Analysis of Chain Reaction Between Two Stock Indices Fluctuations by Statistical Physics Systems

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Abstract: In this paper, we consider the statistical properties of chain reaction of stock indices. The theory of interacting systems and statistical physics are applied to describe and study the fluctuations of two stock indices in a stock market, and the properties of the interacting reaction of the two indices are investigated in the present paper. In this work, stochastic analysis and the two random paths model are used to study the probability distribution for the chain reaction of stock indices, further we show the asymptotical behavior of probability measures of the fluctuations for the two stock indices model. In the last part, we discuss the convergence of the finite dimensional probability distributions for the financial model.

Key–Words: Stock Index, Chain Reaction, Statistical Analysis, Fluctuation, Statistical Physics, Gibbs Probability Measure

1 Introduction

As the stock markets are becoming deregulated worldwide, the modelling of the dynamics of the forwards prices is becoming a key problem in the risk management, physical assets valuation, and derivatives pricing. Meanwhile it can be seen from a lot of phenomena and research results in securities markets that there is indeed a notable correlation among the fluctuations of the securities indices all over the world. For example, we often see that, the chain reaction between Hang Seng Index (Hong Kong Stock Exchange) and Shanghai Composite Index (Shanghai Stock Exchange) shows a notable positive correlation, since there are many mainland companies of China are listed in the Hong Kong Stock Exchange. About 30 years' reformation and opening in economy of China, now the capital market economy of China plays an important role in the national economy, and the chain reaction between the Chinese stock markets and the foreign stock markets is becoming more obviously.

The study of fluctuation of stock price has been made great progress in the past ten years, see [2,3,5,9-14,17-27,31,32]. Recently, some research work has been done in the field of applying the theory of statistical physics dynamic systems to investigate the statistical properties of fluctuations of stock prices, the power stock market is modelled by the dynamics of power spot prices, and the corresponding valuation and hedging of contingent claims for this price process model are also studied, for example see [5,9,10,14,17,20,27,31,32]. For example, by applying the percolation model, Makowiec et al. [17], Tanaka [27] studied the market fluctuations. They constructed a price model by the lattice percolation theory, according to local interaction of percolation, the local interaction or influence among traders in one stock market is constructed, and a cluster of percolation is used to define the cluster of traders sharing the same opinion about the market. Let p_c denote the critical point of influence rate in percolation model, around or at this critical point, [27] show the existence of fat tails for return processes, where the properties of percolation clusters are applied. Here, the critical phenomena of percolation model is used to illustrate the herd behavior of stock market participants. In their study, they assume that the information in the stock market leads to the stock price fluctuation and the investors in stock market follow the effect of sheep flock. There are also some work that has been done by applying Ising type models to a financial model, see [10,22]. In [10,22], the interacting dynamic system is applied to model a financial price model, and the corresponding statistical properties is analyzed. In this paper, we apply the Ising type model to study the chain reaction of stock indices. The original attempt of this work is to study the financial phenomena by statistical physics systems, and it is also important to understand the statistical properties of fluctuations of stock indices in globalized securities markets.

In the present paper, the stochastic process theory and the two random paths model (a statistical physics model) are used to construct a financial price model, which describes the fluctuations of stock indices. In the work of modelling the two indices financial model, we consider the fluctuations of paths in a two random paths model, where the "path" of the random model is considered as a stock index process. The study on the paths of the two random paths model is approximation relevant to the two interfaces problems of the one-dimensional Solid-on-Solid (*SOS*) model. Assume that there is a specified value of the large area in the intermediate region of the two random indices (where the financial meaning of the large area will be given in Section 2), we discuss the statistical properties of the two random indices, and we study the statistical limiting properties of the two random indices model. The statistical analysis method of this paper is mainly based on the research work of [8].

2 Modeling Two Stock Indices by Interacting and Dynamic Systems

In this section, interacting particle system (see [4,6-8,15,16,29,30]) is used to construct a financial model, the particle is considered as a investor or trader, and the path (or interface) is considered as a stock index process. The aim of this paper is to study the limiting properties of the financial model and the volatility function and the drift function of the model.

First we give the construction of this financial model. For a stock market, we consider a single index, and assume that there are m (m large enough) traders in this stock market, and each trader can trade unit number of stocks at each time t. At each time t, the behavior of stock index process is determined by the number of traders x_t^+(with buying positions) and x_t^-(with selling positions). Let '$+$', '$-$' and '0' denote that traders take buying positions, selling positions and neutral positions respectively. If the number of traders in buying positions is bigger than the number of traders in selling positions, it implies that the index price is considered to be low by the market participants, and the index price auctions higher searching for buyers, similarly for the opposite case. Traders with buying positions or selling positions are called market participants.

Let $x_t(r)$ be the position of trader $r(1 \leq r \leq m)$ at time t, and

$$x_t = (x_t(1), \cdots, x_t(m))$$

be the configuration of positions for m traders. A space of all configurations of positions for m traders from time 1 to n is given by

$$\mathcal{X} = \{x = (x_1, \cdots, x_n)\}$$

and let $x_0 = 0$. For a given configuration

$$x = (x_1, \cdots, x_n) \in \mathcal{X}$$

and $t \in \{1, \cdots, n\}$, let

$$A(x_t) = \begin{cases} x_t^+ - x_t^- - h_0 & \text{if} \quad x_t^+ - x_t^- > h_0 \\ 0 & \text{if} \ \mid x_t^+ - x_t^- \mid \leq h_0 \\ -(x_t^- - x_t^+ - h_0) & \text{if} \quad x_t^- - x_t^+ > h_0 \end{cases} \tag{1}$$

where h_0 is a positive integer. If $A(x_t) > 0$, there are more buyers than sellers and the index price is auctioned up. From above definitions and [2,12,21,24,28], we define the index price of the model at time $t(t = 1, 2, \cdots)$ as following

$$S_t = e^{\alpha A(x_t)} S_{t-1}$$

where $\alpha > 0$, and let S_0 be the price at time $t = 0$. Then we have

$$S_t = S_0 \exp\{\alpha \sum_{k=1}^{t} A(x_k)\}. \tag{2}$$

In this paper, we consider the chain reaction of two stock indices, so we let S_t^1 (or $A(x^1)$) and S_t^2 (or $A(x^2)$) to denote the two indices prices respectively. Then we can obtain the corresponding definitions as above (1)(2).

Next we study a two random paths model, the Hamiltonian of the model on the horizontal set of

$$t \in L_n = \{1, \cdots, n\}$$

is given by

$$H_n(A(x^1), A(x^2)) = \sum_{t \in L_n} \Big(|A(x_t^1) - A(x_{t-1}^1)| + |A(x_t^2) - A(x_{t-1}^2)| \Big). \tag{3}$$

For the two random paths models, let

$$\mathcal{X}_n = \{(A(x^1), A(x^2)) : A(x_t^1), A(x_t^2) \in \mathbf{Z}, \text{ for } t \in L_n\}$$

be the corresponding configuration pace. The Gibbs measure associated with the Hamiltonian $H_n(A(x^1), A(x^2))$ is defined as

$$P(A(x^1), A(x^2)) = \frac{1}{Z_{n,\beta}} \exp[-\beta H_n(A(x^1), A(x^2))] \tag{4}$$

where β is a positive parameter called an inverse temperature, and

$$Z_{n,\beta} = \sum_{(A(x^1),A(x^2))\in\mathcal{X}_n} \exp[-\beta H_n(A(x^1), A(x^2))]$$

is a partition function.

According to (1), the integer-valued random variables $A(x_1), A(x_2), \cdots$, are independent and have the same probability distribution. For $t \in L_n$ and $t \geq 1$, let

$$\xi_t = A(x_t^1) - A(x_{t-1}^1)$$
$$\eta_t = A(x_t^2) - A(x_{t-1}^2)$$

where $\xi_0 = 0, \eta_0 = 0$. Let

$$\xi = \{\xi_t, t \in L_n\}$$
$$\eta = \{\eta_t, t \in L_n\}$$

then rewrite above partition function as

$$Z_{n,\beta} = \sum_{\xi,\eta} \exp[-\beta H_n(\xi, \eta)],$$

where $H_n(\xi, \eta)$ is the Hamiltonian function for (ξ, η). From above definitions, $\xi = \{\xi_t, t \in L_n\}$ and $\eta = \{\eta_t, t \in L_n\}$ can be seen as the sequences of i.i.d. random variables respectively. So, the two random paths model has two independent random *SOS* paths, that is, the model corresponds to the ensemble of two independent self-avoiding paths in $[0, n] \times \mathbf{Z}$ starting from $(0, 0)$ and ending at sites z in the line $\{t = n\}$(where $z = (t, y)$), which do not go back in the horizontal direction. Next we introduce the generating function of the hight of the endpoints for one 'step', that is, for a fixed $t \in L_n$, let G_t be the set of real (u, v) such that

$$Q(u, v) = E[e^{u\beta\xi_t + v\beta\eta_t}] < \infty \tag{5}$$

where (u, v) is in some neighbourhood of the origin. Considering the random walk (describing the fluctuations of the index price process) as

$$X_0^\xi = 0, \quad X_t^\xi = \sum_{k=1}^{t} \xi_k, \quad t = 1, 2, \cdots, \tag{6}$$

we define the index price process as a random polygonal function $l_n^\xi(s)$

$$l_n^\xi(s) = X_{[ns]}^\xi + \{ns\}\xi_{[ns]+1}, \quad \text{for } s \in [0, 1],$$

where $[ns]$ denote the integral part of a real number ns, and $\{ns\}$ denote its fractional part. For a fixed n, let

$$a_n^\xi = \frac{1}{n}\sum_{t=0}^{n-1} X_t^\xi = \sum_{t=1}^{n}(1 - \frac{t}{n})\xi_t$$

be a new random variable which denote the area under the index price line $l_n^\eta(s)$. Similarly to above definitions, we can obtain the corresponding definitions $\{X_0^\eta, X_t^\eta, t = 1, 2, \cdots, \}$, $l_n^\eta(s)$ and a_n^η. Let

$$\bar{\xi} = \sum_{t=1}^{n} \xi_t, \quad \bar{\eta} = \sum_{t=1}^{n} \eta_t.$$

For $(u, v) \in \mathbf{R} \times \mathbf{R}$, we define

$$\begin{aligned}\varphi(u, v) \;\; &= \;\; \lim_{n\to\infty} \frac{1}{n} \ln\Big(\sum_{\xi,\eta} \exp[\beta u\bar{\xi} + \beta v\bar{\eta}] \\ &\qquad \times \exp[-\beta H_n(\xi, \eta)]\Big/ Z_{n,\beta}\Big)\end{aligned} \tag{7}$$

by [8], it is known that the limit exists if (u, v) is in some neighbourhood of the origin.

The aim of this paper is to study the asymptotes of fluctuations of the two random paths conditioned by fixing a large area between the two random paths. Denote by $a_n^{\eta-\xi} = a_n^\eta - a_n^\xi$ representing the area of the intermediate layer between the two random paths. For a real $\bar{\zeta}_0$ and $0 \leq s \leq 1$, assume that

$$\begin{aligned}&F(\bar{\zeta}_0, \beta, s) \\ &= \frac{d}{d\zeta_0}\varphi\Big(-(1 - s)\zeta_0, (1 - s)\zeta_0\Big)\Big|_{\zeta_0=\bar{\zeta}_0}, \\ &\frac{1}{\beta}\int_0^1 F(\bar{\zeta}_0, \beta, s)ds = a\end{aligned} \tag{8}$$

where $a > 0$ is some constant. Then we state the main results of this paper.

Main Result: Assume that for some $\delta(\beta) > 0$ and $a > 0$, there exists a real $\bar{\zeta}_0$ satisfying above condition (8) and $|\bar{\zeta}_0| < \delta(\beta)$, then the process

$$\begin{aligned}&Y_n(t) = \\ &\frac{1}{\sqrt{n}}\left\{X_n^\eta(t) - X_n^\xi(t) - \frac{n}{\beta}\int_0^t F(\bar{\zeta}_0, \beta, s)ds\right\}\end{aligned} \tag{9}$$

under $P_{n,\beta}(\cdot \mid a_n^{\eta-\xi} = \lfloor an \rfloor)$, converges weakly to the process

$$Y(t) = \frac{1}{\beta}\int_0^t \sqrt{\varphi''\Big(-(1 - s)\bar{\zeta}_0, (1 - s)\bar{\zeta}_0\Big)}dB(s) \tag{10}$$

conditioned that

$$\int_0^1 Y(t)dt = 0$$

where $\{B(s)\}_{s\geq0}$ is the one dimensional standard Brownian motion, $\varphi(u, v)$ is defined in (6) and $\lfloor an \rfloor$ is the integer part of an.

Remark 1: From above definitions, the stock price S_t^1 can be rewritten as

$$S_t^1 = S_0^1 \exp\{\alpha X_t^\xi\}$$

and S_t^2 can be rewritten as

$$S_t^2 = S_0^2 \exp\{\alpha X_t^\eta\}$$

for $t = 1, 2, \cdots, n$. In a stock market, here the 'area' $a_n^{\eta-\xi}$ may represent the information or the situation on these two stock indices during the time from 0 to n, including the estimation for these stock indices, positive or negative news, trends, political event and economic policy, etc. If the 'area' is positive and has a large positive value, then the index price S_t^2 lies above the index price S_t^1 during the time $t = 1, 2, \cdots, n$ (here we suppose that $S_0^1 = S_0^2$), this implies that there may have a positive influence or positive news on the index S_t^2, so that the market participants are likely to take buying positions for the index S_t^2. This means that, comparing the price of the index S_t^1 and the price of the index S_t^2, the index price S_t^2 is more likely to increase (or go up) than the index price S_t^1. If the 'area' is negative and has a large negative value, then the index price S_t^1 lies above the index price S_t^2 during the time $t = 1, 2, \cdots, n$ (here we suppose that $S_0^1 = S_0^2$), this implies that the index price S_t^2 is more likely to decrease (or go down) than the index price S_t^1. In this paper, we only consider the case that the 'area' is positive, and we can use the similar methods to discuss the opposite case.

3 Estimate of Fluctuations for Two Random Financial Stock Indices Model

From definitions (6), we begin discussing the area between the two random paths. Now we define the areas of $a_n^\xi, a_n^\eta, a_n^{\eta-\xi}$ as followings,

$$a_n^\xi = \frac{1}{n}\sum_{t=1}^n X_t^\xi = \sum_{t=1}^n (1-\frac{t}{n})\xi_t$$

$$a_n^\eta = \frac{1}{n}\sum_{x=1}^n X_t^\eta = \sum_{t=1}^n (1-\frac{t}{n})\eta_t. \qquad (11)$$

$$\begin{aligned} a_n^{\eta-\xi} &= a_n^\eta - a_n^\xi \\ &= \sum_{t=1}^n (1-\frac{t}{n})(\eta_t - \xi_t). \end{aligned} \qquad (12)$$

By the independence of $\{\xi_t, t \in L_n\}$ and $\{\eta_t, t \in L_n\}$, the generation function of the area $a_n^{\eta-\xi}$ is defined by

$$\begin{aligned} &Q_{a_n^{\eta-\xi}}(\zeta) = \\ &\frac{1}{Z_{n,\beta}}\sum_{\xi,\eta} \exp\{\beta\zeta a_n^{\eta-\xi}\}\exp\{-\beta H_n(\xi,\eta)\} \\ &= \frac{1}{Z_{n,\beta}}\prod_{t=1}^n \Big(\sum_{\xi_t,\eta_t}\exp\{\beta\zeta(1-\frac{t}{n})(\eta_t-\xi_t) \\ &\quad -\beta(|\xi_t|+|\eta_t|)\}\Big) \\ &= \prod_{t=1}^n Q\Big(-\zeta(1-\frac{t}{n}), \zeta(1-\frac{t}{n})\Big). \end{aligned} \qquad (13)$$

Let q be a natural number, and let $\{t_i, 1 \le i \le q\}$ be any set of real numbers, such that

$$0 < t_1 < \cdots < t_q \le 1.$$

Set a random vector as

$$\begin{aligned} &\hat{X}_n^{(q)}(t_1, \cdots, t_q) = \\ &(a_n^{\eta-\xi}, A(x^2_{\lfloor t_1 n\rfloor}) - A(x^1_{\lfloor t_1 n\rfloor}), \cdots, \\ &A(x^2_{\lfloor t_q n\rfloor}) - A(x^1_{\lfloor t_q n\rfloor})). \end{aligned} \qquad (14)$$

Then for

$$\underline{\zeta} = (\zeta_0, \zeta_1, \cdots, \zeta_q) \in \mathbf{R}^{q+1},$$

it is similar to (13), we have

$$\begin{aligned} &\frac{1}{Z_{n,\beta}}\sum_{\xi,\eta} e^{\beta\underline{\zeta}\cdot\hat{X}_n^{(q)}(t_1,\cdots,t_q)}e^{-\beta H_n(\xi,\eta)} \\ &= \prod_{x=1}^n Q\Big(-\zeta_n(t;\underline{\zeta}), \zeta_n(t;\underline{\zeta})\Big) \end{aligned} \qquad (15)$$

where

$$\zeta_n(t;\underline{\zeta}) = \zeta_0(1-\frac{t}{n}) + \sum_{i=1}^q \zeta_i 1_{[0,t_i n]}(t).$$

For the real $\bar{\zeta}_0$ defined in (8) and some small constant $\alpha_1 > 0$, let $\underline{\zeta} \in \mathbf{R}^{q+1}$ satisfy the following conditions

$$\begin{aligned} D_{\alpha_1,\bar{\zeta}_0} &= \{\underline{\zeta}: \quad -\alpha_1 < \zeta_0 < \bar{\zeta}_0 + \alpha_1, \\ &\quad |\zeta_i| < \alpha_1, \quad i = 1, \cdots, q\}. \end{aligned} \qquad (16)$$

Next we introduce the corresponding quadratic form, a $(q+1)\times(q+1)$ matrix $V_n(\underline{\zeta})$ denote by

$$\begin{aligned} &V_n(\underline{\zeta}) = \\ &\frac{1}{\beta^2 n}\text{Hess } \ln\Big(\frac{1}{Z_{n,\beta}}\sum_{\xi,\eta} e^{\beta\underline{\zeta}\cdot\hat{X}_n^{(q)}(t_1,\cdots,t_q)} \\ &\times e^{-\beta H_n(\xi,\eta)}\Big), \end{aligned} \qquad (17)$$

where $V_n(\underline{\zeta})$ is analytic in $D_{\alpha_1,\bar{\zeta}_0}$. Assume that $\underline{\zeta} \in D_{\alpha_1,\bar{\zeta}_0}$, and according to definition (17), then uniformly in $\underline{\zeta}$ and

$$\underline{y} = (y_0, \cdots, y_q) \in \mathbf{R}^{q+1}$$

such that $|\ \underline{y}\ |= 1$, we have

$$\underline{y} \cdot V_n(\underline{\zeta}) \cdot \underline{y} \longrightarrow \underline{y} \cdot V(\underline{\zeta}) \cdot \underline{y}, \qquad \text{as } n \to \infty \quad (18)$$

where

$$V(\underline{\zeta}) = \frac{1}{\beta^2} \text{Hess} \int_0^1 \ln Q(-\zeta(s), \zeta(s)) ds, \quad (19)$$

and

$$\zeta(s) = \zeta_0(1-s) + \sum_{i=1}^{q} \zeta_i 1_{[0,t_i]}(s),$$

for $0 \leq s \leq 1$. Let $\hat{P}_n^{(q)}$ be the probability distribution of

$$\hat{X}_n^{(q)}(t_1, \cdots, t_q)$$

under $P_{n,\beta}$, and $\hat{P}_{n,\underline{\zeta}}^{(q)}$ be given by

$$\hat{P}_{n,\underline{\zeta}}^{(q)}(\underline{z}) = e^{\beta \underline{\zeta} \cdot \underline{z}} \hat{P}_n^{(q)}(\underline{z}) \Big/ E_{n,\beta}(e^{\beta \underline{\zeta} \cdot \hat{X}_n^{(q)}(t_1, \cdots, t_q)}) \quad (20)$$

for all $\underline{\zeta} \in D_{\alpha_1,\bar{\zeta}_0}$ and

$$\underline{z} \in \mathcal{Z}_n^{(q)} = (n^{-1}\mathbf{Z}) \times \mathbf{Z}^q.$$

Denote by $\hat{E}_{n,\underline{\zeta}}^{(q)}(\cdot)$ the corresponding expectation function for $\hat{P}_{n,\underline{\zeta}}^{(q)}(\cdot)$. By the uniform boundedness of the family of analytical functions $V_n(\underline{\zeta})$ for all L and all $\underline{\zeta}$ in $D_{\alpha_1,\bar{\zeta}_0}$, according to Lemma 2.7 and Proposition 2.7 in [8], we have the following two results.

(I) Let $\underline{\zeta}_n, \underline{\zeta} \in D_{\alpha_1,\bar{\zeta}_0}$, and $\underline{\zeta}_n \to \underline{\zeta}$ as $n \to \infty$. Then the random vector

$$\hat{Y}_n^{(q)}(t_1, \cdots, t_q) = \frac{1}{\sqrt{n}} (\hat{X}_n^{(q)}(t_1, \cdots, t_q) - \hat{E}_{n,\underline{\zeta}_n}^{(q)} \hat{X}_n^{(q)}(t_1, \cdots, t_q)) \quad (21)$$

converges weakly to a Gaussian random vector

$$\hat{Y}^{(q)}(t_1, \cdots, t_q)$$

of which covariance matrix is given by $V(\underline{\zeta})$.

Let $g_{\underline{\zeta}}$ be the density function of the Gaussian vector $\hat{Y}^{(q)}(t_1, \cdots, t_q)$ given in (I).

(II) Let $\mathcal{Z}_n^{(q)} = (n^{-1}\mathbf{Z}) \times \mathbf{Z}^q$, then for each $\underline{z}_n \in \mathcal{Z}_n^{(q)}$ and $\underline{\zeta}_n \in D_{\alpha_1,\bar{\zeta}_0}$, define

$$\underline{y}_n = \frac{1}{\sqrt{n}} (\underline{z}_n - \hat{E}_{n,\underline{\zeta}_n}^{(q)} \hat{X}_n^{(q)}(t_1, \cdots, t_q)). \quad (22)$$

Then we have

$$n^{(q+3)/2} \hat{P}_{n,\underline{\zeta}_n}^{(q)}(\underline{z}_n) - g_{\underline{\zeta}_n}(\underline{y}_n) \to 0 \quad (23)$$

as $n \to \infty$, and uniformly in $\underline{z}_n \in \mathcal{Z}_n^{(q)}$ and $\underline{\zeta}_n \in D_{\alpha_1,\bar{\zeta}_0}$.

4 Convergence Behaviors of Chain Reaction Between Two Stock Indices

In this section, we discuss the limiting properties of the random vector (14) and show the proofs of the main results in (9)(10).

In the definition (14), the random vector

$$\hat{X}_n^{(q)}(t_1, \cdots, t_q)$$

is given. First we consider the convergence of the finite-dimensional distribution of the random vector

$$\hat{Y}_n^{(q)}(t_1, \cdots, t_q)$$

defined in (21), see [1]. Let $\underline{\zeta}_n^\circ, \underline{\zeta}^\circ$ be a special sequence in $D_{\alpha_1,\bar{\zeta}_0}$, such that

$$\begin{aligned} \underline{\zeta}_n^\circ &= (\zeta_{n,0}, 0, \cdots, 0) \\ \underline{\zeta}^\circ &= (\bar{\zeta}_0, 0, \cdots, 0) \end{aligned} \quad (24)$$

where $\bar{\zeta}_0$ is defined in (8), and $\zeta_{n,0}$ satisfies the following condition(see (13))

$$\frac{d}{d\zeta_0} \ln Q_{a_n^{\eta-\xi}}(\zeta_0) \Big|_{\zeta_0 = \zeta_{n,0}} = \lfloor an \rfloor, \quad (25)$$

by (8)(13) and above (25), it can be proved that $\underline{\zeta}_n^\circ \to \underline{\zeta}^\circ$ as $n \to \infty$. Let

$$\varphi_n(\underline{\zeta}; t_1, \cdots, t_q) = \frac{1}{n} \ln\Big(\sum_{\xi,\eta} e^{\beta \underline{\zeta} \cdot \hat{X}_n^{(q)}(t_1, \cdots, t_q)} e^{-\beta H_n(\xi,\eta)} \Big/ Z_{n,\beta} \Big), \quad (26)$$

and denote by

$$\varphi^{(q)}(\underline{\zeta}; t_1, \cdots, t_q) = \lim_{n \to \infty} \varphi_n(\underline{\zeta}; t_1, \cdots, t_q)$$

for $\underline{\zeta} \in D_{\alpha_1,\bar{\zeta}_0}$. From the definitions (8)(25)(26) and the uniform boundedness of $Hess_{\underline{\zeta}}\varphi_n$, we have

$$\begin{aligned}
&\hat{E}^{(q)}_{n,\underline{\zeta}^\circ_n}\hat{X}^{(q)}_n(t_1,\cdots,t_q)\\
&= (\lfloor an\rfloor, \hat{E}^{(q)}_{n,\underline{\zeta}^\circ_n}(A(x^2_{\lfloor t_1 n\rfloor}) - A(x^1_{\lfloor t_1 n\rfloor})),\cdots,\\
&\quad \hat{E}^{(q)}_{n,\underline{\zeta}^\circ_n}(A(x^2_{\lfloor t_q n\rfloor}) - A(x^1_{\lfloor t_q n\rfloor})))\\
&= \frac{n}{\beta}(\nabla_{\underline{\zeta}}\varphi_n)(\underline{\zeta}^\circ_n; t_1,\cdots,t_q)\\
&= \frac{n}{\beta}(\nabla_{\underline{\zeta}}\varphi^{(q)})(\underline{\zeta}^\circ; t_1,\cdots,t_q) + o(1). \qquad (27)
\end{aligned}$$

By (22)(24), for

$$-\infty < a_j < b_j < \infty, 1 \le j \le q$$

we have

$$\begin{aligned}
&\lim_{n\to\infty}\hat{P}^{(q)}_n(y_j \in [a_j,b_j], 1\le j\le q \mid z_0 = \lfloor an\rfloor)\\
&= \lim_{n\to\infty}\hat{P}^{(q)}_{n,\underline{\zeta}^\circ_n}(y_j \in [a_j,b_j],\\
&\qquad 1\le j\le q \mid z_0 = \lfloor an\rfloor)\\
&= \frac{\int_{[a_1,b_1]\times\cdots\times[a_q,b_q]} g_{\underline{\zeta}^\circ}(0,y_1,\cdots,y_q)dy_1\cdots dy_q}{\int_{\mathbf{R}^q} g_{\underline{\zeta}^\circ}(0,y_1,\cdots,y_q)dy_1\cdots dy_q}. \qquad (28)
\end{aligned}$$

According to the definition (19) and the result of (I) in (21), let

$$\begin{aligned}
&\hat{Y}^{(q)}(t_1,\cdots,t_q)\\
&= (Y_0, Y(t_1), Y(t_2),\cdots,Y(t_q)) \qquad (29)
\end{aligned}$$

be a Gaussian random vector with distribution density $g_{\underline{\zeta}}(y_0,\cdots,y_q)$. Then its covariance matrix is given by

$$\begin{aligned}
&E\left[Y(t_j)Y(t_k)\right] =\\
&\frac{1}{\beta^2}\int_0^{t_j\wedge t_k}\varphi''(-(1-s)\bar{\zeta}_0,(1-s)\bar{\zeta}_0)ds\\
&E\left[Y_0Y(t_j)\right] =\\
&\frac{1}{\beta^2}\int_0^{t_j}\varphi''(-(1-s)\bar{\zeta}_0,(1-s)\bar{\zeta}_0)ds\\
&E\left[Y_0^2\right] =\\
&\frac{1}{\beta^2}\int_0^{1}\varphi''(-(1-s)\bar{\zeta}_0,(1-s)\bar{\zeta}_0)ds \qquad (30)
\end{aligned}$$

for $j,k = 1,\cdots,q$, where $a\wedge b = \min\{a,b\}$. This means that $\{Y_0,\{Y(t)\}_{t\in[0,1]}\}$ is a Gaussian random process with covariance matrix given above for every $q \ge 1$. In above proof, we suppose that

$$\begin{aligned}
&A(x^2_{\lfloor t_i n\rfloor}) - A(x^1_{\lfloor t_i n\rfloor})\\
&= X^\eta_n(\frac{\lfloor t_i n\rfloor}{n}) - X^\xi_n(\frac{\lfloor t_i n\rfloor}{n})
\end{aligned}$$

for $i = 1,\cdots,q$(see (27)). Similarly to Lemma 2.8 in [8], the above argument is also true if we replace

$$X^\eta_n(\frac{\lfloor t_i n\rfloor}{n}) - X^\xi_n(\frac{\lfloor t n_i\rfloor}{n})$$

with

$$X^\eta_n(t_i) - X^\xi_n(t_i)$$

for every $1 \le i \le q$. Then the distribution of

$$\hat{X}^{(q)}_n(t_1,\cdots,t_q)$$

under $P_{n,\beta}(\cdot \mid a^{\eta-\xi}_n = \lfloor an\rfloor)$, converges weakly to the corresponding distribution of Gaussian random vector

$$\hat{Y}^{(q)}(t_1,\cdots,t_q).$$

Secondly, the tightness of above conditional distribution of the random process $Y_n(t)$ should be discussed. Following the similar argument of Section 3 in [8], we can prove a sufficient condition for the tightness of the considered process $Y_n(t)$. Together with the first part of this proof, this completes the proof of main results in (10).

Remark 2: According to the arguments of [8], and with the main results of (9)(10), the probability distribution of the random process

$$\frac{1}{n}(X^\eta_n(t) - X^\xi_n(t))$$

under $P_{n,\beta}(\cdot \mid a^{\eta-\xi}_n = \lfloor an\rfloor)$, converges weakly to the corresponding distribution concentrated on the function

$$\frac{1}{\beta}\int_0^t F(\bar{\zeta}_0,\beta,s)ds.$$

5 Probability Properties of One Stock Index by Chain Reaction

In this section, we study the asymptotical behavior of one stock index

$$\{X^\eta_0, X^\eta_t, t = 1,2,\cdots,\}$$

or $l^\eta_n(s)$ by the chain reaction, see the definitions in Section 2. In Section 3 and Section 4, the convergence properties of

$$\frac{1}{\sqrt{n}}(X^\eta_n(t) - X^\xi_n(t))$$

is studied, here we continue to study the probability distributions of random process

$$\frac{1}{n}X^\eta_n(t)$$

under $P_{n,\beta}(\cdot \mid a_n^{\eta-\xi} = \lfloor an \rfloor)$, as $n \to \infty$.

Corollary: Let

$$\varphi_u'(u,v) = \frac{\partial}{\partial u}\varphi(u,v)$$

and

$$F_u(\bar{\zeta}_0,\beta,s) = -\varphi_u'\Big(-(1-s)\bar{\zeta}_0,(1-s)\bar{\zeta}_0\Big).$$

Suppose that the condition (8) holds, then the probability distribution of the random process

$$\frac{1}{n}X_n^\eta(t)$$

under $P_{n,\beta}(\cdot \mid a_n^{\eta-\xi} = \lfloor an \rfloor)$, converges weakly to the corresponding probability distribution concentrated on the function

$$\frac{1}{\beta}\int_0^t F(\bar{\zeta}_0,\beta,x)dx - \frac{1}{\beta}\int_0^t F_u(\bar{\zeta}_0,\beta,x)dx. \quad (31)$$

Proof of Corollary: The random process

$$(X_n^\eta(t) \mid a_n^{\eta-\xi} = \lfloor an \rfloor)$$

can be written as

$$\begin{aligned}
&\frac{1}{n}(X_n^\eta(t) \mid a_n^{\eta-\xi} = \lfloor an \rfloor)\\
&= \frac{1}{n}(X_n^\eta(t) - X_n^\xi(t) \mid a_n^{\eta-\xi} = \lfloor an \rfloor)\\
&+ \frac{1}{n}(X_n^\xi(t) \mid a_n^{\eta-\xi} = \lfloor an \rfloor). \quad (32)
\end{aligned}$$

For the first term of above equation, under $P_{n,\beta}(\cdot \mid a_n^{\eta-\xi} = \lfloor an \rfloor)$ and by (10) in Section 2 and Remark 2, we have that the probability distribution of the random process

$$\frac{1}{n}(X_n^\eta(t) - X_n^\xi(t) \mid a_n^{\eta-\xi} = \lfloor an \rfloor)$$

converges weakly to the corresponding probability distribution of the function

$$\frac{1}{\beta}\int_0^t F(\bar{\zeta}_0,\beta,x)dx. \quad (33)$$

For the second term of above equation, under $P_{n,\beta}(\cdot \mid a_n^{\eta-\xi} = \lfloor an \rfloor)$, next we show that the probability distribution of the random process

$$\frac{1}{n}(-X_n^\xi(t) \mid a_n^{\eta-\xi} = \lfloor an \rfloor)$$

converges weakly to the corresponding probability distribution of the function

$$\frac{1}{\beta}\int_0^t F_u(\bar{\zeta}_0,\beta,x)dx. \quad (34)$$

Let

$$0 < t_1 < \cdots < t_q \le 1$$

be a set of real numbers, set a random vector as

$$\begin{aligned}
&\hat{X}_n^\xi(t_1,\cdots,t_q) =\\
&(a_n^{\eta-\xi}, -A(x^1_{\lfloor t_1 n \rfloor}),\cdots,-A(x^1_{\lfloor t_q n \rfloor})). \quad (35)
\end{aligned}$$

Let the sequence $\underline{\zeta}_n^\circ, \underline{\zeta}^\circ$ be defined in (24), and with the same condition of (25), we have the corresponding function as following

$$\begin{aligned}
&\varphi_n^1(\underline{\zeta};t_1,\cdots,t_q)\\
&= \frac{1}{n}\ln\Big(\sum_{\xi,\eta} e^{\beta\underline{\zeta}\cdot\hat{X}_n^\xi(t_1,\cdots,t_q)}e^{-\beta H_n(\xi,\eta)}\Big/Z_{n,\beta}\Big)\\
&= \frac{1}{n}\ln\prod_{t=1}^n Q\Big(-\zeta_n^\xi(t;\underline{\zeta}),\zeta_n^\eta(t;\zeta_0)\Big) \quad (36)
\end{aligned}$$

where

$$\zeta_n^\xi(t;\underline{\zeta}) = \zeta_0(1-t/n) + \sum_{i=1}^q \zeta_i 1_{[0,t_i n]}(t),$$

$$\zeta_n^\eta(t;\zeta_0) = \zeta_0(1-x/n),$$

for any

$$\underline{\zeta} = (\zeta_0,\zeta_1,\cdots,\zeta_q) \in D_{\alpha_1,\bar{\zeta}_0}.$$

Let

$$\varphi^1(\underline{\zeta};t_1,\cdots,t_q) = \lim_{n\to\infty}\varphi_n^1(\underline{\zeta};t_1,\cdots,t_q)$$

and $\hat{E}^1_{n,\underline{\zeta}}(\cdot)$ be the expectation for the random vector of

$$\hat{X}_n^\xi(t_1,\cdots,t_q)$$

(see (20)), then we have

$$\begin{aligned}
&\hat{E}^1_{n,\underline{\zeta}_n^\circ}\hat{X}_n^\xi(t_1,\cdots,t_q)\\
&= (\lfloor an \rfloor, \hat{E}^1_{n,\underline{\zeta}_n^\circ}(-A(x^1_{\lfloor t_1 n \rfloor})),\cdots,\\
&\quad \hat{E}^1_{n,\underline{\zeta}_n^\circ}(-A(x^1_{\lfloor t_q n \rfloor})))\\
&= \frac{n}{\beta}(\nabla_{\underline{\zeta}}\varphi_n^1)(\underline{\zeta}_n^\circ;t_1,\cdots,t_q)\\
&= \frac{n}{\beta}(\nabla_{\underline{\zeta}}\varphi^1)(\underline{\zeta}^\circ;t_1,\cdots,t_q) + o(1), \quad (37)
\end{aligned}$$

where for $1 \le i \le q$, and

$$\hat{E}^1_{n,\underline{\zeta}^\circ_n}(-A(x^1_{\lfloor t_i n \rfloor})) = \sum_{t=1}^{n} \frac{\partial}{\partial \zeta_i} \ln Q\Big(-\zeta^\xi_n(t;\underline{\zeta}), \zeta^\eta_n(t;\zeta_0)\Big)\Big|_{\underline{\zeta}=\underline{\zeta}^\circ_n}. \quad (38)$$

For the random vector

$$\hat{X}^\xi_n(t_1, \cdots, t_q)$$

by using the methods of Lemma 2.6 and Proposition 2.7 in [8], we can have the similar results as that of (I) and (II). Then following the steps of (28)∼(30) in Section 4, we can prove that the probability distribution of the random process

$$\frac{1}{\sqrt{n}}(-X^\xi_n(t) - \frac{n}{\beta}\int_0^t F_u(\bar{\zeta}_0, \beta, s)ds), \quad (39)$$

under $P_{n,\beta}(\cdot \mid a^{\eta-\xi}_n = \lfloor an \rfloor)$, converges weakly to some Gaussian distribution. Thus by Remark 2, the probability distribution of the random process

$$\frac{1}{n}(-X^\xi_n(t))$$

converges weakly to the corresponding probability distribution concentrated on the function

$$Y_1(t) = \frac{1}{\beta}\int_0^t F_u(\bar{\zeta}_0, \beta, s)ds. \quad (40)$$

This completes the proof of (34), so that we finish the proof of Corollary.

6 Finite Dimensional Probability Distribution of Extended Stock Indices

In Section 1-4 of the present paper, we discuss the fluctuation properties of two random stock indices which is defined in (6)(9). In this part, we modified these definitions and we discuss the convergence of the corresponding finite dimensional probability distributions.

In this section, we define the stock price S_t on

$$\Omega \times \tilde{\Omega} = \{(\omega, \tilde{\omega}) : \omega \in \Omega, \tilde{\omega} \in \tilde{\Omega}\}$$

with the probability $P(\cdot)$, such that

$$P(\omega, \tilde{\omega}) = P(\omega) \cdot \tilde{P}(\tilde{\omega}).$$

On the space $\tilde{\Omega}$, let $\{B(\tilde{\omega}_k), k = 1, \cdots, t\}$ be the random sequence with the independent and identical distributions. More specifically, from (6)(9), the stock price $S_t (t = 1, 2, \cdots)$ which describes the statistical behavior of two stock prices for all the investors at time t is defined by

$$S_t = S_0 \exp\{\alpha(\sum_{k=1}^{t}(\eta_k - \xi_k) \mid a^{\eta-\xi}_n = \lfloor an \rfloor) + \sum_{k=1}^{t} B(\tilde{\omega}_k)\} \quad (41)$$

where S_0 is initial stock price at time 0, and the parameter $\alpha > 0$ (for the simplicity, let $\alpha = 1$). Now we discuss the price model with the continuous time, which are defined from above (9). The normalized process $C^n_v, v \in [0, 1]$ is defined by

$$\begin{aligned} C^n_v &= A^n_v(\omega) + B^n_v(\tilde{\omega}) \\ &= \frac{1}{\sqrt{n}}[(\sum_{k=1}^{[nv]}(\eta_k - \xi_k) \mid a^{\eta-\xi}_n = \lfloor an \rfloor)] \\ &\quad + \frac{1}{\sqrt{n}}\sum_{k=1}^{[nv]} B(\tilde{\omega}_k). \end{aligned} \quad (42)$$

The normalized process $B^n_v(\tilde{\omega})$, is defined by

$$B^n_v(\tilde{\omega}) = \frac{1}{\sqrt{n}}\sum_{k=1}^{[nv]} B(\tilde{\omega}_k), \quad v \in [0, 1]. \quad (43)$$

In this section, by properly choosing the random sequence $\{B(\tilde{\omega}_k), k = 1, \cdots, t\}$, we suppose that

$$\frac{1}{\sqrt{n}}\left\{\sum_{k=1}^{[nv]} B(\tilde{\omega}_k) + \frac{n}{\beta}\int_0^v F(\bar{\zeta}_0, \beta, s)ds\right\} \quad (44)$$

under $\tilde{P}(\cdot)$, converges weakly to the process

$$\int_0^v \mu(s)ds, \quad v \in [0, 1]. \quad (45)$$

In order to discuss the convergence of the finite dimensional probability distribution for the stock price, let $l \ge 1$, and let

$$0 < v_1 < \cdots < v_l \le 1.$$

For

$$\underline{\zeta} = (\zeta_0, \zeta_1, \cdots, \zeta_l) \in R^{l+1}$$

such that

$$-b < \zeta_0 < \bar{\zeta}_0 + b$$

and

$$|\zeta_q| < c, \quad (q = 1, \cdots, l,\ \bar{\zeta}_0 > 0, b > 0, c > 0)$$

under the probability $P(\cdot)$, the corresponding function for the normalized process $C_v^n, v \in [0,1]$ is given by (see Section 3 and (43))

$$\begin{aligned}
&\psi^{(n,l)}_{v_1,\cdots,v_l}(\zeta_1,\cdots,\zeta_l)\\
&= E[\exp\{\underline{\zeta}\cdot \hat{X}^{(l)}_n(v_1,\cdots,v_l) + \sum_{q=1}^{l}\zeta_q B^n_{v_q}(\tilde{\omega})\}]\\
&= E[\exp\{\underline{\zeta}\cdot \hat{X}^{(l)}_n(v_1,\cdots,v_l)\}]\\
&\quad \times E[\exp\{\sum_{q=1}^{l}\zeta_q B^n_{v_q}(\tilde{\omega})\}]. \qquad (46)
\end{aligned}$$

From (43), we study the limit of the function

$$\psi^{(n,l)}_{v_1,\cdots,v_l}(\zeta_1,\cdots,\zeta_l)$$

(as $n \to \infty$). Following the similar procedure of the proof in Section 3, Section 4 and (45), and according to the similar above argument, then we have

$$\begin{aligned}
&\lim_{n\to\infty}\psi^{(n,l)}_{v_1,\cdots,v_l}(\zeta_1,\cdots,\zeta_l)\\
&= \exp\{\sum_{q=1}^{l}\zeta_q\int_0^{v_q}\mu(v)\mathrm{d}v\\
&+\frac{1}{2}\sum_{q=1}^{l}\sum_{q'=1}^{l}\zeta_q\zeta_{q'}\int^{v_q\wedge v_{q'}}\sigma^2(v)\mathrm{d}v\}.
\end{aligned}$$

Then the finite dimensional probability distribution of the normalized stock price $S_0\exp\{C_v^n\}$ converges to the corresponding finite dimensional probability distribution of random process

$$S_v = S_0\exp\{\int_0^v \mu(u)\mathrm{d}u + \int_0^v \sigma(u)\mathrm{d}B(u)\}. \qquad (47)$$

This completes the proof of the convergence of the finite dimensional probability distribution for the normalized stock price $S_0\exp\{C_v^n\}$.

7 Conclusion

In the present paper, the chain reaction between two stock indices is modelled by the statistical physics dynamic systems in Section 2. In this financial model, an intermediate region of the two random stock indices is constructed, where the region may represent the information or the situation on these two stock indices, including the estimation for these stock indices, positive or negative news, trends, political event and economic policy, etc. This kind of research is a new approach to study the statistical properties of fluctuations of stock market. In Section 3 and Section 4, we show the convergence of probability distributions of the normalized stock indices, where the stochastic process theory is applied to show the results of this paper. Further in Section 5, we study the fluctuation of one stock index which is affected by the chain reaction. In Section 6, we discuss the convergence of the corresponding finite dimensional probability distributions. The research work of the present paper is mainly base on the theory of the statistical physics systems, Gibbs measure and the convergence theory of stochastic process.

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