Demonic fuzzy operators

Huda Alrashidi
King Saud University
Mathematics department
P.O.Box 225899, Riyadh 11324
Saudi Arabia
halrashidi@ksu.edu.sa

Fairouz Tchier
King Saud University
Mathematics department
P.O.Box 22452, Riyadh 11495
Saudi Arabia
ftchier@ksu.edu.sa

Abstract: We deal with a relational algebra model to define a refinement fuzzy ordering (*demonic fuzzy inclusion*) and also the associated fuzzy operations which are fuzzy demonic join ($\sqcup_{fuz}$), fuzzy demonic meet ($\sqcap_{fuz}$) and fuzzy demonic composition ($\square_{fuz}$). We give also some properties of these operations and illustrate them with simple examples. Our formalism is the relational algebra.

Key–Words: Fuzzy sets, demonic operators, demonic fuzzy operators, demonic fuzzy ordering.

1 Relation Algebras

Our mathematical tool is abstract relation algebra [8, 31, 33], which we now introduce.

Definition 1 A *(homogeneous) relation algebra* is a structure $(\mathcal{R}, \cup, \cap, ^-, ^\smile, ;)$ over a non-empty set $\mathcal{R}$ of elements, called *relations*, such that the following conditions are satisfied:

- $(\mathcal{R}, \cup, \cap, ^-)$ is a complete Boolean algebra, with *zero* element Ø, *universal* element L and ordering $\subseteq$.
- *Composition*, denoted by (;), is associative and has an identity element, denoted by I.
- The Schröder rule is satisfied: $P;Q \subseteq R \Leftrightarrow P^\smile;\overline{R} \subseteq \overline{Q} \Leftrightarrow \overline{R};Q^\smile \subseteq \overline{P}$.
- $L;R;L = L \Leftrightarrow R \neq Ø$ (Tarski rule).

The relation $R^\smile$ is called the *converse* of R. The standard model of the above axioms is the set $\wp(S\times S)$ of all subsets of $S \times S$. In this model, $\cup, \cap, ^-$ are the usual *union, intersection* and *complement*, respectively; the relation Ø is the empty relation, the universal relation is $L = S \times S$ and the identity relation is $I = \{(s, s') \mid s' = s\}$. Converse and composition are defined by
$R^\smile = \{(s, s') \mid (s', s) \in R\}$ and $Q;R = \{(s, s') \mid \exists s'' : (s, s'') \in Q \wedge (s'', s') \in R\}$.

Definition 2 *A relation $R \subseteq X \times X$ is :*

(a) reflexive iff $I \subseteq R$, i.e. $(\forall x : (x, x) \in R)$,

(b) transitive iff $R;R \subseteq R$, i.e $(\forall x, y, z : (x, z) \in R$ and $(z, y) \in R \Rightarrow (x, y) \in R)$,

(c) symmetric iff $R \subseteq R^\smile$, i.e. $(\forall x, y : (x, y) \in R \Leftrightarrow (y, x) \in R)$,

(d) antisymmetric iff $R \cap R^\smile \subseteq I$, i.e.. $(\forall x, y : (x, y) \in R$ and $(x, y) \in R^\smile \Rightarrow x = y)$,

(e) equivalence iff R verifies properties (a), (b) and (c),

(f) order iff R verifies properties (a), (b) and (d)

The precedence of the relational operators from highest to lowest is the following: $^-$ and $^\smile$ bind equally, followed by ;, then by $\cap$, and finally by $\cup$. From now on, the composition operator symbol ; will be omitted (that is, we write QR for $Q;R$). From Definition 1, the usual rules of the calculus of relations can be derived (see, e.g., [6, 8, 31]). We assume these rules to be known and simply recall a few of them.

Theorem 3 *Let P, Q, R be relations. Then,*

1. $\overline{\bigcup_{i\in X} R_i} = \bigcap_{i\in X} \overline{R_i}$,
2. $\overline{\bigcap_{i\in X} R_i} = \bigcup_{i\in X} \overline{R_i}$,
3. $\overline{Q \cup R} = \overline{Q} \cap \overline{R}$,
4. $\overline{Q \cap R} = \overline{Q} \cup \overline{R}$,
5. $(Q \cap R) \cup \overline{R} = Q \cup \overline{R}$,
6. $P \cap Q \subseteq R \;\Leftrightarrow\; P \subseteq \overline{Q} \cup R$,
7. $Q \subseteq R \;\Leftrightarrow\; \overline{R} \subseteq \overline{Q}$,
8. $Q(\bigcup_{i\in X} R_i) = \bigcup_{i\in X} QR_i$,
9. $(\bigcup_{i\in X} Q_i)R = \bigcup_{i\in X} Q_iR$,

10. $(P \cup Q)R = PR \cup QR$,
11. $P(Q \cup R) = PQ \cup PR$,
12. $Q(\bigcap_{i \in X} R_i) \subseteq \bigcap_{i \in X} QR_i$,
13. $(\bigcap_{i \in X} Q_i)R \subseteq \bigcap_{i \in X} Q_iR$,
14. $P(Q \cap R) \subseteq PQ \cap PR$,
15. $(P \cap Q)R \subseteq PR \cap QR$,
16. $(\bigcup_{i \in X} R_i)^{\smile} = \bigcup_{i \in X} R_i^{\smile}$,
17. $(Q \cup R)^{\smile} = Q^{\smile} \cup R^{\smile}$,
18. $(\bigcap_{i \in X} R_i)^{\smile} = \bigcap_{i \in X} R_i^{\smile}$,
19. $(Q \cap R)^{\smile} = Q^{\smile} \cap R^{\smile}$,
20. $(QR)^{\smile} = R^{\smile}Q^{\smile}$,
21. $R^{\smile\smile} = R$,
22. $I^{\smile} = I$,
23. $\overline{R}^{\smile} = \overline{R^{\smile}}$,
24. $Q \subseteq R \;\Rightarrow\; PQ \subseteq PR$,
25. $Q \subseteq R \;\Rightarrow\; QP \subseteq RP$.
26. $R\emptyset = \emptyset R = \emptyset$,
27. $RI = IR = R$
28. $\overline{RL}L = \overline{RL}$,
29. $PQ \cap R \subseteq (P \cap RQ^{\smile})(Q \cap P^{\smile}R)$,
30. $PQ \cap R \subseteq P(Q \cap P^{\smile}R)$,
31. $PQ \cap R \subseteq (P \cap RQ^{\smile})Q$,
32. $(P \cap QL)R = PR \cap QL$,
33. $LL = L$,
34. $(\bigcap_{i \in X} R_iL)L = \bigcap_{i \in X} R_iL$,
35. $(QL \cap RL)L = QL \cap RL$,
36. $(\bigcup_{i \in X} R_iL)L = \bigcup_{i \in X} R_iL$,
37. $(QL \cup RL)L = QL \cup RL$,
38. $(P \cap QL)R = PR \cap QL$,
39. $(P \cap LQ^{\smile})R = P(R \cap QL)$,
40. $QLR = QL \cap LR$.
41. $R = (I \cap RR^{\smile})R$.

Definition 4 *A relation R is :*

(a) deterministic *iff $R^{\smile}R \subseteq I$,*

(b) total *iff $L = RL$ (equivalent to $I \subseteq RR^{\smile}$),*

(c) an application iff it is total and deterministic,

(d) injective *iff $R^{\smile}$ is deterministic (i.e. $RR^{\smile} \subseteq I$),*

(e) surjective *iff $R^{\smile}$ is total (i.e. $LR = L$, or also $I \subseteq R^{\smile}R$),*

(f) a partial identity *iff $R \subseteq I$ (sub-identity),*

(g) a vector *iff $R = RL$ (the vectors are usually denoted by the letter v),*

(h) a point *iff $R \neq \emptyset$, $R = RL$ and $RR^{\smile} \subseteq I$.* ∎

A function *is a deterministic relation.*

Theorem 5 *Let P, Q and R be relations.*

(a) Q*deterministic* $\Rightarrow Q(\bigcap_{i \in X} R_i) = \bigcap_{i \in X} QR_i$,

(b) Q *injective* $\Rightarrow (\bigcap_{i \in X} R_i)Q = \bigcap_{i \in X} R_iQ$,

(c) P *deterministic* $\Rightarrow (Q \cap RP^{\smile})P = QP \cap R$,

(d) P *injective* $\Rightarrow P(P^{\smile}Q \cap R) = Q \cap PR$,

(e) Q *total* $\Leftrightarrow \overline{QR} \subseteq Q\overline{R}$,

(f) Q *deterministic* $\Rightarrow Q\overline{R} = QL \cap \overline{QR}$,

(g) Q *application* $\Rightarrow Q\overline{R} = \overline{QR}$,

(h) Q *surjective* $\Leftrightarrow \overline{RQ} \subseteq \overline{R}Q$,

(i) Q *injective* $\Rightarrow \overline{R}Q = LQ \cap \overline{RQ}$,

(j) Q *deterministic* $\Rightarrow Q\overline{R} \cup \overline{QL} = \overline{QR}$,

(k) Q *injective* $\Rightarrow \overline{R}Q \cup \overline{LQ} = \overline{RQ}$,

(l) Q, R *deterministic* $\Rightarrow QR$ *deterministic,*

(m) Q, R *injectives* $\Rightarrow QR$ *injective*,

(n) Q, R *total* $\Rightarrow QR$ *total*,

(o) Q, R *surjective* $\Rightarrow QR$ *surjective*,

(p) $Q \subseteq R, R$ *deterministic and* $RL \subseteq QL \Rightarrow Q = R$,

(q) $Q \subseteq R, R$ *injective and* $LR \subseteq LQ \Rightarrow Q = R$,

(r) R *deterministic* $\Rightarrow Q \cap R$ *deterministic*,

(s) R *injective* $\Rightarrow Q \cap R$ *injective*,

(t) R *total* $\Rightarrow Q \cup R$ *total*,

(u) R *surjective* $\Rightarrow Q \cup R$ *surjective*. ∎

2 Fuzzy Relation

Fuzzy relations are fuzzy subsets of $A \times B$, that is, mapping from $A \to B$. They have been studied by a number of authors, in particular by Zadeh [41],[42], Kaufmann [21] and Rosenfeld [29]. Applications of fuzzy relations are widespread and important.

Definition 6 *Let $A, B \in U$ be universal sets, a fuzzy relation $\tilde{R}$ on $A \times B$ is defined by:* $\tilde{R} = \{((x,y), \mu_{\tilde{R}}(x,y) \mid (x,y) \in A \times B, \mu_{\tilde{R}}(x,y) \in [0,1]\}$ *is called a* fuzzy relation *on* $A \times B$.

Example 7
$\tilde{R} =$ "*x considerably larger than y*", *we have:* ,

$$\tilde{R} = \begin{pmatrix} 0.8 & 1 & 0.1 & 0.7 \\ 0 & 0.8 & 0 & 0 \\ 0.9 & 1 & 0.7 & 0.8 \end{pmatrix}$$

and $\tilde{S} =$ "*y very close to x*"

$$\tilde{S} = \begin{pmatrix} 0.4 & 0 & 0.9 & 0.6 \\ 0.9 & 0.4 & 0.5 & 0.7 \\ 0.3 & 0 & 0.8 & 0.5 \end{pmatrix}.$$

2.1 Basic Operations On Fuzzy Relations

Definition 8 *Let $\tilde{R}$ and $\tilde{S}$ be two fuzzy relations on $A \times B$. Then the following operations are defined:*

- *Union:* $\mu_{\tilde{R}\cup\tilde{S}}(x,y) = \mu_{\tilde{R}}(x,y) \vee \mu_{\tilde{S}}(x,y)$,
- *Intersection:* $\mu_{\tilde{R}\cap\tilde{S}}(x,y) = \mu_{\tilde{R}}(x,y) \wedge \mu_{\tilde{S}}(x,y)$,
- *Max-min composition:* $\tilde{R} \circ \tilde{S} = \{[(x,z), \vee_y\{\mu_{\tilde{R}}(x,y) \wedge \mu_{\tilde{S}}(y,z)\}]\}$.

Example 9

$$\tilde{R} = \begin{pmatrix} 0.8 & 1 & 0.1 \\ 0 & 0.8 & 0 \\ 0.9 & 1 & 0.7 \end{pmatrix}, \tilde{S} = \begin{pmatrix} 0.4 & 0 & 0.9 \\ 0.9 & 0.4 & 0.5 \\ 0.3 & 0 & 0.8 \end{pmatrix}.$$

Then:

- $\tilde{R} \cup \tilde{S} = \begin{pmatrix} 0.8 & 1 & 0.9 \\ 0.9 & 0.8 & 0.5 \\ 0.9 & 1 & 0.8 \end{pmatrix}$,
- $\tilde{R} \cap \tilde{S} = \begin{pmatrix} 0.4 & 0 & 0.1 \\ 0 & 0.4 & 0 \\ 0.3 & 0 & 0.7 \end{pmatrix}$,
- $\tilde{R} \circ \tilde{S} = \begin{pmatrix} 0.9 & 0.4 & 0.8 \\ 0.8 & 0.4 & 0.5 \\ 0.9 & 0.4 & 0.9 \end{pmatrix}$.

Definition 10 *Let $\tilde{R}$ be a fuzzy relation on $A \times A$.*

- $\tilde{R}$ *is reflexive [42] iff* $\mu_{\tilde{R}}(x,x) = 1 \; \forall x \in A$,
- $\tilde{R}$ *is transitive iff* $\mu_{\tilde{R}}(x,z) \geq \mu_{\tilde{R}}(x,y) \wedge \mu_{\tilde{R}}(y,z)$, $\forall x, y, z \in A$,
- $\tilde{R}$ *is symmetric iff* $\tilde{R}(x,y) = \tilde{R}(y,x)$,
- $\tilde{R}$ *is antisymmetric [21] iff for* $x \neq y$ *either* $\mu_{\tilde{R}}(x,y) \neq \mu_{\tilde{R}}(y,x)$ *or* $\mu_{\tilde{R}}(x,y) = \mu_{\tilde{R}}(y,x) = 0$, $\forall x, y \in A$,
- $\tilde{R}$ *is equivalence iff* $\tilde{R}$ *is reflexive, transitive, and symmetric,*
- $\tilde{R}$ *is order iff* $\tilde{R}$ *is reflexive, transitive, and anti-symmtric.*

Example 11

- ($\tilde{R}$ *is reflexive):*

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0.2 & 0.3 \\ 0 & 1 & 0.1 & 1 \\ 0.2 & 0.7 & 1 & 0.4 \\ 0 & 1 & 0.4 & 1 \end{pmatrix} \end{array}$$

- ($\tilde{R}$ *is transitive):*

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.2 & 1 & 0.4 & 0.4 \\ 0 & 0.6 & 0.3 & 0 \\ 0 & 1 & 0.3 & 0 \\ 0.1 & 1 & 1 & 0.1 \end{pmatrix} \end{array}$$

- ($\tilde{R}$ *is symmetric):*

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.1 & 1 & 0.2 & 0.3 \\ 0 & 0.2 & 0.8 & 0.8 \\ 0.1 & 0.3 & 0.8 & 1 \end{pmatrix} \end{array}$$

- *($\tilde{R}$ is antisymmetric):*

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.4 & 0 & 0.7 & 0 \\ 0 & 1 & 0.9 & 0.6 \\ 0.8 & 0.4 & 0.7 & 0.4 \\ 0 & 0.1 & 0 & 0 \end{pmatrix} \end{array}$$

- *($\tilde{R}$ is equivalence):*

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{array} \\ \begin{pmatrix} 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ 0.6 & 0.2 & 0.6 & 0.8 & 0.2 & 1 \end{pmatrix} \end{array}$$

.

The following properties have been proved to hold for fuzzy relations (see [22, 23]);

Theorem 12 *Let $\tilde{R}$, $\tilde{S}$ and $\tilde{T}$ be fuzzy relations. Then:*

(a) $\tilde{R}(\tilde{S}\tilde{T}) = (\tilde{R}\tilde{S})\tilde{T}$,

(b) $\tilde{R}(\tilde{S} \cup \tilde{T}) = (\tilde{R}\tilde{S}) \cup (\tilde{R}\tilde{T})$,

(c) $\tilde{R}(\tilde{S} \cap \tilde{T}) \subseteq (\tilde{R}\tilde{S}) \cap (\tilde{R}\tilde{T})$,

(d) $\tilde{S} \subseteq \tilde{T} \Longrightarrow \tilde{R}\tilde{S} \subseteq \tilde{R}\tilde{T}$,

(e) $\tilde{S} \subseteq \tilde{T} \Longrightarrow \tilde{S}\tilde{R} \subseteq \tilde{T}\tilde{R}$,

(f) $\tilde{R}\tilde{I} = \tilde{I}\tilde{R} = \tilde{R}$ *for all fuzzy relation* $\tilde{R}$,

(g) $(\tilde{R}\tilde{S})^{\smile} = \tilde{S}^{\smile}\tilde{R}^{\smile}$,

(h) $\tilde{R}^{\smile\smile} = \tilde{R}$,

(i) $(\tilde{R} \cup \tilde{S})^{\smile} = \tilde{R}^{\smile} \cup \tilde{S}^{\smile}$,

(j) $(\tilde{R} \cap \tilde{S})^{\smile} = \tilde{R}^{\smile} \cap \tilde{S}^{\smile}$,

(k) $\tilde{R} \subseteq \tilde{S} \Longrightarrow \tilde{R}^{\smile} \subseteq \tilde{S}^{\smile}$.

3 A demonic order refinement

We will give the definition of our ordering.

Definition 13 *We say that a relation Q refines a relation R [25], denoted by $Q \sqsubseteq R$, iff*

$$RL \subseteq QL \text{ and } Q \cap RL \subseteq R,$$

or equivalently, iff

$$Q \cup \overline{QL} \subseteq R \cup \overline{RL} \text{ and } \overline{QL} \subseteq \overline{RL}.$$

■

Example 14

- $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \sqsubseteq \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,
- $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \not\sqsubseteq \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Theorem 15 *The relation $\sqsubseteq$ is a partial order.*

Proof. From Definition(13), it easily follows that the refinement relation is antisymmetric:

$$\begin{array}{ll} & Q \sqsubseteq R \text{ and } R \sqsubseteq Q \\ \Longleftrightarrow & \quad \{ \text{Definition (13)} \} \\ & Q \cup \overline{QL} = R \cup \overline{RL} \text{ and } \overline{QL} = \overline{RL} \\ \Longrightarrow & \quad \{ \overline{QL} = \overline{RL} \Longleftrightarrow QL = RL \} \\ & (Q \cup \overline{QL}) \cap QL = (R \cup \overline{RL}) \cap RL \\ \Longrightarrow & \quad \{ \text{Boolean law.} \} \\ & Q = R \end{array}$$

Consequently, $\sqsubseteq$ is reflexive and transitive (due to the fact that the inclusion $\subseteq$ has these properties); since it is also antisymmetric, it is partial order.

3.1 Demonic operators

In this subsection, we will present demonic operators and also some of their properties. For more details see [4, 5, 7, 13]. To clarify the ideas, take two relations Q and R:

- Their supremum is

$$Q \sqcup R = (Q \cup R) \cap QL \cap RL,$$

and satisfies

$$(Q \sqcup R)L = QL \cap RL.$$

Then, $Q \sqcup R$ is exactly the relational expression of the *demonic union* as defined by [4, 5] (which explains the word *demonic* of $\sqcup$-semilattice $(\mathcal{B}_R, \sqsubseteq))$.

- Their infimum, if it exists, is

$$\begin{aligned} Q \sqcap R &= (Q \cup \overline{QL}) \cap (R \cup \overline{RL}) \cap (QL \cup RL) \\ &= Q \cap R \cup Q \cap \overline{RL} \cup R \cap \overline{QL}, \end{aligned}$$

and it satisfies

$$(Q \sqcap R)L = QL \cup RL.$$

The operator $\sqcap$ is called *demonic intersection*. For $Q \sqcap R$ to exist, we have to verify $L \subseteq ((Q \cup \overline{QL}) \cap (R \cup \overline{RL}))L$. This condition is equivalent to $QL \cap RL \subseteq (Q \cap R)L$, which can be interpreted as follows: the existence condition simply means on the intersection of their domains, Q and R have to agree for at least one value.

Example 16

- $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sqcup \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,
- $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sqcap \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

In what follows, we will give the definition of demonic composition [4, 5, 6].

Definition 17 *The binary operator $\triangleright$, called relative implication, is defined as follows :*

$$Q \triangleright R \stackrel{def}{=} \overline{Q\overline{R}}.$$

Definition 18 *The demonic composition of relations Q and R is*

$$Q \,\square\, R = QR \cap Q \triangleright RL.$$

■

Example 19

- $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \square \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

3.2 Properties of demonic operators

The demonic operators $\sqcap, \sqcup$ and $\square$ have the same properties as $\cap, \cup$ and (;), but the demonic intersections have to be defined. Let us give some of them.

Theorem 20 *Let P, Q and R be relations. Then,*

(a) $P \sqcap (Q \sqcup R) = (P \sqcap Q) \sqcup (P \sqcap R)$,

(b) $P \sqcup (Q \sqcap R) = (P \sqcup Q) \sqcap (P \sqcup R)$,

(c) $R \,\square\, I = I \,\square\, R = R$,

(d) $Q \sqsubseteq R \Rightarrow P \,\square\, Q \sqsubseteq P \,\square\, R$,

(e) $P \sqsubseteq Q \Rightarrow P \,\square\, R \sqsubseteq Q \,\square\, R$,

(f) $P \,\square\, (Q \sqcup R) = P \,\square\, Q \sqcup P \,\square\, R$,

(g) $(P \sqcup Q) \,\square\, R = P \,\square\, R \sqcup Q \,\square\, R$,

(h) $P \,\square\, (Q \sqcap R) \sqsubseteq P \,\square\, Q \sqcap P \,\square\, R$,

(i) $P \,\square\, (Q \,\square\, R) = (P \,\square\, Q) \,\square\, R$,

(j) $(P \sqcap Q) \,\square\, R \sqsubseteq P \,\square\, R \sqcap Q \,\square\, R$. ■

Proposition 21

(a) Q *deterministic* $\Rightarrow Q \,\square\, R = QR$,

(b) P *deterministic* $\Rightarrow P \,\square\, (Q \sqcap R) = PQ \sqcap PR$,

(c) R *total* $\Rightarrow Q \,\square\, R = QR$,

(d) $PL \cap QL = \emptyset \Rightarrow (P \cup Q) \,\square\, R = P \,\square\, R \cup Q \,\square\, R$,

(e) $PL \cap QL = \emptyset \Rightarrow P \sqcap Q = P \cup Q$.

4 A demonic fuzzy order refinement

We will give the definition of domain of fuzzy relations $\tilde{R}$.

Definition 22 *The domain of $\tilde{R}$ is supremum of value in first row of the matrix, and the image of $\tilde{R}$ is supremum of value in first column of the matrix. Formally,*

$$dom(\tilde{R}) \stackrel{def}{=} sup_{y \in B}\{((x,y), \mu_{\tilde{R}}(x,y)) \mid \forall x \in A\},$$

$$img(\tilde{R}) \stackrel{def}{=} sup_{x \in A}\{((x,y), \mu_{\tilde{R}}(x,y)) \mid \forall y \in B\}.$$

- *The vectors $\tilde{R}\tilde{L}$ and $\tilde{R}^{\smile}\tilde{L}$ are particular vectors characterizing respectively the domain and codomain of $\tilde{R}$.*

Now, we will give the definition of fuzzy ordering.

Definition 23 *We say that a fuzzy relation $\tilde{Q}$ fuzzy refines a fuzzy relation $\tilde{R}$, denoted by $\tilde{Q} \sqsubseteq_{fuz} \tilde{R}$, iff*

$$\tilde{R}\tilde{L} \subseteq \tilde{Q}\tilde{L} \text{ and } \tilde{Q} \cap \tilde{R}\tilde{L} \subseteq \tilde{R}$$

i.e $(\vee_{y \in B}\{\mu_{\tilde{R}}(x,y)\} \leq \vee_{y \in B}\{\mu_{\tilde{Q}}(x,y)\})$ *and* $(\mu_{\tilde{Q}}(x,y) \wedge (\vee_{y \in B}\{\mu_{\tilde{R}}(x,y)\}) \leq \mu_{\tilde{R}}(x,y))$.

In other words, $\tilde{Q}$ refines $\tilde{R}$ if and only if the prerestriction of $\tilde{Q}$ to the domain of $\tilde{R}$ is included in $\tilde{R}$. This means that $\tilde{Q}$ must not produce results not allowed by $\tilde{R}$ for those states that are in the domain of $\tilde{R}$.

Example 24

- $\begin{pmatrix} 0.3 & 0.2 & 0.4 \\ 0.7 & 0.8 & 0.8 \\ 0.3 & 0.5 & 0.6 \end{pmatrix} \sqsubseteq_{fuz} \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.4 & 0.5 & 0.9 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$
- $\begin{pmatrix} 0.1 & 0.2 & 0.4 \\ 0.5 & 0.7 & 0.9 \end{pmatrix} \not\sqsubseteq_{fuz} \begin{pmatrix} 0.2 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.8 \end{pmatrix}$.

Theorem 25 *The relation $\sqsubseteq$ is a partial order.*

4.1 Fuzzy Demonic operators

In this subsection, we will present fuzzy demonic operators and also some of their properties.

To clarify the ideas, take two relations $\tilde{Q}$ and $\tilde{R}$:

- Their supremum is

$$\tilde{Q} \sqcup_{fuz} \tilde{R} = (\tilde{Q} \vee \tilde{R}) \wedge \tilde{Q}\tilde{L} \wedge \tilde{R}\tilde{L},$$

$\Longleftrightarrow$

$$\mu_{(\tilde{Q} \sqcup_{fuz} \tilde{R})}(x,y) = min\{max\{\mu_{\tilde{Q}}(x,y), \mu_{\tilde{R}}(x,y)\}, max_y(\mu_{\tilde{Q}}(x,y)), max_y(\mu_{\tilde{R}}(x,y))\}$$

and satisfies

$$(\tilde{Q} \sqcup_{fuz} \tilde{R})\tilde{L} = \tilde{Q}\tilde{L} \cap \tilde{R}\tilde{L}.$$

Then, $\tilde{Q} \sqcup_{fuz} \tilde{R}$ is exactly the relational expression of the *fuzzy demonic union*.

- Their infimum, if it exists, is

$$\tilde{Q} \sqcap_{fuz} \tilde{R} = (\tilde{Q} \wedge \tilde{R}) \vee (\tilde{Q} \wedge 1 - \tilde{R}\tilde{L}) \vee (\tilde{R} \wedge 1 - \tilde{Q}\tilde{L})$$

$\Longleftrightarrow$

$$\mu_{(\tilde{Q} \sqcap_{fuz} \tilde{R})}(x,y) = max\{min\{\mu_{\tilde{Q}}(x,y), \mu_{\tilde{R}}(x,y)\}, min\{\mu_{\tilde{Q}}(x,y), 1 - max_y(\mu_{\tilde{R}}(x,y))\}, min\{\mu_{\tilde{R}}(x,y), 1 - max_y(\mu_{\tilde{Q}}(x,y))\}\}$$

and it satisfies

$$(\tilde{Q} \sqcap_{fuz} \tilde{R})\tilde{L} = \tilde{Q}\tilde{L} \cup \tilde{R}\tilde{L}.$$

The operator $\sqcap_{fuz}$ is called *fuzzy demonic intersection*. For $\tilde{Q} \sqcap_{fuz} \tilde{R}$ to exist, we have to verify $\tilde{L} \subseteq (\tilde{Q} \cup \bar{\tilde{Q}\tilde{L}} \cap \tilde{R} \cup \bar{\tilde{R}\tilde{L}})$. This condition is equivalent to $\tilde{Q}\tilde{L} \cap \tilde{R}\tilde{L} \subseteq (\tilde{Q} \cap \tilde{R})\tilde{L}$, which can be interpreted as follows: the existence condition simply means that on the intersection of their domains, $\tilde{Q}$ and $\tilde{R}$ have to agree for at least one value.

In what follows, we will give the definition of the fuzzy demonic composition.

Definition 26 *The fuzzy demonic composition of relations $\tilde{Q}$ and $\tilde{R}$ is*

$$\tilde{Q} \,\square_{fuz} \tilde{R} = \tilde{Q}\tilde{R} \wedge 1 - \tilde{Q}\overline{\tilde{R}\tilde{L}}$$

$\Longleftrightarrow$

$$\mu_{(\tilde{Q} \,\square_{fuz} \tilde{R})}(x,y) = min[max_y\{min\{\mu_{\tilde{Q}}(x,y), \mu_{\tilde{R}}(y,z)\}\}, 1 - max_y\{min\{\mu_{\tilde{Q}}(x,y), 1 - max_y(\mu_{\tilde{R}}(x,y))\}\}]$$

Example 27

$$\tilde{Q} = \begin{pmatrix} 0.1 & 0 & 0.2 \\ 0.3 & 0.8 & 1 \\ 0 & 1 & 0.7 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0.3 & 0.5 & 0.4 \\ 0.9 & 0.7 & 0.2 \end{pmatrix}.$$

Then:

- $\tilde{Q} \sqcup_{fuz} \tilde{R} = \begin{pmatrix} 0.1 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.5 \\ 0.9 & 0.9 & 0.7 \end{pmatrix}$
- $\tilde{Q} \sqcap_{fuz} \tilde{R} = \begin{pmatrix} 0 & 0.8 & 0 \\ 0.3 & 0.5 & 0.5 \\ 0 & 0.7 & 0.2 \end{pmatrix}$
- $\tilde{Q} \,\square_{fuz} \tilde{R} == \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.4 \end{pmatrix}$

4.2 Properties of fuzzy demonic operators

The fuzzy demonic operators $\sqcap_{fuz}, \sqcup_{fuz}$ and $\square_{fuz}$, have the same properties as $\sqcap, \sqcup$ and $\square$, but the fuzzy demonic intersections have to be defined. Let us give some of them.

Theorem 28 *Let $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ be fuzzy relations. Then,*

- $\tilde{P} \sqcap_{fuz} (\tilde{Q} \sqcup_{fuz} \tilde{R}) = (\tilde{P} \sqcap_{fuz} \tilde{Q}) \sqcup_{fuz} (\tilde{P} \sqcap_{fuz} \tilde{R})$,
- $\tilde{P} \sqcup_{fuz} (\tilde{Q} \sqcap_{fuz} \tilde{R}) = (\tilde{P} \sqcup_{fuz} \tilde{Q}) \sqcap_{fuz} (\tilde{P} \sqcup_{fuz} \tilde{R})$,
- $\tilde{R} \,\square_{fuz} \tilde{I} = \tilde{I} \,\square_{fuz} \tilde{R} = \tilde{R}$,
- $\tilde{Q} \sqsubseteq_{fuz} \tilde{R} \Rightarrow \tilde{P} \,\square_{fuz} \tilde{Q} \sqsubseteq_{fuz} \tilde{P} \,\square_{fuz} \tilde{R}$,
- $\tilde{P} \sqsubseteq_{fuz} \tilde{Q} \Rightarrow \tilde{P} \,\square_{fuz} \tilde{R} \sqsubseteq_{fuz} \tilde{Q} \,\square_{fuz} \tilde{R}$,
- $\tilde{P} \,\square_{fuz} (\tilde{Q} \sqcup_{fuz} \tilde{R}) = \tilde{P} \,\square_{fuz} \tilde{Q} \sqcup_{fuz} \tilde{P} \,\square_{fuz} \tilde{R}$,
- $(\tilde{P} \sqcup_{fuz} \tilde{Q}) \,\square_{fuz} \tilde{R} = \tilde{P} \,\square_{fuz} \tilde{R} \sqcup_{fuz} \tilde{Q} \,\square_{fuz} \tilde{R}$,
- $\tilde{P} \,\square_{fuz} (\tilde{Q} \sqcap_{fuz} \tilde{R}) \sqsubseteq_{fuz} \tilde{P} \,\square_{fuz} \tilde{Q} \sqcap_{fuz} \tilde{P} \,\square_{fuz} \tilde{R}$,
- $\tilde{P} \,\square_{fuz} (\tilde{Q} \,\square_{fuz} \tilde{R}) = (\tilde{P} \,\square_{fuz} \tilde{Q}) \,\square_{fuz} \tilde{R}$,
- $(\tilde{P} \sqcap_{fuz} \tilde{Q}) \,\square_{fuz} \tilde{R} \sqsubseteq_{fuz} \tilde{P} \,\square_{fuz} \tilde{R} \sqcap_{fuz} \tilde{Q} \,\square_{fuz} \tilde{R}$.

Proposition 29

- $\tilde{Q}$ *deterministic* $\Rightarrow \tilde{Q} \,\square_{fuz} \tilde{R} = \tilde{Q}\tilde{R}$,
- $\tilde{P}$ *deterministic* $\Rightarrow \tilde{P} \,\square_{fuz} (\tilde{Q} \sqcap_{fuz} \tilde{R}) = \tilde{P}\tilde{Q} \sqcap_{fuz} \tilde{P}\tilde{R}$,

- $\tilde{R}$ *total* $\Rightarrow \tilde{Q} \,\square_{fuz} \tilde{R} = \tilde{Q}\tilde{R}$,
- $\tilde{P}\tilde{L} \sqcap_{fuz} \tilde{Q}\tilde{L} = \emptyset \Rightarrow (\tilde{P} \sqcup_{fuz} \tilde{Q}) \,\square_{fuz} \tilde{R} = \tilde{P} \,\square_{fuz} \tilde{R} \cup \tilde{Q} \,\square_{fuz} \tilde{R}$,
- $\tilde{P}\tilde{L} \sqcap_{fuz} \tilde{Q}\tilde{L} = \emptyset \Rightarrow \tilde{P} \sqcap_{fuz} \tilde{Q} = \tilde{P} \cup \tilde{Q}$.

There are many properties achieved for relations, but not fulfilled for fuzzy relations. For instance, if $\tilde{Q}$ and $\tilde{R}$ are fuzzy relations, then:

(a) $\overline{\tilde{Q} \sqcup_{fuz} \tilde{R}} \neq \overline{\tilde{Q}} \sqcap_{fuz} \overline{\tilde{R}}$,

(b) $\overline{\tilde{Q} \sqcap_{fuz} \tilde{R}} \neq \overline{\tilde{Q}} \sqcup_{fuz} \overline{\tilde{R}}$,

(c) $(\tilde{Q} \sqcap_{fuz} \tilde{R}) \sqcup_{fuz} \tilde{R} \neq \tilde{Q} \sqcup_{fuz} \overline{\tilde{R}}$,

(d) $\tilde{Q} \sqsubseteq_{fuz} \tilde{R} \not\Rightarrow \overline{\tilde{R}} \not\sqsubseteq_{fuz} \overline{\tilde{Q}}$.

Example 30

$$\tilde{Q} = \begin{pmatrix} 0.1 & 0 \\ 1 & 0.2 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.8 \end{pmatrix}.$$

- $\overline{\tilde{Q} \sqcup_{fuz} \tilde{R}} = \begin{pmatrix} 0.9 & 0.9 \\ 0.2 & 0.2 \end{pmatrix}$ *but* $\overline{\tilde{Q}} \sqcap_{fuz} \overline{\tilde{R}} = \begin{pmatrix} 0.8 & 0.7 \\ 0.2 & 0.6 \end{pmatrix}$,
- $\overline{\tilde{Q} \sqcap_{fuz} \tilde{R}} = \begin{pmatrix} 0.8 & 0.7 \\ 0.4 & 0.8 \end{pmatrix}$ *but* $\overline{\tilde{Q}} \sqcup_{fuz} \overline{\tilde{R}} = \begin{pmatrix} 0.8 & 0.8 \\ 0.4 & 0.4 \end{pmatrix}$,
- $(\tilde{Q} \sqcap_{fuz} \tilde{R}) \sqcup_{fuz} \tilde{R} = \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.2 \end{pmatrix}$ *but* $\tilde{Q} \sqcup_{fuz} \overline{\tilde{R}} = \begin{pmatrix} 0.1 & 0.1 \\ 0.4 & 0.2 \end{pmatrix}$,
- $\tilde{Q} = \begin{pmatrix} 0.1 & 0 \\ 1 & 0.2 \end{pmatrix}, \ \tilde{R} = \begin{pmatrix} 0.1 & 0.1 \\ 0.4 & 0.4 \end{pmatrix}$ *then* $\tilde{Q} \sqsubseteq_{fuz} \tilde{R}$ *but* $\overline{\tilde{R}} \not\sqsubseteq_{fuz} \overline{\tilde{Q}}$. *because*

$$\overline{\tilde{Q}}\tilde{L} = \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} \not\subseteq \begin{pmatrix} 0.9 \\ 0.6 \end{pmatrix} = \overline{\tilde{R}}\tilde{L}.$$

References:

[1] R. J. R. Back. : On the correctness of refinement in program development. Thesis, Department of Computer Science, University of Helsinki, 1978.

[2] R. J. R. Back and J. von Wright.: Combining angels, demons and miracles in program specifications. *Theoretical Computer Science*,100, 1992, 365–383.

[3] Backhouse, R. C. and van der Woude, J.: Demonic Operators and Monotype Factors. *Mathematical Structures in Comput. Sci.*, **3(4)**, 417–433, Dec. (1993). Also: Computing Science Note 92/11, Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands, 1992.

[4] Berghammer, R.: Relational Specification of Data Types and Programs. Technical report 9109, Fakultät für Informatik, Universität der Bundeswehr München, Germany, Sept. 1991.

[5] Berghammer, R. and Schmidt, G.: Relational Specifications. In C. Rauszer, editor, *Algebraic Logic*, **28** of *Banach Center Publications*. Polish Academy of Sciences, 1993.

[6] Berghammer, R. and Zierer, H.: Relational Algebraic Semantics of Deterministic and Nondeterministic Programs. *Theoretical Comput. Sci.*, **43**, 123–147 (1986).

[7] Boudriga, N., Elloumi, F. and Mili, A.: On the Lattice of Specifications: Applications to a Specification Methodology. *Formal Aspects of Computing*, **4**, 544–571 (1992).

[8] Chin, L. H. and Tarski, A.: Distributive and Modular Laws in the Arithmetic of Relation Algebras. *University of California Publications*, **1**, 341–384 (1951).

[9] Conway, J. H.: *Regular Algebra and Finite Machines*. Chapman and Hall, London, 1971.

[10] Davey, B. A. and Priestley, H. A.: *Introduction to Lattices and Order*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1990.

[11] J. Desharnais, B. Möller, and F. Tchier. Kleene under a demonic star. *8th International Conference on Algebraic Methodology And Software Technology (AMAST 2000)*, May 2000, Iowa City, Iowa, USA, *Lecture Notes in Computer Science*, Vol. 1816, pages 355–370, Springer-Verlag, 2000.

[12] Desharnais, J., Belkhiter, N., Ben Mohamed Sghaier, S., Tchier, F., Jaoua, A., Mili, A. and Zaguia, N.: Embedding a Demonic Semilattice in a Relation Algebra. *Theoretical Computer Science*, 149(2):333–360, 1995.

[13] Desharnais, J., Tchier, F.: Demonic relational semantics of sequential programs. Rapport de recherche DIUL-RR-9406, Departement d'Informatique, Universite Laval, Quebec, Canada, decembre 1995.

[14] Desharnais, J., Jaoua, A., Mili, F., Boudriga, N. and Mili, A.: A Relational Division Operator: The Conjugate Kernel. *Theoretical Comput. Sci.*, **114**, 247–272 (1993).

[15] Dilworth, R. P.: Non-commutative Residuated Lattices. *Trans. Amer. Math. Sci.*, **46**, 426–444 (1939).

[16] E. W. Dijkstra. : *A Discipline of Programming*. Prentice-Hall, Englewood Cliffs, N.J., 1976.

[17] H. Doornbos. : A relational model of programs without the restriction to Egli-Milner monotone constructs. *IFIP Transactions*, A-56:363–382. North-Holland, 1994.

[18] C. A. R. Hoare and J. He. : The weakest prespecification. *Fundamenta Informaticae IX*, 1986, Part I: 51–84, 1986.

[19] C. A. R. Hoare and J. He. : The weakest prespecification. *Fundamenta Informaticae IX*, 1986, Part II: 217–252, 1986.

[20] C. A. R. Hoare and al. : Laws of programming. *Communications of the ACM*, 30:672–686, 1986.

[21] Kaufmann, A. .: Intriduction to the Theory of Fuzzy Subsets. *Vol. I, New York, San Francisco, London*, 1975.

[22] Kaufmann, A. .: Theorie des sous-ensembles flous (Fuzzy Set Theory). *Vol. 1, 2nd. ed., Masson, Paris*, 1977: 219–232.

[23] Kawahara, Y. and Furusawa, H. .: An algebraic formatisation of fuzzy relations. Presented at second International Seminar on Relational Methods in Computer Science, Rio, Brazil, July 1995 .

[24] R. D. Maddux. : Relation-algebraic semantics. *Theoretical Computer Science*, 160:1–85, 1996.

[25] Mili, A., Desharnais, J. and Mili, F.: Relational Heuristics for the Design of Deterministic Programs. *Acta Inf.*, **24(3)**, 239–276 (1987).

[26] Mills, H. D., Basili, V. R., Gannon, J. D. and Hamlet,R. G.: *Principles of Computer Programming. A Mathematical Approach*. Allyn and Bacon, Inc., 1987.

[27] Nguyen, T. T.: A Relational Model of Demonic Nondeterministic Programs. *Int. J. Foundations Comput. Sci.*, **2(2)**, 101–131 (1991).

[28] D. L. Parnas. A Generalized Control Structure and its Formal Definition. *Communications of the ACM*, 26:572–581, 1983

[29] Rosenfeld.: A fuzzy graph. In Zedah et al., 1975, 77-96.

[30] Schmidt, G.: Programs as Partial Graphs I: Flow Equivalence and Correctness. *Theoretical Comput. Sci.*, **15**, 1–25 (1981).

[31] Schmidt, G. and Ströhlein, T.: *Relations and Graphs*. EATCS Monographs in Computer Science. Springer-Verlag, Berlin, 1993.

[32] Sekerinski, E.: A Calculus for Predicative Programming. In R. S. Bird, C. C. Morgan, and J. C. P. Woodcock, editors, *Second International Conference on the Mathematics of Program Construction*, volume 669 of *Lecture Notes in Comput. Sci.* Springer-Verlag, 1993.

[33] Tarski, A.: On the calculus of relations. *J. Symb. Log. 6*, 3, 1941, 73–89.

[34] F. Tchier.: Sémantiques relationnelles démoniaques et vérification de boucles non déterministes. Theses of doctorat, Département de Mathématiques et de statistique, Université Laval, Canada, 1996.

[35] F. Tchier.: Demonic semantics by monotypes. *International Arab conference on Information Technology* (Acit2002),University of Qatar, Qatar, 16-19 December 2002.

[36] F. Tchier.: Demonic relational semantics of compound diagrams. In: Jules Desharnais, Marc Frappier and Wendy MacCaull, editors. Relational Methods in computer Science: The Québec seminar, pages 117-140, Methods Publishers 2002.

[37] F. Tchier.: While loop d demonic relational semantics monotype/residual style. *2003 International Conference on Software Engineering Research and Practice (SERP03)*, Las Vegas, Nevada, USA, 23-26, June 2003.

[38] F. Tchier.: Demonic Semantics: using monotypes and residuals. IJMMS 2004:3 (2004) 135-160. (International Journal of Mathematics and Mathematical Sciences)

[39] M. Walicki and S. Medal.: Algebraic approches to nondeterminism: An overview. *ACM computong Surveys,29(1)*, 1997, 30-81.

[40] L.Xu, M. Takeichi and H. Iwasaki.: Relational semantics for locally nondeterministic programs. *New Generation Computing 15*, 1997, 339-362.

[41] Zadeh, L. A. .: Fuzzy Sets. *Inform and Control 8* 1965, 338–353.

[42] Zadeh, L. A. .: Similarity relations and fuzzy orderings. *Information Science 3* 1971, 177–206.

[43] Yeh, R. T., and Bang, S.Y.: Fuzzy relations,fuzzy graphs and their applictions to clustering analysis. *In Zedah et al.*, 1975.:125-150.