V_1^{m-1,k}) - f(\cdot, V_1^{m,k}) \]

and

\[ V_1(0, mh) \geq \phi(0, kh, mh) \]

under assumption (21). Then the inequality (20) can be proved by induction with respect to \( m. \) Indeed, let \( q^{m,k} = V_1 - w^{m,k}. \) Then \( q^{m,k}(0) \geq 0 \) and

\[ 0 \geq L_{m,k}(V_1) = q^{m,k}_\eta - \frac{1}{h} (q^{m,k} - q^{m-1,k}) \]

\[ + w^{m-1,k} \frac{1}{h} (q^{m,k} - q^{m-1,k}) \]

\[ + f(\cdot, q^{m,k}) - f(\cdot, w^{m,k}) \]

\[ \geq q^{m,k}_\eta - \frac{1}{h} (q^{m,k} - q^{m-1,k}) \]

\[ + w^{m-1,k} \frac{1}{h} (q^{m,k} - q^{m-1,k}) + c_2 q^{m,k} \]

(22)

Let \( q_1^{m,k} = e^{c_2 h} q^{m,k}. \) Then

\[ 0 \geq q_1^{m,k}_\eta - \frac{1}{h} (q_1^{m,k} - q_1^{m-1,k}) \]

\[ + w^{m-1,k} \frac{1}{h} (q_1^{m,k} - q_1^{m-1,k}) \]

\[ + \alpha e^{c_2 h} w^{m-1,k} + c_2 q_1^{m,k} \]

\[ \geq q_1^{m,k}_\eta - \frac{1}{h} (1 - w^{m-1,k}) - c_2 q_1^{m,k} + \frac{1}{h} q_1^{m-1,k} \]
where $0 < h' < h$. By (23), if we choose $\alpha = \alpha(h)$ large enough, then it is easily to know that $q_{m,k}^\eta$ can not attain negative minimum in interior of $(0, R)$ by maximal principle.

$$q_{m,k}^\eta|_{\eta=0} = e^{\alpha m h} q_{m,k}^\eta|_{\eta=0} \geq 0$$

$$q_{m,k}^\eta|_{\eta=R} = e^{\alpha m h} q_{m,k}^\eta|_{\eta=R} = 0,$$

so (18) is true.

Under the condition (21), let us show that there is a positive $T_0$ such that for $mh \leq T_0$ there exist function $V_1$ satisfying the desired inequality. Let $\varphi_1(s)$ be a smooth function such that for $\eta > \frac{1}{2} R$,

$$\varphi_1(s) = R - \eta,$$

for $\frac{1}{4} R \leq s \leq \frac{1}{2} R$,

$$\frac{R}{2} \leq \varphi_1 \leq R,$$

for $\eta < \frac{1}{4} R$,

$$\varphi_1(s) = R.$$

Set

$$V_1 = M \varphi_1(\eta) \varphi_2(\beta_1 \eta) e^{\beta_2 m h},$$

(24)

where $\varphi_2$ is a smooth function such that for $0 \leq s \leq R$,

$$\varphi_2(s) = 4 - e^{\frac{1}{2} s},$$

for $s \geq 2 R$,

$$\varphi_2(s) = 1,$$

for $R \leq s \leq 2 R$

$$1 \leq \varphi_2(s) \leq 3.$$

The constant $M$ is chosen from the condition

$$V_1(\eta, 0, 0) \geq \frac{1}{2}. \quad \text{The positive constants } \beta_1, \beta_2 \text{ will be specified shortly.}$$

Clearly

$$L(V_1) = \frac{M e^{\beta_2 m h} (\varphi_1(\eta)) \varphi_2(\beta_1 \eta)) \eta}{h} + \frac{M \varphi_1(\eta) \varphi_2(\beta_1 \eta) (e^{\beta_2 m h} - e^{\beta_2 (m-1) h})}{h} - f(\cdot, w_{m,k}).$$

For a given small positive number $\delta$, if $R - \eta < \delta$, we can choose $\beta_1$ such that $\beta_1 \eta \geq 2 R$ then

$$\varphi_1(\eta) \varphi_2(\beta_1 \eta) = R - \eta,$$

so choosing $\beta_2$ large enough, $mh \leq T_0$ small enough

$$L(V_1) = - \frac{1}{h} M(R-\eta)(e^{\beta_2 m h} - e^{\beta_2 (m-1) h}) + c_1(w_{m,k}^2 + c_2)$$

$$\leq - \frac{1}{h} M(R-\eta)(e^{\beta_2 m h} - e^{\beta_2 (m-1) h}) \leq \frac{1}{h} M(R-\eta)(e^{\beta_2 m h})^p$$

$$+ c_1 M(R-\eta)(e^{\beta_2 m h})^p$$

$$\leq M(R-\eta)(e^{\beta_2 m h} - e^{3/2 (m-1) h})$$

$$+ c_3 M e^{\beta_2 m h} + c_4(\eta) \leq 0.$$
and for \( k = 0, m \geq 1 \)

\[
\Phi^{m,k}(\eta) = \left( \frac{m^{m,k} - m^{m-1,k}}{h} \right)^2. 
\]  

(26)

Also, we need to define \( \Phi^{0,k}(\eta) = \left( \frac{w^{0,k} - w^{0,k-1}}{h} \right)^2 \). So let \( w^{-1,k} = w^{-1,k}(\eta, kh) \) be a bounded function such that

\[
\frac{w^{0,k} - w^{0,k-1}}{h} = \frac{w^{0,k} + w^{-1,k}w^{0,k} - w^{0,k-1}}{h} - f(\cdot, w^{0,k}). 
\]  

(27)

\( w^{-1,k} \) is well defined because \( \frac{1}{h} + \frac{w^{0,k}}{h} > 0 \). Clearly, on account of \( w_0 \) has bounded first order derivatives and \( w_0(\eta) \) is bounded, \( \frac{w^{0,k} - w^{0,k-1}}{h} \) is uniformly bounded with respect to \( h \), so \( |\Phi^{0,k}| \leq c \).

Let

\[
r^{m,k} = \frac{w^{m,k} - w^{m-1,k}}{h}, \quad r^{m,k} = \frac{w^{m,k} - w^{m-1,k}}{h}. 
\]

Now we will deduce the equation for \( \Phi^{m,k}(\eta) \). To this end, we subtract from equation (15) for \( w^{m,k} \) equation (13) for \( w^{m-1,k} \) and multiply the result by \( \frac{2r^{m,k}}{h} \) to get the first equation; from (15) for \( w^{m,k} \) we subtract (13) for \( w^{m,k-1} \) and multiply the result by \( \frac{2r^{m,k}}{h} \) to get the second equation. We find the equations for \( \Phi^{m,k}(\eta) \) with \( k = 0, m \geq 1 \) by taking only the first equation.

In order to derive the equation for \( \Phi^{m,k}(\eta) \) with \( m = 1 \), we utilize the relation (27) which defines the values of \( w^{-1,k} \). Taking the sum of the three equations just obtained we get the equation for \( \Phi^{m,k}(\eta), k \geq 1 \).

Say, we have

\[
((15)^{m,k} - (15)^{m-1,k}) \frac{2r^{m,k}}{h} = 2r^{m,k}r^{m,k} - \frac{2r^{m,k}}{h}(r^{m,k} - r^{m-1,k}) + \frac{2r^{m,k}}{h}(w^{m-1,k}r^{m,k} - w^{m-1,k}r^{m-1,k}) - \frac{2r^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m-1,k})); 
\]

\[
(18)^{m,k} - (18)^{m-1,k}) \frac{2r^{m,k}}{h} = 2r^{m,k}r^{m,k} - \frac{2r^{m,k}}{h}(r^{m,k} - r^{m-1,k}) + \frac{2r^{m,k}}{h}(w^{m-1,k}r^{m,k} - w^{m-1,k}r^{m-1,k}) - \frac{2r^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m-1,k})); 
\]

\[
((18)^{m,k} - (18)^{m-1,k}) \frac{2r^{m,k}}{h} = 2r^{m,k}r^{m,k} - \frac{2r^{m,k}}{h}(r^{m,k} - r^{m-1,k}) + \frac{2r^{m,k}}{h}(w^{m-1,k}r^{m,k} - w^{m-1,k}r^{m-1,k}) - \frac{2r^{m,k}}{h}(f(\cdot, w^{m,k}) - f(\cdot, w^{m-1,k})); 
\]

By (12)-(13), we have

\[
\Phi^{m,k} - \frac{1}{h}(\Phi^{m,k} - \Phi^{m-1,k}) + \frac{w^{m-1,k}}{h}(\Phi^{m,k} - \Phi^{m-1,k}) 
\]
\[ \begin{align*}
& \geq 2(\rho_{m,k}^2) + 2(r_{\eta}^2) - \frac{1}{h}(\rho_{m,k}^2 + (r_{m,k})^2) \\
& + \frac{1}{h}(\rho_{m-1,k}^2 + (r_{m-1,k})^2) \\
& + \frac{w_{m-1,k}}{h}[((\rho_{m,k})^2 - (\rho_{m,k-1})^2) \\
& + \frac{w_{m-1,k}}{h}[(r_{m,k})^2 - (r_{m,k-1})^2] \\
& + \frac{2\rho_{m,k}}{h}(\rho_{m,k} - \rho_{m-1,k}) \\
& - \frac{2\rho_{m,k}}{h}[w_{m-1,k}(\rho_{m,k} - \rho_{m,k-1})] \\
& - \frac{2\rho_{m,k}}{h}[(w_{m-1,k} - w_{m-1,k-1})\rho_{m-1,k}] \\
& + \frac{2c_1}{h}(\rho_{m,k}^2 + \frac{2r_{\eta}}{h}(r_{m,k} - r_{m,k-1}) \\
& - \frac{2c_1}{h}(w_{m-1,k} - w_{m-1,k-1},r_{m,k-1}) \\
& + \frac{2c_1}{h}(\rho_{m,k}^2 + (r_{m,k})^2) \\
& + \frac{1}{h}[(\rho_{m,k}^2) + (r_{m,k})^2] \\
& \geq 2(\rho_{m,k}^2) + 2(r_{\eta}^2) + \frac{(\rho_{m,k})^2}{h} + (r_{m,k})^2) \\
& + \frac{1}{h}[(\rho_{m,k}^2) + (r_{m,k})^2] \\
& + \frac{1}{h}[(\rho_{m,k}^2) + (r_{m,k})^2] \\
& - \frac{2\rho_{m,k}}{h}[(w_{m-1,k} - w_{m-1,k-1})\rho_{m-1,k}] \\
& - \frac{2\rho_{m,k}}{h}w_{m-1,k-1} + \frac{2r_{\eta}}{h}w_{m-1,k-1} - \alpha > 0,
\end{align*} \]

If we choose \( \alpha = \alpha(h) \geq \beta = \beta(h) \) large enough, such that \( \beta - \frac{w_{m-1,k}}{h} > 0 \), then by Cauchy inequality, we have

\[ \begin{align*}
& \Phi_{m,k}^\eta - \frac{1}{h}(\Phi_{m,k} - \Phi_{m-1,k}) \\
& + \frac{w_{m-1,k}}{h}(\Phi_{m,k} - \Phi_{m,k-1}) - \alpha \Phi_{m,k} + \beta \Phi_{m,k-1} \\
& = \Phi_{m,k}^\eta - \frac{1}{h}(\Phi_{m,k} - \Phi_{m-1,k}) \\
& - (\beta - \frac{w_{m-1,k}}{h})(\Phi_{m,k} - \Phi_{m,k-1}) \\
& - (\alpha - \beta) \Phi_{m,k} > 0.
\end{align*} \] (28)

Now, we have two cases. The first case is that at the maximal value point of \( \Phi_{m,k}, \Phi_{m,k} - \Phi_{m-1,k} \geq 0, \) then by the maximal principle, \( \Phi_{m,k} \) can not attain its maximum in the interior of \((0, R)\). The second case is that at the maximal point of \( \Phi_{m,k}, \Phi_{m,k} - \Phi_{m-1,k} \leq 0, \) let \( \Phi = e^{-\gamma h} \Phi \). Then by (28)

\[ \begin{align*}
& \Phi_{m,k}^\eta - \frac{1}{h}(\Phi_{m,k} - \Phi_{m-1,k}) - \gamma e^{-\gamma h} \Phi_{m,k} \\
& - e^{-\gamma h}(\beta - \frac{w_{m-1,k}}{h})(\Phi_{m,k} - \Phi_{m,k-1}) - (\alpha - \beta) \Phi_{m,k} \\
& \geq 0
\end{align*} \]

and

\[ \begin{align*}
& \Phi_{m,k}^\eta - \frac{1}{h}(\Phi_{m,k} - \Phi_{m-1,k}) - \gamma e^{-\gamma h} \Phi_{m,k} \\
& - e^{-\gamma h}(\beta - \frac{w_{m-1,k}}{h})(\Phi_{m,k} - \Phi_{m,k-1}) - (\alpha - \beta) \Phi_{m,k} \\
& \geq 0
\end{align*} \]

where \( h_1 < h \). If we choose \( \gamma = \gamma(h) \) large enough, then \( \Phi_{m,k} \) can not attain its maximum in the interior of \((0, R)\). \( \Phi(\eta) = e^{-\gamma h} \Phi(\eta) \) also can not attain its maximum in the interior of \((0, R)\).

(iii). If at the maximal value point of \( \Phi_{m,k}, \Phi_{m,k} - \Phi_{m-1,k} \leq 0, \) let \( \Phi = \Phi + 1 \). Then

\[ \begin{align*}
& \Phi_{m,k}^\eta - \frac{1}{h}(\Phi_{m,k} - \Phi_{m-1,k}) \\
& - \alpha \Phi_{m,k} + \beta \Phi_{m,k-1} \\
& \leq 2(\rho_{m,k}^2) + 2(r_{\eta}^2) + (\alpha + \beta - \frac{1}{h}) \\
& + \frac{2c_1 + 1 - \frac{w_{m-1,k}}{h}}{h}[(\rho_{m,k}^2) + (r_{m,k}^2)] \\
& + (\beta - \frac{w_{m-1,k}}{h})((\rho_{m,k}^2) + (r_{m,k}^2)] \\
& + \frac{1}{h}[(\rho_{m,k}^2) + (r_{m,k}^2)] \\
& + \frac{2\rho_{m,k}}{h}[(w_{m-1,k} - w_{m-1,k-1})\rho_{m-1,k}] \\
& - \frac{2\rho_{m,k}}{h}w_{m-1,k-1} + \frac{2r_{\eta}}{h}w_{m-1,k-1} - \alpha > 0.
\end{align*} \]

If we choose \( \alpha = \alpha(h), \beta = \beta(h) \) large enough, such that \( 2\rho_{m,k} + \frac{w_{m-1,k}}{h} = \alpha > 0, \) \( \alpha \leq \beta \), which implies that \( \beta - \frac{w_{m-1,k}}{h} > 0 \), then by Cauchy inequality, we have

\[ \begin{align*}
& \Phi_{m,k}^\eta - \frac{1}{h}(\Phi_{m,k} - \Phi_{m-1,k}) \\
& - \alpha \Phi_{m,k} + \beta \Phi_{m,k-1} \\
& = \Phi_{m,k}^\eta - \frac{1}{h}(\Phi_{m,k} - \Phi_{m-1,k}) \\
& - (\alpha - \beta)(\Phi_{m,k} - \Phi_{m,k-1}) - \beta \Phi_{m,k} > 0.
\end{align*} \]

By the maximal principle, \( \Phi_{m,k}^\eta \) can not attain its maximum in the interior of \((0, R)\). Thus \( \Phi_{m,k} \) can not attain its maximum in the interior of \((0, R)\) too.

When \( \eta = R, \Phi_{m,k} = 0 \). When \( \eta = 0 \), because \( \phi(\eta, \xi, \tau) \) is a smooth function on \( \Omega \times (0, T) \), clearly we have \( \Phi_{m,k}(0) \leq c \), so

\[ \Phi_{m,k}(\eta) \leq c, \forall \eta \in [0, R]. \] (29)
Now by (15), $|w_{m,k}^n| \leq c$. This implies that $|w_{m,k}| \leq c$. So Lemma 2 is proved.

**Theorem 4** Under the assumption of Lemma 2 and Lemma 3, the problem (9)-(11) admits a solution $w$ with the following properties: $w$ is continuous,

$$|w| \leq K_1(R - \eta)$$

(30)

in the domain $(0, T) \times \Omega$, $w$ has bounded weak derivatives $w_\eta, w_\xi, w_\tau$.

$$|w_\xi| \leq c(R - \eta), |w_\tau| \leq c(R - \eta);$$

(31)

the weak derivative $w_\eta$, $w_{\eta\eta}$ exists and bounded, equation (15) holds almost everywhere in the same domain.

**Proof** : First, let us prove the uniqueness of the solution. Assume the contrary, that $w_1$ and $w_2$ are two solutions of problems (9)-(11). Then, the function $z = w_1 - w_2$ satisfies the following equation

$$z_{\eta\eta} + z_{\xi} + (w_1 w_2 + w_2 w_1)z = f(\cdot, w_1) - f(\cdot, w_2),$$

$$z(\eta, \xi, 0) = 0, z|_{|0, T) \times \Omega} = 0.$$ Let $z_1 = e^{-\alpha t}z$. If we choose $\alpha$ large enough then it is easily to prove that $z \equiv 0$ by the maximal principle.

Now, we will prove the existence of the solution of (9)-(11). The solutions $w_{m,k}^n$ of problem (15)-(16) should be linearly extended to the domain $(0, T) \times \Omega$. First, when $(k - 1)h < \xi \leq kh$, $k = 1, 2, \cdots, k(h), k(h) = [Nh]$, let

$$w_{m,k}^n(\eta, \xi) = w_{m,k}^n(\eta, (k - 1)h)\lambda + (1 - \lambda)hkh$$

(32)

$$= (1 - \lambda)w_{m,k}^n(\eta) + \lambda w_{m,k}^{n-1}(\eta).$$

Secondly, when $(m - 1)h < \tau < mh$, $m = 1, 2, \cdots, m(h), m(h) = [Th]$, let

$$w_{h}(\tau, \xi, \eta) = w_{h}(\eta, \xi, mh(1 - \sigma) + (m - 1)h\sigma)$$

$$= (1 - \sigma)w_{h}(\eta, \xi) + \sigma w_{h}^{m-1}(\eta, \xi).$$

(33)

According to Lemma 2, Lemma 3, the functions $w_h(\eta, \xi, \tau)$ from this family satisfy the Lipschitz condition with respect to $\xi, \tau$, and have uniformly (in $h$) bounded first derivative in $\eta$ for $0 \leq \xi \leq N, 0 \leq \eta \leq R$. By the Arzela Theorem, there is a sequence $h_i \to 0$ such that $w_h$ uniformly converge to some $w(\eta, \xi, \tau)$. It follows from Lemma 2, Lemma 3 that $w(\eta, \xi, \tau)$ has bounded weak derivatives $w_\tau, w_\xi, w_{\eta\eta}$ in $(0, T) \times \Omega$. Moreover,

$$|w_\xi| \leq c(R - \eta), |w_\tau| \leq c(R - \eta).$$

The sequence $w_{hi}$ may be assumed such that the derivatives $w_\tau, w_\xi, w_{\eta\eta}, w_{\eta\eta}$ in the domain $(0, T) \times \Omega$ coincide with weak limits in $L^2((0, T) \times \Omega)$ of the respective functions

$$w_{hi}(\eta, \xi, \tau + h) - w_{hi}(\eta, \xi, \tau),$$

$$w_{hi}(\eta, \xi + h, \tau) - w_{hi}(\eta, \xi, \tau),$$

$$w_{hi,\eta}, w_{hi,\eta\eta}.$$ Denoting $w_{m,k}^n = w_h(\eta, \xi, \tau) = w(\eta, kh, mh)$, by (15),

$$w_{hi,k}^m = \frac{w_{m,k}^n - w_{m-1,k}^n}{h} + w_{m-1,k}^m w_{m,k}^n - w_{m,k-1}^n,$$

(34)

$$- f(\cdot, w_{m,k}^n) = 0.$$ Now, suppose that $\phi(\eta, \xi, \tau)$ be a smooth function, which support set is compact in $(0, T) \times \Omega$. Let

$$\phi_{m,k}(\eta) = \phi(\eta, kh, mh).$$

Let us multiply with $h\phi_{m,k}(\eta)$ at the two side of (34), integrating the resulting equation in $\eta$ from 0 to $R$, and taking the sum over $k, m$ from 1 to $k(h), m(h)$ respectively, we obtain

$$\sum_{k,m}^h R \int \phi_{m,k}(\eta)\phi_{m,k}^n - \frac{w_{m,k}^n - w_{m-1,k}^n}{h} + w_{m-1,k}^n w_{m,k}^n - w_{m,k-1}^n$$

$$= \int f(\cdot, w_{m,k}^n)\phi_{m,k}^n d\eta = 0.$$ (35)

Denote the function $f^\bar{\tau}(\tau, \xi, \eta)$ on $[0, T) \times \Omega$ as: for $(m - 1)h < \tau < mh, (k - 1)h < \xi \leq kh$,

$$f^\bar{\tau}(\xi, \eta) = f(\eta, kh, mh),$$

(36)

and denote

$$\frac{\Delta w_h}{h}^m = \frac{w_{m,k}^n - w_{m-1,k}^n}{h},$$

$$\frac{\Delta w_h}{h}^k = \frac{w_{m,k}^n - w_{m,k-1}^n}{h}.$$ Then we can rewrite (35) to

$$\int_0^T \int (\bar{w}_{h\eta\eta}^\bar{\tau} - \frac{\Delta w_h}{h})^n \phi^\bar{\tau} + \frac{\Delta w_h}{h}^2 \bar{\phi}_h$$

$$- f(\cdot, w_{m,k}^n)\phi^\bar{\tau} d\tau d\xi d\eta = 0.$$ (37)

Since

$$|\bar{w} - w| = |\bar{w} - w_h| + |w_h - w| \leq Mh + |w_h - w|,$$

when $h \to 0$, $\bar{w} \to w$. Just likely, $\phi \to \phi^\bar{\tau}, f(\cdot, w_{m,k}^n) \phi \to g(\cdot, \phi)\phi$. At the same time,

$$\frac{\Delta w_h}{h} \to w_\tau, \frac{\Delta w_h}{h}^2 \to w_\xi, \bar{w}_{h\eta\eta} \to w_{\eta\eta},$$

$$\bar{w}_{h\eta\eta} \to w_{\eta\eta}.$$
in $L^2((0, T) \times (0, R) \times (0, N))$, so, if let $h \to 0$ in (37), then
\[
\int_0^T \int_0^R \int_0^N (w_{\eta \eta} - w_z + w \xi - g(z, w)) \varphi d\tau d\xi d\eta = 0.
\]
By the arbitrary of $\varphi$, we get our result.

3 Parallel results
Consider the following initial boundary problem

\[
w_{\eta \eta} - w_z + w \xi = f(\eta, \xi, \tau, u), (\xi, \eta, t) \in \Omega \times (0, T)
\]

\[w(\eta, \xi, 0) = w_0(\eta, \xi), (\xi, \eta) \in \Omega \tag{38}\]

\[w |_{\{\eta = 0\} \times [0, T]} = \phi(0, \xi, t), w |_{\{\eta = R\} \times [0, T]} = 0, \tag{39}\]

where $\Omega = (0, N) \times (0, R)$, $w_0(0, \xi) = \phi(0, \xi, 0)$ as before, $\phi$ is compatible with the functions $w_0, \phi$, which satisfy the condition (14), and $f$ satisfies the condition $(12)-(13)$. In addition, we must assume that $K_2 \leq \frac{1}{2R}$.

\[|\varphi(\eta, \tau)| \leq K_2(R - \eta), \varphi \leq \frac{1}{2}. \tag{42}\]

Instead of equation (38)-(41), let us consider the following system of ordinary differential equations:

\[w_{\eta \eta} - w_z + w \xi = -f(\cdot, w_{\eta \eta}), \]

\[w \big|_{\eta = 0} = \phi(0, k\xi, mh), \]

where

\[w_{0, k} = w_{0, k}(0, \xi), \]

\[w_{0, k} |_{\eta = 0} = \varphi((0, kh, mh)), \]

\[m = 1, \ldots, [Th]; k = 0, 1, \ldots, [Nh]. \tag{44}\]

Similarly, we have

Lemma 5 Under the conditions of (11), (12) and (42), the problem (43)-(44) admits a unique solution for $kh \leq N_0$ and small enough $h$, where $N_0$ is a suitable small positive number. The solution satisfies the following estimate

\[V_0(\eta, kh) \leq w_{\eta \eta} \leq V_1(\eta, kh), \tag{45}\]

where $V_0, V_1$ are continuous functions, positive in $(0, R)$, $V_1 \leq \frac{1}{2}$ and such that

\[V_0 \equiv K_0(R - \eta), V_1 \equiv K_1(R - \eta), K_1 \leq \frac{1}{2R}. \tag{46}\]

In a neighborhood of $\eta = R$.

Lemma 6 Assume that the conditions of Lemma 5 are fulfilled, $w_0$ has bounded first order derivatives, $w_0|_{\eta \eta}$ is bounded, then

\[w_{\eta \eta} \leq \frac{1}{h}(w_{\eta \eta} - w_{\eta \eta}^{m,k}), \frac{1}{h}(w_{\eta \eta}^{m,k} - w_{\eta \eta}^{m,k-1}), w_{\eta \eta}^{m,k} \]

are bounded for $kh \leq N_1$ and $h \leq h_0$, uniformly with respect to $h$, where the positive constants $N_1 \leq N_0$.

Theorem 7 Under the assumption of Lemma 5 and Lemma 6, problem (38)-(41) admits a solution $w$ with the following properties: $w$ is continuous,

\[|w| \leq K_1(R - \eta) \tag{47}\]

in the domain $(0, T) \times \Omega$, $w$ has bounded weak derivatives $w_{\eta}, w_\xi, w_\tau$,

\[|w_{\xi}| \leq c(R - \eta), |w_\tau| \leq c(R - \eta); \tag{48}\]

the weak derivative $w_{\eta}, w_{\eta \eta}$ exists and bounded, equation (38) holds almost everywhere in the same domain.

4 The proof of Theorem 1
Let $p = \partial_x u$ and differential (1) with respect to $x$. Then we get

\[\partial_{xx}p + p\partial_y u + u\partial_y p - \partial_y p = \frac{\partial}{\partial u} f(\cdot, u)p. \tag{49}\]

Consider the following problem: for $z = (t, x, y) \in (0, T) \times \Omega$,

\[\partial_{xx}p + u\partial_y p - \partial_y p = \frac{\partial}{\partial u} f(\cdot, u)p - p\partial_y g = g(\cdot, p), \tag{49}\]

\[p(0, \cdot) = p_0(x, y) = u_{0y}(x, y), \tag{50}\]

\[p |_{\{x = 0\} \times [0, T]} = u_{1y}(0, y, t), \quad p |_{\{x = R\} \times [0, T]} = 0. \tag{51}\]

where $g(\cdot, p) = p(\frac{\partial}{\partial u} f(\cdot, u) - \partial_y u)$. By Theorem 4, we can assume that $g(\cdot, p)$ is a Lipschitz function and satisfies with (11). Just like the discussion of section 2, on account of (27)-(28), we are able to get the boundedness of the weak first order derivatives of $p$. Then $\partial_{xx}u = \partial_{xx}p, \partial_{yx}u = \partial_{yx}p, \partial_{xx}u = \partial_{xx}p$ are bounded. Which means that $\partial_{xx}u, \partial_{xx}u, \partial_{yx}u, \partial_y u$ are actually continuous functions. So (1)-(3) has the solution in classical sense.

Similarly, one is able to prove the problem (38)-(41) has the solution in classical sense, and so the theorem is got.
The global solution of Cauchy problem

However, the known results in the equation (1) are all in local solutions, say, when $T$ is suitably small positive constant. The methods used in these papers seem difficult to be generalized to the cases when $T$ is an any given positive constant. The equation (1) is a degenerate parabolic equation, so it only has the weak global solution in general. To consider the weak global solution of the Cauchy problem of (1)-(2) (certainly, the $\Omega$ in (1) should be changed to $\mathbb{R}^2$), let us firstly consider the following nonlinear degenerate parabolic equation of the form

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(a^{ij}(u)\frac{\partial u}{\partial x^j}) = c(t,x,u), \quad (t,x) \in Q_T = (0,T) \times \mathbb{R}^n$$

where

$$a^{ij}\xi_i\xi_j \geq 0, \forall \xi = (\xi_1, \xi_2, \cdots, \xi_N) \in \mathbb{R}^N,$$ (54)

and pairs of equal indices imply the summation from 1 up to $N$. Equation (52) arises in many applications, including two phase flow in porous media (cf. [14] and references cited therein), sedimentation-consolidation processes (cf. [15] and references cited therein).

We notice that (1) is a very special case of (52), say, $a^{ij}$ in (1) has the form

$$(a^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ (53)

We had got the posedness of (52) in [16] and [17], so we can get the existence and uniqueness of the global weak solution for the Cauchy problem (1)-(2) from [17] etc. In details, we can discuss the global weak solution as follows.

Following reference [18], if $u \in BV(Q_T), Q_T = \mathbb{R}^N \times (0,T)$ if and only if $u \in L_{loc}^1(Q_T)$ and

$$\int_0^T \int_{B_\rho} |u(x_1 + h_1, \cdots, x_N + h_N, t + h_{N+1}) - u(x,t) | \, dxdt \leq K \mid h \mid,$$ (55)

where

$$B_\rho = \{x \in \mathbb{R}^N; \mid X \mid < \rho\}, \quad h = (h_1, h_2, \cdots, h_N, h_{N+1})$$

and $K$ is a positive constant. This is equivalent to that the generalized derivatives of every function in $BV(Q_T)$ are regular Radon measures on $Q_T$.

Let $S_\eta(s) = \int^\eta_0 h_\eta(\tau) \, d\tau$ for small $\eta > 0$, where $h_\eta(s) = \frac{2}{\eta}(1 - \frac{|s|^\eta}{\eta})_+$. Obviously $h_\eta(s) \in C(R)$ and

$$h_\eta(s) \geq 0, \quad \mid s h_\eta(s) \mid \leq 1, \quad \mid S_\eta(s) \mid \leq 1;$$

$$\lim_{\eta \rightarrow 0} S_\eta(s) = \text{sgn}(s), \quad \lim_{\eta \rightarrow 0} S_\eta'(s) = 0,$$ (57)

where sgn represents the sign function.

According to the idea of [16] and [17], we can introduce the following notion.

**Definition 8** A function $u \in BV(Q_T) \cap L^\infty(Q_T)$ is said to be a weak solution of the problem (52)-(53) if

1. there exist the functions $g^i \in L^2_{loc}(Q_T), i = 1, 2, \cdots, N$ such that

$$\int_0^T \int_{Q_T} \phi(t, x)r^{ij}(u)\frac{\partial u}{\partial x_i} \, dxdt = \int_0^T \int_{Q_T} \phi(t, x)g^i(t, x) \, dxdt.$$ (58)

for any $\phi \in C^0_Q(Q_T)$, where $(r^{ij}(u))$ is the square root of the matrix $(a^{ij}(u)),$ and $r^{ij}(u)$ is the composite mean value of $r^{ij}$ and $u$ as usual.

2. $u$ satisfies

$$\int_0^T \int_{Q_T} \{I_\eta(u - k)\varphi_t - B_\eta(u, k)\varphi_x + A_\eta(u, k)\varphi + c(t, x, u)S_\eta(u - k)\varphi - \sum_{i=1}^N S_\eta'(u - k)(g_i)^2 \varphi \} \, dx \, dy \, dt \geq 0, (59)$$

for any $\varphi \in C^2(Q_T), \varphi \geq 0, \forall u \in R, \eta > 0$.

3. \[
\lim_{\eta \rightarrow 0} \int_\Omega |u(t, x, y) - u_0(x, y)| \, dx \, dy = 0, \quad \text{for any } \rho \in R, \eta > 0.
\]

where

$$I_\eta(u - k) = \int_0^{u-k} S_\eta(s - k) \, ds,$$

$$B_\eta(u, k) = \int_0^u b_s S_\eta(s - k) \, ds,$$ (61)

$$A_\eta(u, k) = \int_k^u a^{ij}(s) S_\eta(s - k) \, ds.$$ (62)

Review the following general definition of the entropy solution for Cauchy problem of (52) (cf. [18] etc.),

\[
\int_0^T \int_{Q_T} |u - k| \varphi_t - \text{sgn}(u - k)(b_k - b_\eta)\varphi_x + \text{sgn}(u - k)(A^{ij}(u) - A^{ij}(k))\varphi_{x_ix_j}
\]

\[
+ c(t, x, u)\varphi \, dx \, dt \geq 0,
\]

(63)
where 

\[ A^{ij}(u) = \int_0^u a^{ij}(s)ds, \]

on account of the arbitrariness of the constant \( k \), from (5.13), it is well known that (1.1) is true in the sense of distribution, i.e. for any \( \phi(x, t) \in C^0_\infty(Q_T) \),

\[
\int \int_{Q_T} \frac{\partial u}{\partial t} \phi(t, x) = \int \int_{Q_T} A^{ij}(u) \phi_{x,x} dxdt + \int \int_{Q_T} b_i \phi_{x} dxdt + \int \int_{Q_T} c(\cdot, u) \phi dxdt. \tag{64}
\]

This fact means that the entropy solution is stronger than the general weak solution.

Clearly, (60) implies (64), the entropy solution defined in Definition 8 is stronger than the general one defined as (63). At the same time, (60) implies (64) certainly.

Considering the problem (1)-(2), the dimension of the space variables is \( N = 2 \), we can simply denote \( x = (x_1, x_2) = (x, y) \). Then we quote the following

**Definition 9** A function \( u \in BV(Q_T) \cap L^\infty(Q_T) \) is said to be a weak solution of the problem (1)-(2) if \( u \) satisfies (58), (60) and

\[
\int \int_{Q_T} \{ I_{\eta}(u-k) \varphi_t - B_{\eta}(u,k) \varphi_y + I_{\eta}(u-k) \varphi_{xx} - f(t,x,y,u) S_{\eta}(u-k) \varphi - S_{\eta}'(u-k)(\partial_xu)^2 \varphi \} dx dy dt \geq 0,
\]

for any \( \varphi \in C^2_0(Q_T) \), \( \varphi \geq 0 \), any \( k \in R, \eta > 0 \). Here

\[ B_{\eta}(u,k) = -\int_k^u s S_{\eta}(s-k) ds. \]

Immediately, by [16], [17], we have

**Theorem 10** Suppose that

\[ u_0(x, y) \in L^\infty(R^2), \]

\( f_r(t, x, y, r) \) is bounded, and \( u_0(x, y), f(t, x, y, r) \) are suitably smooth. Then problem (1)-(2) has a general solution in the sense of Definition 9.

The outline of the proof of Theorem 10: Consider the following regularized equation

\[ \partial_{xx} u + u \partial_y u - \partial_t u - \epsilon \Delta u = f(\cdot, u), (t,x,y) \in (0,T] \times \Omega, \]

with the initial value (2). As in [18] we can prove

\[ |u_e| \leq M, \tag{66} \]

\[
\int_{Q_T} (A^{ij}(u_e) u_{ex}, u_{ex}) + \epsilon | \nabla u_e |^2 \omega_\lambda(x) dx dt \leq C
\]

\[
\int_{Q_T} (| \frac{\partial u_e}{\partial t} | + | \nabla u_e |) \omega_\lambda(x) dx dt \leq C
\]

where \( C, M \) are constants independent of \( \epsilon \), and

\[ \omega_\lambda(x, y) = \exp\{-\lambda \sqrt{1 + x^2 + y^2}\}, \]

for a given positive constant \( \lambda \).

Thus there exists a subsequence \( \{ u_{e_n} \} \) of \( \{ u_e \} \) and a function \( u \in BV(Q_T) \cap L^\infty(Q_T) \) such that

\[ u_{e_n} \to u, \ a.e. \ in \ Q_T. \]

We now prove that \( u \) is a generalized solution of (1)-(2). From (67), we have

\[
\int_{Q_T} | r_{ij} \frac{\partial u}{\partial x_j} |^2 \omega_\lambda(x) dx dt \leq C, \ i = 1, \cdots, N,
\]

where \( \{ r_{ij} \} \) is the square root of the matrix \( \{ a^{ij} \} \) generally, while for our special case, \( r_{ij} = a^{ij} \). This means that \( r_{ij} \frac{\partial u}{\partial x_j} \) is weakly compact in \( L^2_{loc}(Q_T) \).

Without loss of generality, we may assume that \( r_{ij} \frac{\partial u}{\partial x_j} \) itself converges weakly in \( L^2_{loc}(Q_T) \) to a function \( g^i \in L^2_{loc}(Q_T) \). For any \( \phi \in C^1_0(Q_T) \)

\[
\int_{Q_T} \phi g^i dx dt = \lim_{\epsilon \to 0} \int_{Q_T} \phi \frac{\partial u}{\partial x_j} \cdot r_{ij}(s) ds dx dt
\]

\[
= \int_{Q_T} \phi \int_0^u \int_{r_{ij}(s)} I_{\eta}(u-k) \varphi_x dx ds dx dt
\]

\[
= \int_{Q_T} \phi \int_0^u \int_{r_{ij}(s)} I_{\eta}(u-k) \varphi_x dx ds dx dt
\]

\[
= \int_{Q_T} \phi \int_0^u \int_{r_{ij}(s)} I_{\eta}(u-k) \varphi_x dx ds dx dt
\]

This implies \( u \) satisfies (58) in Definition 8.

Let \( \phi \in C^1_0(Q_T) \), \( \phi \geq 0 \), \( k \in R, \eta > 0 \). Multiply (9) by \( \phi S_{\eta}(u - k) \) and integrate over \( Q_T \), we obtain

\[
- \int_{Q_T} I_{\eta}(u - k) \varphi_t - B_{\eta}(u, k) \varphi_y + I_{\eta}(u - k) \varphi_{xx} - f(t, x, u) S_{\eta}(u - k) \varphi - S_{\eta}'(u - k)(\partial_xu)^2 \varphi \leq 0
\]

\[
\int_{Q_T} (u_e - u)(u_e - u) + \epsilon | \nabla (u - u_e) |^2 \omega_\lambda(x) dx dt \leq C
\]

\[
\int_{Q_T} (| \frac{\partial u_e}{\partial t} | + | \nabla u_e | - | \nabla u_e |) \omega_\lambda(x) dx dt \leq C
\]
\[ -\varepsilon \int_{Q_T} S'_\eta(u_e - k) \frac{\partial u_e}{\partial x_i} \varphi \phi dx dt = 0. \tag{69} \]

Notice that, on the left hand side, the seventh term and the eighth term tend to zero as \(\varepsilon \to 0\), the last term is nonnegative, and by (69), on account of that,

\[
\lim_{\varepsilon \to 0} \inf \int_{Q_T} S'_\eta(u_e - k) a^{ij}(u_e) \frac{\partial u_e}{\partial x_i} \frac{\partial u_e}{\partial x_j} \phi dx dt \geq \int_{Q_T} S'_\eta(u_e - k) g^i g^j \phi dx dt
\]

Letting \(\varepsilon \to 0\) in (69), we get (65).

**Theorem 11** Let \(u, v\) be solutions of (1.1)-(1.2) with initial values \(u_0(x), v_0(x) \in L^\infty(R^2)\) respectively. Then

\[
\int_{\Omega} |u(x,y,t) - v(x,y,t)| \omega(x,y) dx dy \leq C \int_{\Omega} |u_0 - v_0 | \omega(x,y) dx dy. \tag{70}
\]

The **outline of the proof of Theorem 11**: Let \(\Gamma_u\) be the set of all jump points of \(u \in BV(Q_T), \nu\) the unit normal vector of \(\Gamma_u\) at \(X = (x,y,t), \nu^+(X)\) and \(\nu^-(X)\) the approximate limits of \(u\) at \(X \in \Gamma_u\) with respect to \((\nu, Y - X) > 0\) and \((\nu, Y - X) < 0\) respectively.

Let \(u\) be a solution of (52)-(53). Then we can prove that (see [16][17])

\[
\int_{u^-}^{u^+} \gamma^{ij}(s,t,x) dv = 0, \quad \text{a.e.} (t,x) \in \Gamma_u, j = 1, 2. \tag{71}
\]

Let \(u, v\) be two generalized solutions of (1) with initial values

\[ u(x_1, x_2, 0) = u_0(x_1, x_2), \quad v(y_1, y_2, 0) = v_0(y_1, y_2). \]

By Definition 9, we have for any \(\varphi \in C^2_0(Q_T), \varphi \geq 0, k, l \in R,\)

\[
\int_{Q_T} \{ I_\eta(u - k) \varphi_l - B_\eta(u,k) \varphi x_1x_1 \\
- \sum_{n=1}^{N} S'_\eta(u - k) g^1 g^1 \varphi \\
- f(t,x_1,x_2,u) \varphi S_\eta(u - k) \} dx dt \geq 0, \tag{72}
\]

\[
\int_{Q_T} \{ I_\eta(v - l) \varphi_r - B_\eta(v,l) \varphi x_2 + I_\eta(v - l) \varphi y_1y_1 \\
- \sum_{n=1}^{N} S'_\eta(v - l) g^2 g^2 \varphi \\
- f(t,y_1,y_2,v) \varphi S_\eta(v - l) \} dy dt \geq 0. \tag{73}
\]

For simplicity, we denote \(x = (x_1, x_2), \) \(y = (y_1, y_2).\) Let \(\psi(t,x,\tau,\varphi) \geq 0, \psi \in C^2(Q_T \times Q_T), \text{supp} \psi \subset Q_T \text{ if } (\tau, \varphi) \in Q_T, \text{supp} \psi \subset Q_T \text{ if } (\tau, \varphi) \in Q_T.\)

Choose \(k = v(t,\tau,\varphi), l = u(t,x), \varphi = \psi(t,x,\tau,\varphi)\) in (72) (73) and integrate over \(Q_T,\) to get

\[
\int_{Q_T} \int_{Q_T} \{ I_\eta(u - v) \psi_1 + B_\eta(u,v) \psi x_2 + B_\eta(v, u) \psi y_2 \\
+ I_\eta(u - v) \psi x_1x_1 + I_\eta(v - u) \psi y_1y_1 \\
- S'_\eta(u - v) \sum_{n=1}^{N} (g^1 g^1 + g^2 g^2) \psi \\
+ \psi f(t,x,u) S_\eta(u - v) + f(\tau, y, v) S_\eta(v - u) \} dx dt dy d\tau \geq 0. \tag{74}
\]

Choose \(\psi(t,x,\tau,\varphi) = \phi(t,x) j_h(t - \tau, x - y),\)

\[
\omega(h) = \frac{1}{h} \omega(s), \quad \omega(s) \in C^\infty_0(R), \omega(s) \geq 0, \omega(s) = 0 \text{ if } |s| > 1,
\]

\[
\int_{-\infty}^{\infty} \omega(s) ds = 1.
\]

Clearly

\[
\frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0, i = 1, \cdots, N;
\]

\[
\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \psi}{\partial x_i} j_h + \frac{\partial \psi}{\partial y_i} = \frac{\partial \psi}{\partial x_i} j_h.
\]

Then (74) becomes

\[
\int_{Q_T} \int_{Q_T} \{ I_\eta(u - v) \psi_1 + B_\eta(u,v) \psi x_2 + B_\eta(v, u) \psi y_2 \\
+ I_\eta(u - v) \psi x_1x_1 + I_\eta(v - u) \psi y_1y_1 \\
- S'_\eta(u - v) \sum_{n=1}^{N} (g^1 g^1 + g^2 g^2) \psi \\
+ \psi f(t,x,u) S_\eta(u - v) + f(\tau, y, v) S_\eta(v - u) \} dx dt dy d\tau \geq 0. \tag{75}
\]

Letting \(\eta \to 0, h \to 0\) in (75), by (71), we are able to get

\[
\int_{Q_T} \{ |u(x,t) - v(x,t)| \phi e^{-2sgn(u-v)(u^2-v^2)} \phi x_2 \\
+ |u(x,t) - v(x,t)| \phi x_1x_1 \\
+ sgn(u-v)(f(t,x,u) - f(t,x,v)) \} \geq 0. \tag{76}
\]

Let

\[
\eta(t) = \int_{\tau - t}^{\tau - t} \alpha_\epsilon(\sigma) d\sigma, \quad \epsilon < \min\{\tau, T - s\},
\]
where $\alpha_\epsilon(t)$ is the kernel of mollifier with $\alpha_\epsilon(t) = 0$ for $t \notin (-\epsilon, \epsilon)$. By approximation, we can choose $\phi$ with $\phi(x, t) = \omega_\lambda(x)\eta(t)$ in (76), where $\omega_\lambda(x)$ is the function as before. Using the estimates
\[
|\nabla \omega_\lambda| \leq C_\lambda \omega_\lambda(x), \quad |\Delta \omega_\lambda(x)| \leq C_\lambda \omega_\lambda(x),
\]
and letting $\epsilon \to 0$ in (76), we obtain
\[
\int_{\mathbb{R}^N} |u(s, x) - v(s, x)|\omega_\lambda(x) dx \\
\leq \int_{\mathbb{R}^N} |u(\tau, x) - v(\tau, x)|\omega_\lambda(x) dx \\
+C \int_{\tau}^{s} \int_{\mathbb{R}^N} |u(t, x) - v(t, x)|\omega_\lambda(x) dx dt.
\]
Hence by Gronwall lemma, we obtain
\[
\int_{\mathbb{R}^N} |u(s, x) - v(s, x)|\omega_\lambda(x) dx \\
\leq C \int_{\mathbb{R}^N} |u(\tau, x) - v(\tau, x)|\omega_\lambda(x) dx.
\]
Letting $\tau \to 0$, the proof of Theorem 11 is completed.

At the last of the paper, we would like to point that the uniqueness of the initial boundary problem of (52) is still an open problem. So, it also seem very difficult to solve the posedness of the global weak solution of (1)-(3) for the time being.

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References: