Multidimensional Unreplicated Linear Functional Relationship Model with Single Slope and Its Coefficient of Determination

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Abstract: - Multidimensional unreplicated linear functional relationship model (MULFR) with single slope is considered where p-dimensional measurement errors are introduced. When the ratio of error variances is known, the parameters' estimation can be considered as a generalization of the unreplicated linear functional relationship model. However, investigation on unbiased property of the estimators are not strict-forward. Taylor approximation is applied to show the intercept and slope estimators are approximately unbiased. The consistency property is discussed using Fisher Information Matrix. The coefficient of determination for MULFR model and its properties are also studied. A simulation study is carried out to evaluate the proposed estimators of the intercept and slope, and the coefficient of determination. This coefficient of determination provides a useful analysis tool for many image processing applications. A numerical example for JPEG compressed image quality assessment is explained.

Key-Words: - Coefficient of determination; Image quality; JPEG compression; Measurement errors; Unreplicated linear functional relationship

1 Introduction

Over the centuries, linear regression model has become the focus of study for many applications to investigate the relationship between a response variable and a set of explanatory variables. In certain applications such as engineering, economics, psychology, chemistry and biology, a situation arises where this relationship is obscured by random fluctuations associated with both variables (Sprent, 1969). Fuller (1987) made the same comment where the assumption that the explanatory variable can be measured exactly may not be realistic in many situations. Such experience had lead to the development of a new type of linear relationship when both variables are subject to error or so called functional relationship although other names have also been used such as 'law-like relationship', 'regression with errors in x', 'errors-in-models' and 'measurement error models'.

Adcock (1877, 1878) is the first to investigate the problem of fitting a linear relationship when both dependent variable and independent variable are subject to error (Sprent, 1990). However, this bivariate functional relationship is not appropriate in many real situations where multivariate data is considered. A number of multivariate unreplicated linear functional relationship models have been proposed to fit data. Sprent (1969), for example discussed the multidimensional (multiple) functional relationship with a single linear functional relationship given by $Y_i = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_p X_{pi}$. In Sprent's model, there is at least one or more independent linear relationships or replication, each represents a space of $p-1$ dimensions. Chan & Mak (1983, 1984) considered a multivariate linear functional relationship, in which the error variances and covariances need not be homogeneous; $\begin{bmatrix} \boldsymbol{x}_i \\ \boldsymbol{y}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{a} \end{bmatrix} + \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{B} \end{bmatrix} \boldsymbol{X}_i + \begin{bmatrix} \boldsymbol{\delta}_i \\ \boldsymbol{\varepsilon}_i \end{bmatrix}$ where

$$\boldsymbol{x}_i = \begin{bmatrix} x_{1i} \\ \vdots \\ x_{pi} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{1i} \\ \vdots \\ X_{pi} \end{bmatrix} + \begin{bmatrix} \delta_{1i} \\ \vdots \\ \delta_{pi} \end{bmatrix}$$

and

$$\boldsymbol{y}_i = \begin{bmatrix} y_{1i} \\ \vdots \\ y_{qi} \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_0 \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1p} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2p} \\ \vdots & \vdots & & \vdots \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{bmatrix} \begin{bmatrix} X_{1i} \\ \vdots \\ X_{pi} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \vdots \\ \varepsilon_{qi} \end{bmatrix}.$$

There was also interest in considering several simultaneous linear relationships between p variates subject to error by Gleser & Watson (1973). These simultaneous relationships frequently occurred in economics and physical sciences where it often involves large measurement errors. An excellent work on functional data analysis had also been done by Ramsay and Silverman (1997). They had considered a wide range of functional linear models which includes functional canonical correlation analysis, relationship of the response y and the covariate x is modelled by $\hat{y}_i(t) = \alpha(t) + \int_{\Upsilon_X} x_i(t)\beta(s,t)\,ds$, functional responses with multivariate covariates, functional linear models for scalar responses and functional linear models for functional responses.

This paper considers a new functional linear relationship model where the $(\boldsymbol{X}_i, \boldsymbol{Y}_i)$ observations are p-dimensional with single slope. It can be considered as a generalization of the unreplicated linear functional relationship (ULFR) model proposed by Adcock (1877) where the bivariate observations are only one-dimensional. The essential difference of the proposed model and the multivariate model of Chan & Mak (1983, 1984) is the new model considers different elements in the intercept vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)'$ and the slope matrix $\boldsymbol{B}$ is replaced by a single slope β. These ULFR and MULFR models and their coefficient of determination are particular useful in many image processing applications where the relationship or similarity between two images are concerned, such as full reference image quality assessment, performance evaluation and feature matching in pattern recognition. For example, Chang et al. (2008) applied the coefficient of determination for ULFR model to assess the quality of JPEG compressed image and to evaluate the performance of various de-noising filters.

The remaining parts of this paper are organized as follows. Section 2 discusses the parameters estimation using maximum likelihood approach. The properties of the estimated intercept and slope are also investigated. We show that the maximum likelihood estimators are approximately unbiased using Taylor approximation and they are consistent estimators. Section 3 derives the coefficient of determination for the proposed model and its properties are investigated. A simulation study is carried out to evaluate the proposed estimators and coefficient of determination in Section 4. The application of the proposed coefficient of determination in assessing the quality of JPEG compressed image will be discussed in Section 5. Lastly, conclusion will be drawn in Section 6.

2 Multidimensional Unreplicated Linear Functional Relationship Model with Single Slope

Suppose that $\boldsymbol{Y}_i = (Y_{1i}, Y_{2i}, \ldots, Y_{pi})'$ and $\boldsymbol{X}_i = (X_{1i}, X_{2i}, \ldots, X_{pi})'$ are two linearly related unobservable true values of two variables with p-dimensions such that

$$\boldsymbol{Y}_i = \boldsymbol{\alpha} + \beta \boldsymbol{X}_i,\ i = 1, \ldots, n \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)'$ are intercepts and β is the slope of the linear function. The two corresponding random vectors $\boldsymbol{y}_i = (y_{1i}, y_{2i}, \ldots, y_{pi})'$ and $\boldsymbol{x}_i = (x_{1i}, x_{2i}, \ldots, x_{pi})'$ are observed with errors $\boldsymbol{\delta}_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{pi})'$ and $\boldsymbol{\varepsilon}_i = (\varepsilon_{1i}, \varepsilon_{2i}, \ldots, \varepsilon_{pi})'$ such that,

$$\left.\begin{aligned} \boldsymbol{x}_i &= \boldsymbol{X}_i + \boldsymbol{\delta}_i \\ \boldsymbol{y}_i &= \boldsymbol{Y}_i + \boldsymbol{\varepsilon}_i \end{aligned}\right\} i = 1, \ldots, n. \tag{2}$$

Both $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ can be observed in such a way that they are from two independent processes, especially in image processing. Assuming both error vectors are mutually and independently normally distributed with

(i) $E(\boldsymbol{\delta}_i) = \boldsymbol{0} = E(\boldsymbol{\varepsilon}_i)$

(ii) $var(\delta_{ki}) = \sigma^2$ and $var(\varepsilon_{ki}) = \tau^2$ for $k = 1, \ldots, p;\ \ i = 1, \ldots, n$

(iii) $cov(\delta_{ki}, \delta_{kj}) = 0 = cov(\varepsilon_{ki}, \varepsilon_{kj})$, for all $i \neq j;\ \ i, j = 1, \ldots, n$

$cov(\delta_{ki}, \delta_{hi}) = 0 = cov(\varepsilon_{ki}, \varepsilon_{hi})$, for all $h \neq k;\ \ h, k = 1, \ldots, p$, $i = 1, \ldots, n$

and $cov(\delta_{ki}, \varepsilon_{hj}) = 0$ for all $i, j = 1, \ldots, n$ and $h, k = 1, \ldots, p$

That is $\boldsymbol{\delta}_i \sim IND(\mathbf{0},\boldsymbol{\Omega}_{22})$ and $\boldsymbol{\varepsilon}_i \sim IND(\mathbf{0},\boldsymbol{\Omega}_{11})$

where $\boldsymbol{\Omega}_{11} = \begin{pmatrix} \tau^2 & 0 & 0 & 0 \\ 0 & \tau^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \tau^2 \end{pmatrix} = \tau^2 \boldsymbol{I}$,

$\boldsymbol{\Omega}_{22} = \begin{pmatrix} \sigma^2 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma^2 \end{pmatrix} = \sigma^2 \boldsymbol{I}$ and let $\boldsymbol{v}_i = \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \boldsymbol{\delta}_i \end{pmatrix}$,

then $cov(\boldsymbol{v}_i, \boldsymbol{v}_i) = \boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12} \\ \boldsymbol{\Omega}_{21} & \boldsymbol{\Omega}_{22} \end{pmatrix}$ are diagonal variance-covariance matrices, $\boldsymbol{\Omega}_{12} = \boldsymbol{\Omega}_{21} = \mathbf{0}$, $\boldsymbol{\Omega}_{11}$ and $\boldsymbol{\Omega}_{22}$ are positive definite.

2.1 Maximum Likelihood Estimators

We start with the joint probability density function of $\boldsymbol{\delta}_i$ and $\boldsymbol{\varepsilon}_i$

$$f(\boldsymbol{x}_i, \boldsymbol{y}_i) = \frac{1}{(2\pi)^{r/2} |\boldsymbol{\Omega}|^{1/2}} \times \exp\left[-\frac{1}{2} \left\{ \left[(\boldsymbol{y}_i - \boldsymbol{Y}_i)' \quad (\boldsymbol{x}_i - \boldsymbol{X}_i)' \right] \boldsymbol{\Omega}^{-1} \begin{pmatrix} \boldsymbol{y}_i - \boldsymbol{Y}_i \\ \boldsymbol{x}_i - \boldsymbol{X}_i \end{pmatrix} \right\} \right] \tag{3}$$

where $r = 2p$, $E(\boldsymbol{x}_i) = E(\boldsymbol{X}_i + \boldsymbol{\delta}_i) = \boldsymbol{X}_i$ and $E(\boldsymbol{y}_i) = E(\boldsymbol{Y}_i + \boldsymbol{\varepsilon}_i) = \boldsymbol{Y}_i$. The likelihood function for Equation (3) is

$$L = \frac{1}{\mathrm{K} |\boldsymbol{\Omega}|^{n/2}} \exp\left[-\frac{1}{2} \left\{ \sum_{i=1}^{n} \left[(\boldsymbol{x}_i - \boldsymbol{X}_i)' \boldsymbol{\Omega}_{22}^{-1} (\boldsymbol{x}_i - \boldsymbol{X}_i) + (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i)' \boldsymbol{\Omega}_{11}^{-1} (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i) \right] \right\} \right]$$

where $\mathrm{K} = (2\pi)^{rn/2}$ and the log-likelihood function is

$$L^* = -\ln \mathrm{K} - \frac{n}{2} \ln |\boldsymbol{\Omega}| - \frac{1}{2} \sum_{i=1}^{n} \left[(\boldsymbol{x}_i - \boldsymbol{X}_i)' \boldsymbol{\Omega}_{22}^{-1} (\boldsymbol{x}_i - \boldsymbol{X}_i) + (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i)' \boldsymbol{\Omega}_{11}^{-1} (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i) \right] \tag{4}$$

To overcome the unbounded problem of Equation (4), an additional constraint following (Kendall & Stuart, 1979) will be added as follow

(iv) $\boldsymbol{\Omega}_{11} = \lambda \boldsymbol{\Omega}_{22} \Leftrightarrow \boldsymbol{\Omega}_{11}^{-1} = \frac{1}{\lambda} \boldsymbol{\Omega}_{22}^{-1} \Leftrightarrow \tau^2 = \lambda \sigma^2$

where the ratio of error variances λ is a known constant. In this case, Equation (4) becomes

$$L^* = -\ln \mathrm{K} - \frac{n}{2} \ln \lambda^p - n \ln |\boldsymbol{\Omega}_{22}| - \frac{1}{2} \sum_{i=1}^{n} \left[(\boldsymbol{x}_i - \boldsymbol{X}_i)' \boldsymbol{\Omega}_{22}^{-1} (\boldsymbol{x}_i - \boldsymbol{X}_i) + \frac{1}{\lambda} (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i)' \boldsymbol{\Omega}_{22}^{-1} (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i) \right] \tag{5}$$

where

$$|\boldsymbol{\Omega}| = \left| \begin{pmatrix} \boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12} \\ \boldsymbol{\Omega}_{21} & \boldsymbol{\Omega}_{22} \end{pmatrix} \right| = |\boldsymbol{\Omega}_{11} \boldsymbol{\Omega}_{22}| = |\lambda \boldsymbol{\Omega}_{22} \boldsymbol{\Omega}_{22}| = \lambda^p |\boldsymbol{\Omega}_{22}|^2 .$$

There are $(np + p + 2)$ parameters to be estimated, which are $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_p$, $\boldsymbol{\alpha}$, β, and σ^2.

From the vector derivative formula for quadratic matrix equation evaluating to a scalar, we have

$$\begin{aligned} \frac{\partial L^*}{\partial (\beta \boldsymbol{X}_i - \boldsymbol{y}_i)} &= -\frac{1}{2\lambda} \sum_{i=1}^{n} \left\{ (\boldsymbol{\Omega}_{22}^{-1}) + (\boldsymbol{\Omega}_{22}^{-1})' \right\} \times \left[(\beta \boldsymbol{X}_i - \boldsymbol{y}_i) + \boldsymbol{\alpha} \right] \\ &= -\frac{1}{2\lambda} \sum_{i=1}^{n} 2 \boldsymbol{\Omega}_{22}^{-1} \left[(\beta \boldsymbol{X}_i - \boldsymbol{y}_i) + \boldsymbol{\alpha} \right] \qquad \left(\because (\boldsymbol{\Omega}_{22}^{-1})' = \boldsymbol{\Omega}_{22}^{-1} \right) \\ &= -\frac{1}{\lambda} \sum_{i=1}^{n} \boldsymbol{\Omega}_{22}^{-1} \left[\beta \boldsymbol{X}_i - \boldsymbol{y}_i + \boldsymbol{\alpha} \right] \end{aligned}$$

and the tangent vector to curve $(\beta \boldsymbol{X}_i - \boldsymbol{y}_i) : \mathbb{R} \to \mathbb{R}^n$ is $\frac{\partial (\beta \boldsymbol{X}_i - \boldsymbol{y}_i)}{\partial \beta} = \boldsymbol{X}_i$.

By using the Chain rule, we have

$$\begin{aligned} \frac{\partial L^*}{\partial \beta} &= \frac{\partial L^*}{\partial (\beta \boldsymbol{X}_i - \boldsymbol{y}_i)'} \frac{\partial (\beta \boldsymbol{X}_i - \boldsymbol{y}_i)}{\partial \beta} \\ &= -\frac{1}{\lambda} \sum_{i=1}^{n} (\beta \boldsymbol{X}_i - \boldsymbol{y}_i + \boldsymbol{\alpha})' \boldsymbol{\Omega}_{22}^{-1} \boldsymbol{X}_i \\ &= \frac{1}{\lambda} \sum_{i=1}^{n} (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i)' \boldsymbol{\Omega}_{22}^{-1} \boldsymbol{X}_i \end{aligned}$$

Therefore, differentiate Equation (5) with respect to β and set the result equal to zero $\left(\frac{\partial \boldsymbol{L}^*}{\partial \beta} = 0 \right)$, yields

$$\sum_{i=1}^{n} \frac{1}{\sigma^2} (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i)' \boldsymbol{I} (\boldsymbol{X}_i) = 0 \qquad \left(\because \boldsymbol{\Omega}_{22} = \sigma^2 \boldsymbol{I} \right)$$

$$\sum_{i=1}^{n} \boldsymbol{y}_i' \boldsymbol{X}_i - \boldsymbol{\alpha}' \sum_{i=1}^{n} \boldsymbol{X}_i - \beta \sum_{i=1}^{n} \boldsymbol{X}_i' \boldsymbol{X}_i = 0$$

$$\therefore \hat{\beta} = \frac{\sum_{i=1}^{n} \boldsymbol{y}_i' \hat{\boldsymbol{X}}_i - \hat{\boldsymbol{\alpha}}' \sum_{i=1}^{n} \hat{\boldsymbol{X}}_i}{\sum_{i=1}^{n} \hat{\boldsymbol{X}}_i' \hat{\boldsymbol{X}}_i} \tag{6}$$

Similarly, differential Equation (5) with respect $\boldsymbol{\alpha}$, $\boldsymbol{X}_i$ and σ give the following results

$$\frac{\partial L^*}{\partial \boldsymbol{X}_i} = -\frac{1}{2}\left\{ \begin{array}{l} -2\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right) \\ +\frac{2}{\lambda}\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)\left(-\beta\right) \end{array} \right\} = \mathbf{0}$$

$$\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right) + \frac{1}{\lambda}\beta\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right) = \mathbf{0}$$

$$-\left(\lambda + \beta^2\right)\boldsymbol{X}_i + \left(\lambda\boldsymbol{x}_i + \beta\boldsymbol{y}_i - \beta\boldsymbol{\alpha}\right) = \mathbf{0}$$

$$\therefore \hat{\boldsymbol{X}}_i = \frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2} \tag{7}$$

$$\frac{\partial L^*}{\partial \boldsymbol{\alpha}} = -\frac{1}{2}\sum_{i=1}^{n}\frac{-2}{\lambda}\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right) = \mathbf{0}$$

$$\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right) = \mathbf{0}$$

$\because \boldsymbol{\Omega}_{22}$ is positive definite and diagonal and $\boldsymbol{\Omega}_{22} = \sigma^2\boldsymbol{I}$

$$\sum_{i=1}^{n}\boldsymbol{y}_i - n\boldsymbol{\alpha} - \beta\sum_{i=1}^{n}\boldsymbol{X}_i = \mathbf{0}$$

$$\therefore \hat{\boldsymbol{\alpha}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{y}_i - \hat{\beta}\frac{1}{n}\sum_{i=1}^{n}\hat{\boldsymbol{X}}_i$$

Substitute Equation (5.7) yields

$$\hat{\boldsymbol{\alpha}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{y}_i - \hat{\beta}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right]$$

$$\hat{\boldsymbol{\alpha}} = \overline{\boldsymbol{y}} - \frac{1}{\lambda + \hat{\beta}^2}\left(\lambda\hat{\beta}\overline{\boldsymbol{x}} + \hat{\beta}^2\overline{\boldsymbol{y}} - \hat{\beta}^2\hat{\boldsymbol{\alpha}}\right)$$

$$\hat{\boldsymbol{\alpha}} - \frac{\hat{\beta}^2\hat{\boldsymbol{\alpha}}}{\lambda + \hat{\beta}^2} = \left(1 - \frac{\hat{\beta}^2}{\lambda + \hat{\beta}^2}\right)\overline{\boldsymbol{y}} - \frac{\lambda\hat{\beta}\overline{\boldsymbol{x}}}{\lambda + \hat{\beta}^2}$$

$$\hat{\boldsymbol{\alpha}} = \overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}} \tag{8}$$

where $\overline{\boldsymbol{y}} = \left[\overline{y}_1\ \overline{y}_2 \cdots \overline{y}_p\right]'$ and $\overline{\boldsymbol{x}} = \left[\overline{x}_1\ \overline{x}_2 \cdots \overline{x}_p\right]'$.

$$\frac{\partial L^*}{\partial \sigma} = -\frac{2n}{\sigma} + \sigma^{-3}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right)'\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)'\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)\right\} = 0$$

$$\frac{2n}{\sigma} = \frac{1}{\sigma^3}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right)'\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)'\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)\right\}$$

$$\therefore \hat{\sigma}^2 = \frac{1}{2n}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)\right\} \tag{9}$$

Since $\hat{\sigma}^2$ is a inconsistent estimator of σ^2 (Kendall & Stuart, 1979), we multiply Equation (9) by $\frac{2n}{n-2}$ yields the consistent estimator

$$\therefore \hat{\sigma}^2 = \frac{1}{n-2}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)\right\} \tag{10}$$

Substitute Equations (7) and (8) into Equation (6) yields

$$\hat{\beta} = \frac{\sum_{i=1}^{n}\boldsymbol{y}_i'\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right) - \hat{\boldsymbol{\alpha}}'\sum_{i=1}^{n}\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)}{\sum_{i=1}^{n}\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)'\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)}$$

$$= \frac{\left(\lambda + \hat{\beta}^2\right)\left\{\lambda S_{xy} + \hat{\beta}S_{yy} + \lambda n\hat{\beta}\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}} + n\hat{\beta}^3\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}\right\}}{\lambda^2\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i + 2\lambda\hat{\beta}S_{xy} + \hat{\beta}^2S_{yy} + 2n\lambda\hat{\beta}^2\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}} + n\hat{\beta}^4\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}}$$

where $S_{xx} = \sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}$, $S_{yy} = \sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{y}}'\overline{\boldsymbol{y}}$ and $S_{xy} = \sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}}$.

This implies that

$$\hat{\beta}^2S_{xy} + \hat{\beta}\left(\lambda S_{xx} - S_{yy}\right) - \lambda S_{xy} = 0 \tag{11}$$

Solving the quadratic Equation (11) yields

$$\hat{\beta} = \frac{-\left(\lambda S_{xx} - S_{yy}\right) \pm \sqrt{\left(\lambda S_{xx} - S_{yy}\right)^2 + 4\lambda S_{xy}^2}}{2S_{xy}}$$

$$\hat{\beta} = \frac{-\left(\lambda S_{xx} - S_{yy}\right) + \sqrt{\left(\lambda S_{xx} - S_{yy}\right)^2 + 4\lambda S_{xy}^2}}{2S_{xy}} \tag{12}$$

The positive sign is used in Equation (12) because it gives a maximum to the likelihood function in Equation (5) as shown below. From the previous result, we have

$$\frac{\partial L^*}{\partial \beta} = \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i\right)' \boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i$$

$$= \frac{1}{\lambda}\left(\sum \boldsymbol{y}_i{}'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i - \boldsymbol{\alpha}'\sum \boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i - \beta\sum \boldsymbol{X}_i{}'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right)$$

and the second order derivative yields

$$\frac{\partial^2 L^*}{\partial \beta^2} = \frac{-1}{\lambda}\sum \boldsymbol{X}_i{}'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i = \frac{-1}{\lambda\sigma^2}\sum \boldsymbol{X}_i{}'\boldsymbol{X}_i \ .$$

$$\left(\because \boldsymbol{\Omega}_{22} = \sigma^2 \boldsymbol{I}\right)$$

Since $\sum \boldsymbol{X}_i{}'\boldsymbol{X}_i > 0$ (practically $\boldsymbol{X} \neq \boldsymbol{0}$) and $\lambda > 0$, this implies that $\frac{\partial^2 L^*}{\partial \beta^2} < 0$. The $\hat{\beta}$s are local maximum points. Now, we let

$$\hat{\beta} = \frac{\left(S_{yy} - \lambda S_{xx}\right) \pm \sqrt{\left(S_{yy} - \lambda S_{xx}\right)^2 + 4\lambda S_{xy}^2}}{2S_{xy}} = \frac{\Delta}{2S_{xy}}$$

Furthermore, it could be shown that $\Delta = 2\hat{\beta}S_{xy} \geq 0$ must be non-negative and therefore the positive square root must always be taken.

Result 1: Given the Multidimensional ULFR model with single slope defined by Equations (1) and (2). The maximum likelihood estimators of $\boldsymbol{\alpha}$, β, $\boldsymbol{X}_i$ and σ_k^2 are

$$\hat{\boldsymbol{\alpha}} = \overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}$$

$$\hat{\beta} = \frac{-\left(\lambda S_{xx} - S_{yy}\right) + \sqrt{\left(\lambda S_{xx} - S_{yy}\right)^2 + 4\lambda S_{xy}^2}}{2S_{xy}}$$

$$\hat{\boldsymbol{X}}_i = \frac{\lambda \boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}$$

and

$$\hat{\sigma}^2 = \frac{1}{n-2}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)\right\}$$

where λ is the ratio of error variances, and $S_{xx} = \sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}$, $S_{yy} = \sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{y}}'\overline{\boldsymbol{y}}$ and $S_{xy} = \sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}}$.

2.2 Unbiased Estimators

The following two sections discuss the properties of $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$, i.e. their unbiasedness and consistency.

Result 2: Given the MULFR model stated in Equations (1) and (2), then the maximum likelihood estimators of $\boldsymbol{\alpha}$ and β are approximate unbiased estimators, i.e.

$$E\left(\hat{\beta}\right) \doteq \beta \text{ and } E\left(\hat{\boldsymbol{\alpha}}\right) \doteq \boldsymbol{\alpha}$$

Proof. Note that Equation (12) can be written as $\hat{\beta} = \theta + \sqrt{\theta^2 + \lambda}$ where $\theta\left(\boldsymbol{x}_i, \boldsymbol{y}_i\right) = \frac{S_{yy} - \lambda S_{xx}}{2S_{xy}}$.

Thus, the expected value of $\hat{\beta}$ is

$$E\left(\hat{\beta}\right) = E\left(\theta + \sqrt{\theta^2 + \lambda}\right) = E\left(\theta\right) + E\left(\sqrt{\theta^2 + \lambda}\right) \quad (13)$$

Since Equation (13) cannot be solved explicitly, we solve it by using the first order Taylor approximations (or Delta method) (Bain & Engelhardt, 1992) for the mean of $\theta\left(\boldsymbol{x}_i, \boldsymbol{y}_i\right)$. The first expected value in Equation (13) can be obtained by the following

$$\theta\left(\boldsymbol{x}_i, \boldsymbol{y}_i\right) = \theta\left(\boldsymbol{X}_i + \boldsymbol{\delta}_i, \boldsymbol{Y}_i + \boldsymbol{\varepsilon}_i\right) \quad \text{(from Equation (2))}$$

$$\doteq \theta\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right) + \boldsymbol{\delta}_i{}'\left.\frac{\partial\theta}{\partial \boldsymbol{x}_i}\right|_{\boldsymbol{x}_i = \boldsymbol{X}_i} + \boldsymbol{\varepsilon}_i{}'\left.\frac{\partial\theta}{\partial \boldsymbol{y}_i}\right|_{\boldsymbol{y}_i = \boldsymbol{Y}_i}$$

$$= \theta\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right) + \boldsymbol{\delta}_i{}'\left.\theta_{\boldsymbol{x}_i}\right|_{\boldsymbol{x}_i = \boldsymbol{X}_i} + \boldsymbol{\varepsilon}_i{}'\left.\theta_{\boldsymbol{y}_i}\right|_{\boldsymbol{y}_i = \boldsymbol{Y}_i} \quad (14)$$

where the partial derivatives are evaluated at the mean $\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right)$. Since

$$E\left(\boldsymbol{\delta}_i{}'\left.\theta_{\boldsymbol{x}_i}\right|_{\boldsymbol{x}_i = \boldsymbol{X}_i}\right) = E\left[\sum_{k=1}^{p}\delta_{ik}\left.\theta_{x_{ik}}\right|_{x_{ik} = X_{ik}}\right]$$

$$= \sum_{k=1}^{p}\left.\theta_{x_{ik}}\right|_{x_{ik} = X_{ik}} E\left(\delta_{ik}\right) = 0$$

$$\left(\because E\left(\boldsymbol{\delta}_i\right) = \boldsymbol{0} \Rightarrow E\left(\delta_{ik}\right) = 0\right)$$

Similarly, we have $E\left(\boldsymbol{\varepsilon}_i{}'\left.\theta_{\boldsymbol{y}_i}\right|_{\boldsymbol{y}_i = \boldsymbol{Y}_i}\right) = 0$. Therefore, Equation (14) becomes

$$E\left[\theta\left(\boldsymbol{x}_i, \boldsymbol{y}_i\right)\right] \doteq E\left[\theta\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right)\right] + \mathrm{E}\left(\boldsymbol{\delta}_i{}'\left.\theta_{\boldsymbol{x}_i}\right|_{\boldsymbol{x}_i = \boldsymbol{X}_i}\right) + \mathrm{E}\left(\boldsymbol{\varepsilon}_i{}'\left.\theta_{\boldsymbol{y}_i}\right|_{\boldsymbol{y}_i = \boldsymbol{Y}_i}\right)$$

$$= \theta\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right) = \frac{S_{YY} - \lambda S_{XX}}{2S_{XY}} \quad (15)$$

where $S_{YY} = \sum \boldsymbol{Y}_i'\boldsymbol{Y}_i - n\overline{\boldsymbol{Y}}'\overline{\boldsymbol{Y}}$, $S_{XX} = \sum \boldsymbol{X}_i'\boldsymbol{X}_i - n\overline{\boldsymbol{X}}'\overline{\boldsymbol{X}}$ and $S_{XY} = \sum \boldsymbol{X}_i'\boldsymbol{Y}_i - n\overline{\boldsymbol{X}}'\overline{\boldsymbol{Y}}$.

Now let $\varphi(\boldsymbol{x}_i,\boldsymbol{y}_i)=\sqrt{\theta^2(\boldsymbol{x}_i,\boldsymbol{y}_i)+\lambda}$. This implies that $\frac{\partial\varphi}{\partial\boldsymbol{x}_i}=(\theta^2+\lambda)^{-1/2}\theta\frac{\partial\theta}{\partial\boldsymbol{x}_i}$. We have

$$\varphi(\boldsymbol{x}_i,\boldsymbol{y}_i)=\varphi(\boldsymbol{X}_i+\boldsymbol{\delta}_i,\boldsymbol{Y}_i+\boldsymbol{\varepsilon}_i)$$

$$\doteq\theta(\boldsymbol{X}_i,\boldsymbol{Y}_i)+\boldsymbol{\delta}_i'\left.\frac{\partial\varphi}{\partial\boldsymbol{x}_i}\right|_{\boldsymbol{x}_i=\boldsymbol{X}_i}+\boldsymbol{\varepsilon}_i'\left.\frac{\partial\varphi}{\partial\boldsymbol{y}_i}\right|_{\boldsymbol{y}_i=\boldsymbol{Y}_i}$$

$$=\theta(\boldsymbol{X}_i,\boldsymbol{Y}_i)+(\theta^2+\lambda)^{-1/2}\theta\boldsymbol{\delta}_i'\,\theta_{\boldsymbol{x}_i}\big|_{\boldsymbol{x}_i=\boldsymbol{X}_i}$$
$$+(\theta^2+\lambda)^{-1/2}\theta\boldsymbol{\varepsilon}_i\,\theta_{\boldsymbol{y}_i}\big|_{\boldsymbol{y}_i=\boldsymbol{Y}_i}$$

Hence, the second expected value in Equation (5.13) is

$$E\left[\varphi(\boldsymbol{x}_i,\boldsymbol{y}_i)\right]\doteq E\left[\varphi(\boldsymbol{X}_i,\boldsymbol{Y}_i)\right]+\theta(\theta^2+\lambda)^{-1/2}\times$$
$$\mathrm{E}\left(\boldsymbol{\delta}_i'\,\theta_{\boldsymbol{x}_i}\big|_{\boldsymbol{x}_i=\boldsymbol{X}_i}\right)+\theta(\theta^2+\lambda)^{-1/2}\,\mathrm{E}\left(\boldsymbol{\varepsilon}_i'\,\theta_{\boldsymbol{y}_i}\big|_{\boldsymbol{y}_i=\boldsymbol{Y}_i}\right)$$
$$=\varphi(\boldsymbol{X}_i,\boldsymbol{Y}_i)+\theta(\theta^2+\lambda)^{-1/2}(0)+\theta(\theta^2+\lambda)^{-1/2}(0)$$
$$=\sqrt{\theta^2(\boldsymbol{X}_i,\boldsymbol{Y}_i)+\lambda}$$
$$=\sqrt{\left(\frac{S_{YY}-\lambda S_{XX}}{2S_{XY}}\right)^2+\lambda}\qquad(16)$$

From the Equations (15) and (16), hence Equation (13) becomes

$$E(\hat{\beta})\doteq\frac{S_{YY}-\lambda S_{XX}}{2S_{XY}}+\sqrt{\left(\frac{S_{YY}-\lambda S_{XX}}{2S_{XY}}\right)^2+\lambda}$$
$$=\frac{(S_{YY}-\lambda S_{XX})+\sqrt{(S_{YY}-\lambda S_{XX})^2+4\lambda S_{xy}^2}}{2S_{XY}}\qquad(17)$$

The next step is to show that $S_{XY}=\beta S_{XX}$ and $S_{YY}=\beta^2S_{XX}=\beta S_{XY}$ as also stated in (Lindley, 1947).

$$S_{XY}=\sum\boldsymbol{X}_i'\boldsymbol{Y}_i-n\bar{\boldsymbol{X}}'\bar{\boldsymbol{Y}}$$
$$=\sum\boldsymbol{X}_i'(\boldsymbol{\alpha}+\beta\boldsymbol{X}_i)-n\bar{\boldsymbol{X}}'(\boldsymbol{\alpha}+\beta\bar{\boldsymbol{X}})$$
$$=\sum(\boldsymbol{X}_i'\boldsymbol{\alpha}+\beta\boldsymbol{X}_i'\boldsymbol{X}_i)-n\bar{\boldsymbol{X}}'\boldsymbol{\alpha}-n\beta\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}$$
$$=\boldsymbol{\alpha}'\sum\boldsymbol{X}_i+\beta\sum\boldsymbol{X}_i'\boldsymbol{X}_i-n\boldsymbol{\alpha}'\bar{\boldsymbol{X}}-n\beta\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}$$
$$=\boldsymbol{\alpha}'\left(\sum\boldsymbol{X}_i-n\bar{\boldsymbol{X}}\right)+\beta\left(\sum\boldsymbol{X}_i'\boldsymbol{X}_i-n\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}\right)$$
$$=\beta S_{XX}$$

and

$$S_{YY}=\sum\boldsymbol{Y}_i'\boldsymbol{Y}_i-n\bar{\boldsymbol{Y}}'\bar{\boldsymbol{Y}}$$
$$=\sum(\boldsymbol{\alpha}+\beta\boldsymbol{X}_i)'(\boldsymbol{\alpha}+\beta\boldsymbol{X}_i)-n(\boldsymbol{\alpha}+\beta\bar{\boldsymbol{X}})'(\boldsymbol{\alpha}+\beta\bar{\boldsymbol{X}})$$
$$=\sum\left(\boldsymbol{\alpha}'\boldsymbol{\alpha}+2\beta\boldsymbol{\alpha}'\boldsymbol{X}_i+\beta^2\boldsymbol{X}_i'\boldsymbol{X}_i\right)$$
$$-n\left(\boldsymbol{\alpha}'\boldsymbol{\alpha}+2\beta\boldsymbol{\alpha}'\bar{\boldsymbol{X}}+\beta^2\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}\right)$$
$$=n\boldsymbol{\alpha}'\boldsymbol{\alpha}+2\beta\boldsymbol{\alpha}'\sum\boldsymbol{X}_i+\beta^2\sum\boldsymbol{X}_i'\boldsymbol{X}_i$$
$$-n\left(\boldsymbol{\alpha}'\boldsymbol{\alpha}+2\beta\boldsymbol{\alpha}'\bar{\boldsymbol{X}}+\beta^2\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}\right)$$
$$=\beta^2\sum\boldsymbol{X}_i'\boldsymbol{X}_i-n\beta^2\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}$$
$$=\beta^2\left(\sum\boldsymbol{X}_i'\boldsymbol{X}_i-n\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}\right)$$
$$=\beta^2S_{XX}=\beta S_{XY}$$

Therefore, Equation (17) can be reduced to

$$E(\hat{\beta})\doteq$$
$$\frac{(\beta^2S_{XX}-\lambda S_{XX})+\sqrt{(\beta^2S_{XX}-\lambda S_{XX})^2+4\lambda\beta^2S_{xx}^2}}{2\beta S_{XX}}$$
$$=\frac{(\beta^2-\lambda)S_{XX}+\sqrt{(\beta^2S_{XX}+\lambda S_{XX})^2}}{2\beta S_{XX}}$$
$$=\frac{(\beta^2-\lambda)+(\beta^2+\lambda)}{2\beta}=\frac{2\beta^2}{2\beta}=\beta$$

From the Equation (7), we have

$$\hat{\boldsymbol{\alpha}}=\bar{\boldsymbol{y}}-\hat{\beta}\bar{\boldsymbol{x}}$$
$$\Rightarrow E(\hat{\boldsymbol{\alpha}})=E(\bar{\boldsymbol{y}}-\hat{\beta}\bar{\boldsymbol{x}})=\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}E(\hat{\beta})\doteq\bar{\boldsymbol{y}}-\beta\bar{\boldsymbol{x}}=\boldsymbol{\alpha}$$

2.3 Consistency Property

2.3.1 Variance of the expected parameters

Result 3: Given that $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ are MLE of $\boldsymbol{\alpha}$ and β, respectively for the MULFR model, then

$$V\hat{a}r(\hat{\beta})=\lambda\hat{\sigma}^2\left[\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i-\frac{1}{n}\left(\sum\hat{\boldsymbol{X}}_i'\right)\left(\sum\hat{\boldsymbol{X}}_i\right)\right]^{-1}$$

$$V\hat{a}r(\hat{\boldsymbol{\alpha}})=\lambda\hat{\sigma}^2\left[n\boldsymbol{I}-\left\{\sum\hat{\boldsymbol{X}}_i\right\}\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1}$$

$$C\hat{o}v(\hat{\boldsymbol{\alpha}},\hat{\beta})=-\lambda\hat{\sigma}^2\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left(\sum\hat{\boldsymbol{X}}_i'\right)\times$$
$$\left[n\boldsymbol{I}-\left\{\sum\hat{\boldsymbol{X}}_i\right\}\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1}$$

Proof. To find the variance of the $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$, we consider the Fisher Information Matrix (FIM) of parameters $\boldsymbol{\alpha}$ and β. The first order partial derivatives for log-likelihood function are given by

$$\frac{\partial L^*}{\partial\boldsymbol{\alpha}}=\frac{1}{\lambda}\sum(\boldsymbol{y}_i-\boldsymbol{\alpha}-\beta\boldsymbol{X}_i)'\,\boldsymbol{\Omega}_{22}^{-1}$$

And $\frac{\partial L^*}{\partial \beta} = \frac{1}{\lambda}\sum\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i\right)' \boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i$

The second order partial derivatives for log-likelihood function and their negative expected values are given by

$\frac{\partial^2 L^*}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}'} = -\frac{n}{\lambda}\boldsymbol{\Omega}_{22}^{-1}$, hence $E\left(-\frac{\partial^2 L^*}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}'}\right) = \frac{n}{\lambda}\boldsymbol{\Omega}_{22}^{-1}$

$\frac{\partial^2 L^*}{\partial \boldsymbol{\alpha}'\partial \beta} = -\frac{1}{\lambda}\sum\left(\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\right)$, hence

$$E\left(-\frac{\partial^2 L^*}{\partial \boldsymbol{\alpha}'\partial \beta}\right) = \frac{1}{\lambda}\sum\left(\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\right)$$

$\frac{\partial^2 L^*}{\partial \beta^2} = -\frac{1}{\lambda}\sum\left(\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right)$, hence

$$E\left(-\frac{\partial^2 L^*}{\partial \beta^2}\right) = \frac{1}{\lambda}\sum\left(\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right)$$

and $\frac{\partial^2 L^*}{\partial \beta\partial \boldsymbol{\alpha}} = -\frac{1}{\lambda}\sum\left(\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right)$, hence

$$E\left(-\frac{\partial^2 L^*}{\partial \beta\partial \boldsymbol{\alpha}}\right) = \frac{1}{\lambda}\sum\left(\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right)$$

Next, we find the estimated FIM for $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ given by

$$F = \begin{bmatrix} \frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1} & \frac{1}{\lambda}\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right) \\ \frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right) & \frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is a $p \times p$ matrix given by $\frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1}$, B is a $p \times 1$ matrix given by $\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)$, C is a $1 \times p$ matrix given by $\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)$ in which $C' = B$, and D is a 1×1 matrix given by $\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)$.

Thus, the inverse of F is

$$F^{-1} = \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & \left(D - CA^{-1}B\right)^{-1} \end{bmatrix}$$

Therefore, we obtained the following results:

$$V\hat{a}r\left(\hat{\beta}\right) = \left(D - CA^{-1}B\right)^{-1}$$

$$= \left[\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right) - \frac{1}{\lambda}\left\{\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)\right\}\left(\frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)^{-1}\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right]^{-1}$$

$$= \lambda\hat{\sigma}^2\left[\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i - \frac{1}{n}\left(\sum\hat{\boldsymbol{X}}_i'\right)\left(\sum\hat{\boldsymbol{X}}_i\right)\right]^{-1} \quad (18)$$

$$\left(\because \boldsymbol{\Omega}_{22} = \sigma^2\boldsymbol{I}\right)$$

$$V\hat{a}r\left(\hat{\boldsymbol{\alpha}}\right) = \left(A - BD^{-1}C\right)^{-1}$$

$$= \lambda\left[\frac{n}{\hat{\sigma}^2}\boldsymbol{I} - \left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i\right\}\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1}$$

$$\left(\because \boldsymbol{\Omega}_{22} = \sigma^2\boldsymbol{I}\right)$$

$$= \lambda\hat{\sigma}^2\left[n\boldsymbol{I} - \left\{\sum\hat{\boldsymbol{X}}_i\right\}\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1} \quad (19)$$

and

$$C\hat{o}v\left(\hat{\boldsymbol{\alpha}}, \hat{\beta}\right) = -D^{-1}C\left(A - BD^{-1}C\right)^{-1}$$

$$= -\lambda\hat{\sigma}^2\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left(\sum\hat{\boldsymbol{X}}_i'\right)\times$$

$$\left[n\boldsymbol{I} - \left\{\sum\hat{\boldsymbol{X}}_i\right\}\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1} \quad (20)$$

$$\left(\because \boldsymbol{\Omega}_{22} = \sigma^2\boldsymbol{I}\right)$$

2.3.2 Consistent estimators

Definition 1: *An estimator $\hat{\theta}_n$ of θ based on a random sample of size n is a consistent estimator of θ if $\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| > \omega\right) = 0$ for every $\omega > 0$.*

It has been shown in Result 2 that estimators $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ are approximately unbiased. From Chebyshev's inequality, we see that

$$P\left(\left|\hat{\beta} - \beta\right| \geq \omega\right) \leq \frac{Var\left(\hat{\beta}\right)}{\omega^2} \quad (21)$$

Without loss of generality, we remove the equality inside the probability in Equation (21) and combined with Definition 1, yields

$$\lim_{n\to\infty} P\left(\left|\hat{\beta} - \beta\right| > \omega\right) \leq \frac{1}{\omega^2}\lim_{n\to\infty} Var\left(\hat{\beta}\right) = 0$$

$$\Leftrightarrow \lim_{n\to\infty} Var\left(\hat{\beta}\right) = 0 \qquad \text{for every } \omega > 0.$$

In order to show that the estimator $\hat{\beta}$ is consistent, we need to indicate $Var\left(\hat{\beta}\right) \to 0$ as $n \to \infty$. This can be easily obtained from the results in Section 4.1 as follows

$$\lim_{n\to\infty} V\hat{a}r\left(\hat{\beta}\right) = \lambda \lim_{n\to\infty}\left[\frac{\hat{\sigma}^2}{\sum \hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i - \frac{1}{n}\left(\sum \hat{\boldsymbol{X}}_i'\right)\left(\sum \hat{\boldsymbol{X}}_i\right)}\right]$$

$$= \lambda \frac{0}{\sum \hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i} = 0$$

Similarly, we can show $Var\left(\hat{\boldsymbol{\alpha}}\right) \to 0$ as $n \to \infty$.

2.3.3 Confidence Interval for $\boldsymbol{\alpha}$ and β

For a p-dimensional ULFR model defined in Equations (1) and (2), we have $2np$-independent observations that are available to estimate $\left(np+p+2\right)$-parameters of the population, i.e. p parameters from $\boldsymbol{\alpha}$ and one from β. Hence, the number of degrees of freedom is $np-\left(p+2\right)$.

Now, we can define the $\left(1-a\right)100\%$ confidence intervals for $\boldsymbol{\alpha}$ and β as

$$\hat{\alpha}_k - t_{\frac{a}{2},np-p-2}\, se\left(\hat{\alpha}_k\right) \le \alpha_k \le \hat{\alpha}_k + t_{\frac{a}{2},np-p-2}\, se\left(\hat{\alpha}_k\right) \quad (22)$$

and

$$\hat{\beta} - t_{\frac{a}{2},np-p-2}\, se\left(\hat{\beta}\right) \le \beta \le \hat{\beta} + t_{\frac{a}{2},np-p-2}\, se\left(\hat{\beta}\right) \quad (23)$$

where a is the level of significance, the standard errors $se\left(\hat{\alpha}_k\right) = \sqrt{V\hat{a}r\left(\hat{\alpha}_k\right)}$ and $se\left(\hat{\beta}\right) = \sqrt{V\hat{a}r\left(\hat{\beta}\right)}$ can be obtained from Result 5.

3 Coefficient of Determination for MULFR Model

Re-write the Equations (1) and (2) as

$$\boldsymbol{y}_i = \boldsymbol{\alpha} + \beta \boldsymbol{X}_i + \boldsymbol{\varepsilon}_i = \boldsymbol{\alpha} + \beta \boldsymbol{x}_i + \left(\boldsymbol{\varepsilon}_i - \beta\boldsymbol{\delta}_i\right) = \boldsymbol{\alpha} + \beta \boldsymbol{x}_i + \boldsymbol{V}_i \quad (24)$$

where the errors of the model

$$\boldsymbol{V}_i = \boldsymbol{\varepsilon}_i - \beta\boldsymbol{\delta}_i = \boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{x}_i,\ i=1,2,\ldots,n \quad (25)$$

is a normally distributed p-dimensional random variable with

$$E\left(\boldsymbol{V}_i\right) = E\left(\boldsymbol{\varepsilon}_i - \beta\boldsymbol{\delta}_i\right) = E\left(\boldsymbol{\varepsilon}_i\right) - \beta E\left(\boldsymbol{\delta}_i\right) = \boldsymbol{0}$$

$$\left(\because E\left(\boldsymbol{\varepsilon}_i\right) = E\left(\boldsymbol{\delta}_i\right) = 0\right)$$

and $Var\left(\boldsymbol{V}_i\right) = Var\left(\boldsymbol{\varepsilon}_i - \beta\boldsymbol{\delta}_i\right)$

$$= Var\left(\boldsymbol{\varepsilon}_i\right) + \beta^2 Var\left(\boldsymbol{\delta}_i\right) - 2Cov\left(\boldsymbol{\varepsilon}_i, \beta\boldsymbol{\delta}_i\right)$$

$$= \boldsymbol{\Omega}_{11} + \beta^2\boldsymbol{\Omega}_{22} \qquad \left(\because Cov\left(\boldsymbol{\varepsilon}_i, \boldsymbol{\delta}_i\right) = \boldsymbol{\Omega}_{12} = \boldsymbol{0}\right)$$

If $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ are estimators of $\boldsymbol{\alpha}$ and β, respectively, then from Equation (25) we have

$$\hat{\boldsymbol{V}}_i = \boldsymbol{y}_i - \hat{\boldsymbol{y}}_i = \boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\boldsymbol{x}_i,\ i=1,2,\ldots,n$$

is the residual of the model. Since

$$Var\left(\hat{\boldsymbol{V}}_i\right) = Var\left(\boldsymbol{\varepsilon}_i - \hat{\beta}\boldsymbol{\delta}_i\right)$$

$$= Var\left(\boldsymbol{\varepsilon}_i\right) + \hat{\beta}^2 Var\left(\boldsymbol{\delta}_i\right) - 2Cov\left(\boldsymbol{\varepsilon}_i, \hat{\beta}\boldsymbol{\delta}_i\right)$$

$$= \boldsymbol{\Omega}_{11} + \hat{\beta}^2\boldsymbol{\Omega}_{22}$$

$$= \lambda\boldsymbol{\Omega}_{22} + \hat{\beta}^2\boldsymbol{\Omega}_{22} \qquad \because \boldsymbol{\Omega}_{11} = \lambda\boldsymbol{\Omega}_{22}$$

$$= \left(\lambda + \hat{\beta}^2\right)\boldsymbol{\Omega}_{22}$$

$$\because Var\left(\boldsymbol{\varepsilon}_i\right) = \boldsymbol{\Omega}_{11},\quad Var\left(\boldsymbol{\delta}_i\right) = \boldsymbol{\Omega}_{22},\quad Cov\left(\boldsymbol{\varepsilon}_i, \hat{\beta}\boldsymbol{\delta}_i\right) = \boldsymbol{0}$$

The residual sum of squares is divided by $\left(\lambda + \hat{\beta}^2\right)$ yields

$$SS_E = \frac{1}{\lambda+\hat{\beta}^2}\sum \hat{\boldsymbol{V}}_i^2 = \frac{1}{\lambda+\hat{\beta}^2}\sum\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\boldsymbol{x}_i\right)^2$$

$$= \frac{1}{\lambda+\hat{\beta}^2}\begin{pmatrix} \sum \boldsymbol{y}_i'\boldsymbol{y}_i - 2\hat{\boldsymbol{\alpha}}'\sum \boldsymbol{y}_i \\ -2\hat{\beta}\sum \boldsymbol{x}_i'\boldsymbol{y}_i + 2\hat{\beta}\hat{\boldsymbol{\alpha}}'\sum \boldsymbol{x}_i \\ +n\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}} + \hat{\beta}^2\sum \boldsymbol{x}_i'\boldsymbol{x}_i \end{pmatrix}$$

$$= \frac{1}{\lambda+\hat{\beta}^2}\left(\sum \boldsymbol{y}_i'\boldsymbol{y}_i - 2n\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\overline{\boldsymbol{y}}\right.$$

$$- 2\hat{\beta}\sum \boldsymbol{x}_i'\boldsymbol{y}_i + 2n\hat{\beta}\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\overline{\boldsymbol{x}}$$

$$\left. + n\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right) + \hat{\beta}^2\sum \boldsymbol{x}_i'\boldsymbol{x}_i\right)$$

$$= \frac{1}{\lambda+\hat{\beta}^2}\begin{pmatrix} \left[\sum \boldsymbol{y}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{y}}'\overline{\boldsymbol{y}}\right] - \\ 2\hat{\beta}\left[\sum \boldsymbol{x}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}}\right] + \hat{\beta}^2\left[\sum \boldsymbol{x}_i'\boldsymbol{x}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}\right] \end{pmatrix}$$

$$= \frac{S_{yy} - 2\hat{\beta}S_{xy} + \hat{\beta}^2 S_{xx}}{\lambda + \hat{\beta}^2}$$

We only consider the case $\lambda = 1$ that is when $\boldsymbol{\Omega}_{11} = \boldsymbol{\Omega}_{22}$. For those cases when $\lambda \ne 1$, we can always reduce it to the case of $\lambda = 1$ by dividing the observed values of y_k by $\sqrt{\lambda_k}$ as the ULFR (Kendall & Stuart, 1979). Hence,

$$SS_E = \frac{S_{yy} - 2\hat{\beta}S_{xy} + \hat{\beta}^2 S_{xx}}{1 + \hat{\beta}^2} \quad (26)$$

Then the coefficient of determination can be defined as

$$R_p^2 = \frac{SS_R}{\boldsymbol{S}_{yy}} = 1 - \frac{SS_E}{\boldsymbol{S}_{yy}} = \frac{\boldsymbol{S}_{yy} - SS_E}{\boldsymbol{S}_{yy}} \quad (27)$$

For the case $\lambda = 1$, Equation (27) becomes

$$R_p^2 = \frac{\hat{\beta} S_{xy}}{S_{yy}} \tag{28}$$

Result 4: Let the ratio of the error variances be known and equals one $(\lambda = 1)$, then the coefficient of determination of the MULFR model is

$$R_p^2 = \frac{SS_R}{S_{yy}} = \frac{\hat{\beta} S_{xy}}{S_{yy}}$$

Proof: we need to show

$$\frac{SS_R}{S_{yy}} = \frac{\hat{\beta} S_{xy}}{S_{yy}} \quad \Leftrightarrow SS_R = \hat{\beta} S_{xy}.$$

By definition,

$$\begin{aligned} SS_R &= S_{yy} - SS_E \\ &= S_{yy} - \left(\frac{S_{yy} - 2\hat{\beta} S_{xy} + \hat{\beta}^2 S_{xx}}{1 + \hat{\beta}^2} \right) \\ &= \frac{\left(S_{yy} + \hat{\beta}^2 S_{yy}\right) - \left(S_{yy} - 2\hat{\beta} S_{xy} + \hat{\beta}^2 S_{xx}\right)}{1 + \hat{\beta}^2} \\ &= \frac{\hat{\beta}^2 S_{yy} + 2\hat{\beta} S_{xy} - \hat{\beta}^2 S_{xx}}{1 + \hat{\beta}^2} \\ &= \frac{\hat{\beta}^2 \left(S_{yy} - S_{xx}\right) + 2\hat{\beta} S_{xy}}{1 + \hat{\beta}^2} \end{aligned} \tag{29}$$

From Equation (11) and $\lambda = 1$, we have

$$S_{xy}\hat{\beta}^2 = \left(S_{yy} - S_{xx}\right)\hat{\beta} + S_{xy} \tag{30}$$

Substitute Equation (30) into Equation (29) yields

$$SS_R = \frac{\hat{\beta}\left\{\left[\left(S_{yy} - S_{xx}\right)\hat{\beta} + S_{xy}\right] + S_{xy}\right\}}{1 + \hat{\beta}^2}$$

$$= \frac{\hat{\beta}\left\{\hat{\beta}^2 S_{xy} + S_{xy}\right\}}{1 + \hat{\beta}^2} = \frac{\hat{\beta} S_{xy}\left(\hat{\beta}^2 + 1\right)}{1 + \hat{\beta}^2} = \hat{\beta} S_{xy}$$

3.1 Properties of Coefficient of Determination when $\lambda = 1$

3.1.1 Confident Interval

Note that $E\left(R_p^2\right) = E\left(\frac{\hat{\beta} S_{xy}}{S_{yy}}\right) = \frac{S_{xy}}{S_{yy}} E\left(\hat{\beta}\right) \doteq \frac{\beta S_{xy}}{S_{yy}}$

and

$$Var\left(R_p^2\right) = Var\left(\frac{\hat{\beta} S_{xy}}{S_{yy}}\right) = \frac{S_{xy}^2}{S_{yy}^2} V\hat{a}r\left(\hat{\beta}\right)$$

Refer to Equation (23) and let $L_\beta = \hat{\beta} - t_{\frac{a}{2}, np-p-1} se\left(\hat{\beta}\right)$ and $U_\beta = \hat{\beta} + t_{\frac{a}{2}, np-p-1} se\left(\hat{\beta}\right)$.

There are four possible cases:

Case I: when $U_\beta \geq L_\beta \geq 0$

Then the $(1-a)100\%$ confidence interval for the population R_p^2 is

$$\hat{\beta}\frac{S_{xy}}{S_{yy}} - t_{\frac{a}{2}, np-p-1}\sqrt{\frac{S_{xy}^2}{S_{yy}^2} V\hat{a}r\left(\hat{\beta}\right)} \leq \beta \frac{S_{xy}}{S_{yy}} \leq \hat{\beta}\frac{S_{xy}}{S_{yy}} + t_{\frac{a}{2}, np-p-1}\sqrt{\frac{S_{xy}^2}{S_{yy}^2} V\hat{a}r\left(\hat{\beta}\right)}$$

$$\left[\hat{\beta} - t_{\frac{a}{2}, np-p-1} se\left(\hat{\beta}\right)\right]\frac{S_{xy}}{S_{yy}} \leq \beta \frac{S_{xy}}{S_{yy}} \leq \left[\hat{\beta} + t_{\frac{a}{2}, np-p-1} se\left(\hat{\beta}\right)\right]\frac{S_{xy}}{S_{yy}}$$

$$\frac{L_\beta S_{xy}}{S_{yy}} \leq \beta \frac{S_{xy}}{S_{yy}} \leq \frac{U_\beta S_{xy}}{S_{yy}} \tag{31}$$

Case II: When $U_\beta > 0 > L_\beta$ and $\left|U_\beta\right| \geq \left|L_\beta\right|$, then Equation (31) holds.

Case III: When $U_\beta > 0 > L_\beta$ and $\left|U_\beta\right| < \left|L_\beta\right|$, then

$$\frac{U_\beta S_{xy}}{S_{yy}} \leq \beta \frac{S_{xy}}{S_{yy}} \leq \frac{L_\beta S_{xy}}{S_{yy}} \tag{32}$$

Case IV: When $0 \geq U_\beta \geq L_\beta$, then Equation (32) holds.

Result 5: Let the ratio of the error covariances be known and equals one $(\lambda = 1)$, then the $(1-a)100\%$ confidence interval for the population R_p^2 is

$$\frac{L_\beta S_{xy}}{S_{yy}} \leq \beta \frac{S_{xy}}{S_{yy}} \leq \frac{U_\beta S_{xy}}{S_{yy}} \text{ if } \left|U_\beta\right| \geq \left|L_\beta\right|$$

$$\frac{U_\beta S_{xy}}{S_{yy}} \leq \beta \frac{S_{xy}}{S_{yy}} \leq \frac{L_\beta S_{xy}}{S_{yy}} \text{ if } \left|U_\beta\right| < \left|L_\beta\right|.$$

3.1.2 Range: $0 \leq R_p^2 \leq 1$

From the regression sum of squares, we have

$$0 \leq SS_R = S_{yy} - SS_E \leq S_{yy}$$

$$0 \le \frac{SS_R}{S_{yy}} \le \frac{S_{yy}}{S_{yy}} = 1$$
$$\therefore 0 \le R_p^2 \le 1$$

3.1.3 Non-symmetry Property

Given the MULFR model defined by Equations (1) and (2) with $\lambda = 1$, we have

$$\hat{\beta} = \frac{\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{xy}}$$

and

$$R_p^2 = \frac{\hat{\beta}S_{xy}}{S_{yy}} = \frac{\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{yy}}$$

$$\Rightarrow \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2} = 2S_{yy}R_p^2 - \left(S_{yy} - S_{xx}\right) \quad (33)$$

Now we consider a MULFR model by replacing Equation (1) with

$$\boldsymbol{X}_i = \boldsymbol{\alpha}^* + \beta^* \boldsymbol{Y}_i, \qquad i = 1, 2, \ldots, n \quad (34)$$

It can be shown that the estimated slope $\left(\hat{\beta}^*\right)$ and coefficient of determination, say $\tilde{R}_p^2$ for the new model when $\lambda = 1$ are

$$\hat{\beta}^* = \frac{\left(S_{xx} - S_{yy}\right) + \sqrt{\left(S_{xx} - S_{yy}\right)^2 + 4S_{xy}^2}}{2S_{xy}}$$

and

$$\tilde{R}_p^2 = \frac{\hat{\beta}^* S_{xy}}{S_{xx}} = \frac{\left(S_{xx} - S_{yy}\right) + \sqrt{\left(S_{xx} - S_{yy}\right)^2 + 4S_{xy}^2}}{2S_{xx}}$$

$$= \frac{-\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{xx}}$$

$$= -\frac{1}{2S_{xx}}\left[\left(S_{yy} - S_{xx}\right) - \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}\right]$$

$$= -\frac{1}{2S_{xx}}\left[\left(S_{yy} - S_{xx}\right) - 2S_{yy}R_p^2 + \left(S_{yy} - S_{xx}\right)\right]$$

from Equation (33)

$$= -\frac{1}{2S_{xx}}\left[2\left(S_{yy} - S_{xx}\right) - 2S_{yy}R_p^2\right]$$

$$= \frac{S_{yy}}{S_{xx}}R_p^2 - \frac{\left(S_{yy} - S_{xx}\right)}{S_{xx}}$$

Let $S_{yy} = kS_{xx}$ and $k > 0$, then

$$\tilde{R}_p^2 = kR_p^2 - k + 1 = k\left(R_p^2 - 1\right) + 1 \quad (35)$$

Result 6: Given $S_{yy} = kS_{xx}$ where $k > 0$. Let R_p^2 and $\tilde{R}_p^2$ be the coefficient of determination for MULFR model with $\lambda = 1$ as defined by Equation (1) and Equation (34), respectively. Then

$$\tilde{R}_p^2 = k\left(R_p^2 - 1\right) + 1.$$

Hence, R_p^2 is symmetric when $k = 1$ and non-symmetric otherwise.

3.1.4 Range of R_p^2 (an improvement of Section 3.1.2)

Let $S_{yy} = kS_{xx}$. Since $S_{yy} \ge 0$ and $S_{xx} \ge 0$, we consider $k > 0$. From Equation (28), we have

$$R_p^2 = \frac{\hat{\beta}S_{xy}}{S_{yy}} = \frac{\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{yy}}$$

$$= \frac{\left(kS_{xx} - S_{xx}\right) + \sqrt{\left(kS_{xx} - S_{xx}\right)^2 + 4S_{xy}^2}}{2kS_{xx}}$$

$$= \frac{\left(k-1\right)S_{xx} + \sqrt{\left(k-1\right)^2 S_{xx}^2 + 4S_{xy}^2}}{2kS_{xx}}$$

$$= \frac{\left(k-1\right)}{2k} + \sqrt{\frac{\left(k-1\right)^2 S_{xx}^2 + 4S_{xy}^2}{4k^2 S_{xx}^2}}$$

$$\ge \frac{\left(k-1\right)}{2k} + \sqrt{\frac{\left(k-1\right)^2 S_{xx}^2}{4k^2 S_{xx}^2}} \qquad \left(\because 4S_{xy}^2 \ge 0\right)$$

$$= \frac{\left(k-1\right)}{2k} + \frac{\left(k-1\right)}{2k} = 1 - \frac{1}{k}$$

Since $0 \le R_p^2 \le 1$, then $0 \le 1 - \frac{1}{k} \le R_p^2 \le 1$. We consider the following two cases:

<u>Case I:</u> when $0 < k \le 1$

As $k \to 0^+$, then $R_p^2 \ge 1 - \frac{1}{k} \to -\infty$. However we have $R_p^2 \ge 0$, this implies that $0 \le R_p^2 \le 1$.

As $k \to 1^-$, then $R_p^2 \ge 1 - \frac{1}{k} \to 0$. Hence, we have $0 \le R_p^2 \le 1$.

As $k = 1$, then $R_p^2 \ge 1 - \frac{1}{k} = 0$. Hence, we have $0 \le R_p^2 \le 1$.

Case II: when $k>1$

As $k\to\infty$, then $R_p^2 \ge 1-\frac{1}{k} \to 1$. Hence, we have $0<1-\frac{1}{k} \le R_p^2 \le 1$.

Result 7: Let $S_{\boldsymbol{yy}} = kS_{\boldsymbol{xx}}$ for $k>0$ and R_p^2 be the coefficient of determination for MULFR model when $\lambda=1$. Then

$$\left(1-\frac{1}{k}\right)^{+} \le R_p^2 \le 1$$

where $c^{+} = \begin{cases} c, & c>0 \\ 0, & c\le 0 \end{cases}$ and $c = 1-\frac{1}{k}$.

3.1.5 The Coefficient of Determination for ULFR Model, R_F^2 is a Special Case of R_p^2 When $p=1$.

From Chang et al. (2007), the coefficient of determination for ULFR model when $\lambda=1$ is

$$R_F^2 = \frac{\left(S_{yy}-\lambda S_{xx}\right)+\left\{\left(S_{yy}-\lambda S_{xx}\right)^2+4\lambda S_{xy}\right\}^{1/2}}{2S_{yy}}$$

where $S_{yy}=\sum y_i^2 - n\bar{y}^2$, $S_{xx}=\sum x_i^2 - n\bar{x}^2$ and $S_{xy}=\sum x_i y_i - n\bar{x}\bar{y}$.

From Result 3 of MULFR model, we have

$$\begin{aligned}
S_{\boldsymbol{xx}} &= \sum_{i=1}^{n} \boldsymbol{x}_i'\boldsymbol{x}_i - n\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}} \\
&= \sum\begin{bmatrix} x_{1i} & x_{2i} & \cdots & x_{pi}\end{bmatrix}\begin{bmatrix} x_{1i} & x_{2i} & \cdots & x_{pi}\end{bmatrix}' \\
&\quad - n\begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_p\end{bmatrix}\begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_p\end{bmatrix}' \\
&= \sum\left(x_{1i}^2 + x_{2i}^2 + \cdots + x_{pi}^2\right) - n\left(\bar{x}_1^2 + \bar{x}_2^2 + \cdots + \bar{x}_p^2\right) \\
&= \left(\sum x_{1i}^2 - n\bar{x}_1^2\right) + \left(\sum x_{2i}^2 - n\bar{x}_2^2\right) + \cdots + \left(\sum x_{pi}^2 - n\bar{x}_p^2\right) \\
&= S_{xx}^1 + S_{xx}^2 + \cdots + S_{xx}^p
\end{aligned}$$

When $p=1$, then $S_{\boldsymbol{xx}} = S_{xx}^1 = S_{xx}$.

Similarly, we have $S_{\boldsymbol{yy}} = S_{yy}^1 = S_{yy}$ and $S_{\boldsymbol{xy}} = S_{xy}^1 = S_{xy}$.

Therefore, $R_p^2 = \dfrac{\left(S_{\boldsymbol{yy}}-S_{\boldsymbol{xx}}\right)+\sqrt{\left(S_{\boldsymbol{yy}}-S_{\boldsymbol{xx}}\right)^2+4S_{\boldsymbol{xy}}^2}}{2S_{\boldsymbol{yy}}}$

$$\Rightarrow R_{p=1}^2 = \frac{\left(S_{yy}-S_{xx}\right)+\sqrt{\left(S_{yy}-S_{xx}\right)^2+4S_{xy}^2}}{2S_{yy}} = R_F^2$$

Result 8: Given that $\lambda=1$. Let R_F^2 and R_p^2 be the coefficient of determination for ULFR model and p-dimensional MULFR model, respectively. When $p=1$, then

$$R_{p=1}^2 = R_F^2$$

4 Simulation study

In this section, we demonstrate some simulation results. Parameter values are chosen to reflect those that may occur in practice. Without loss of generality, it is fixed that $\lambda=1$, $\boldsymbol{\alpha}=[0,0]$ and $\beta=1$ for 2-dimensional model. This resulted in $\boldsymbol{Y}=\boldsymbol{X}$, which is true in many image processing applications that the two images being compared are identical if noise is not present. Some variations from these parameters setting will also be considered, i.e. $\lambda = 0.5, 1.5, 5, 10$ and 100; $\boldsymbol{\alpha} = [-0.5,-1],[-1,1],[0.5,1]$ and $[10,10]$; $\beta = 0.5, 1.5$ and 10. Random errors δ and ε are generated as independent normal distribution using the quadratic transformation method (Pooi, 2003 and Ng, 2006). These random errors are added into the fixed $\boldsymbol{X}$ and $\boldsymbol{Y}$ to obtain the observed values $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. Samples size of $n=10$, 50, 100, 250, 1000 and 4000 are drawn repeatedly. In each case, the number of simulated realizations is 10000.

Table 1 (see Appendix) provides the mean and standard deviation of estimates of the ideal true parameter values $\lambda=1, \beta=1, \boldsymbol{\alpha}=[0,0]$. The columns of $\bar{\lambda}$ and $s(\lambda)$ verified that the two random errors δ and ε are properly generated where the ratio of the error variances close to one with smaller standard deviations when sample size increased. The fourth and fifth columns indicate that the $\hat{\beta}$ is a good estimator of β even when the sample size is 10. The expected standard deviation of $\hat{\beta}$ denoted by $s(\hat{\beta}) = \sqrt{V\hat{a}r(\hat{\beta})}$ is also shown to be consistent when its variance is asymptotically approaching zero. The average $\hat{\boldsymbol{\alpha}}$ in columns seventh and eighth also approaches the true value for large sample size. The R_p^2 is expectably close to its desired value one.

Tables 2, 3 and 4 (see Appendix) display the mean and standard deviation of estimates of the true parameter values which varied from the ideal case. It appears that the estimated parameters are very close to all true value of parameters with reasonably small standard deviations. This suggests that the estimators can still perform for different true parameter values when the ratio of error variances is one.

Many researchers assumed the ratio of error variances is know and equals one. For the case $\lambda \neq 1$, Kendall and Stuart (1979) suggested to reduce it to the case $\lambda = 1$ by dividing the observed values of y_k by $\sqrt{\lambda_k}$. In this study, the effect of $\lambda \neq 1$ is investigated using simulation approach. Table 5 show the simulation results obtained from true parameters value $\beta = 1, \boldsymbol{\alpha} = [0,0]$ and varying λ values. Top row indicates the true λ used in the simulation and the second column gives the average λ calculated from the simulation works. Note that the larger the sample size, the smaller the differences between calculated λ value and the true λ value. Table 5 (see Appendix) shows that there is a drop in performance of the estimators $\hat{\beta}$ and $\hat{\boldsymbol{\alpha}}$, and R_p^2 to achieve the desired true value when the ratio of the error variance λ increased. However, the performance of $\hat{\beta}$ and R_p^2 are still considered good for large deviation of λ not more than 20.

5 Numerical Example: JPEG Compressed Image Quality

In this section, the coefficient of determination, $R_{p=2}^2$ for two-dimensional MULFR model is used to measure the similarity between a reference (original) image and its JPEG compressed image, which in turn reflects the quality of the compressed image. Six standard test images are considered and examples of their compressed images are shown in Figure 1. These standard test images are compressed using JPEG algorithm with compression factor range from 1 to 100.

Two image quality factors, namely image luminance (mean value) and image contrast (variance value) are calculated for both reference image and compressed image. These image quality factors were also used in a well-known image quality metric called mean structural similarity (MSSIM) (Wang et al., 2002) because they carry important physical meaning of an image. The confidence interval for $R_{p=2}^2$ is also displayed where upper limit (red color) and lower limit (blue color) are set to 1 and 0, respectively when $R_{p=2}^2$ value is not computable.

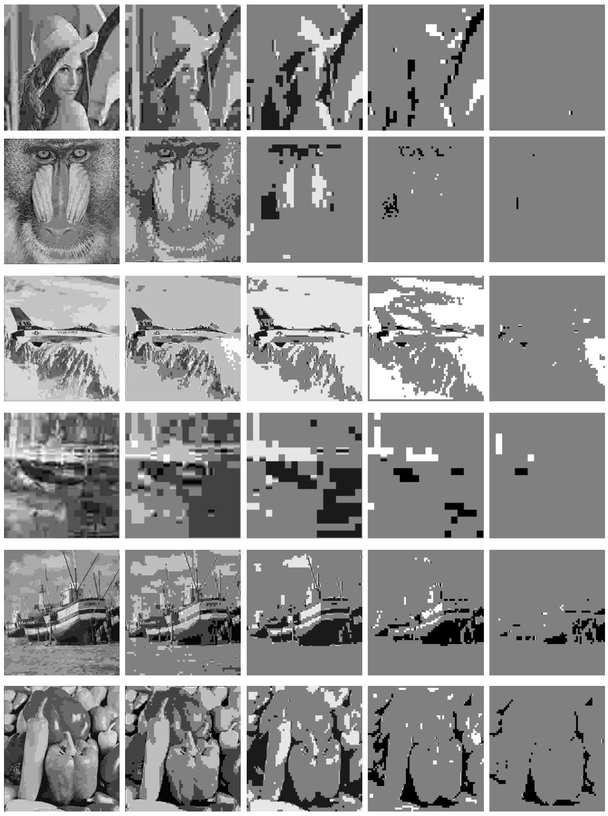

Fig. 1: Standard test images. Top to Bottom: Lena (size 256×256), Baboon (size 256×256), Airplane (size 512×512), Bridge (size 145×145), Boat (size 512×512) and Peppers (size 512×512). Left to Right: original, decompressed images with compression factor 90, 70, 50, 30 and 10.

Figure 2 shows the plots of quality index $\left(Rp2 = R_{p=2}^2\right)$ versus the JPEG compression factor, say Q_i, range from 1 to 100 obtained for various test images. In general, $R_{p=2}^2$ works very well for large compression factors (usually greater than 30) with small confidence intervals. As the JPEG compression factor increases, the quality measure $R_{p=2}^2$ shows an increasing index in decompressed image quality. For example, the decompressed Lena image (see Figure 1) at compression factor $Q_i = 30$ has a quality value of $R_{p=2}^2 = 0.3856$ (confidence interval, $I = [0.3654, 0.4057]$). At compression factor $Q_i = 50$, the Lena decompressed image quality increases to $R_{p=2}^2 = 0.6106$; $\left(I = [0.5884, 0.6327]\right)$. The decompressed quality values increases to $R_{p=2}^2 = 0.7962$ $\left(I = [0.7781, 0.8144]\right)$ and

$R^2_{p=2} = 0.9655 \left(I = [0.9574, 0.9736]\right)$ when the compression factors are $Q_i = 70$ and $Q_i = 90$, respectively (see Fig. 2 for Lena image). It is noted that there are some small fluctuations along the increasing monotonic trend in Figure 2. This phenomenon is mainly caused by the error generated from the decompression process and it is acceptable in any good quality measure (Avcibas, 2002).

It is observed that the measure $R^2_{p=2}$ seems to be not stable for compression factors less than 30. With small compression factors, $R^2_{p=2}$ performs well in some images (e.g. image Airplane, Boat and Peppers) but it performs poorly or cannot produce value for certain images (e.g. Lena, Baboon and Bridge). For those compression factors where $R^2_{p=2}$ cannot produce value, the upper and lower confidence limits are set to one and zero, respectively. Referring back to Figure 1 reveals that most images are badly-compressed at compression factor $Q = 30$ or lower. Some decompressed images at these levels are totally not identifiable from their origin, thus it is not significance to make inferences of the image quality.

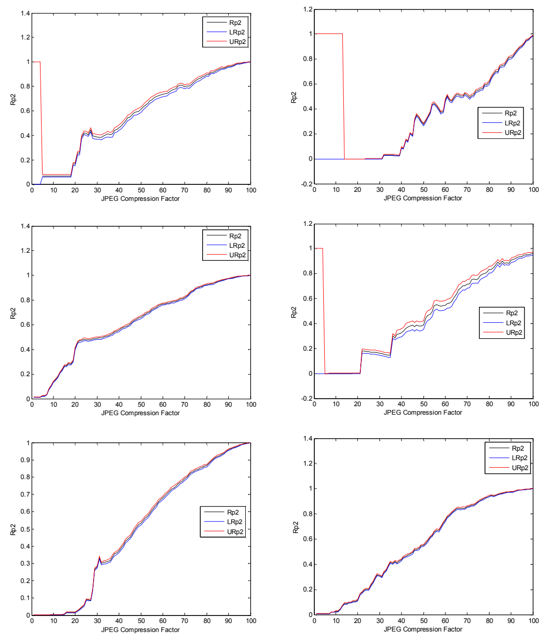

Fig. 2: $\left(Q_i, R^2_{p=2}\right)$ plots and 95% confidence intervals (red line: upper confidence limit. blue line: lower confidence limit) for various standard test images. $R^2_{p=2}$ was computed using luminance and contrast factors. (From left to right) Top: Lena, Baboon. Middle: Airplane, Bridge. Bottom: Boat, Peppers.

5.1 Mean Opinion Score at Low Compression Factors

Another subjective test is carried out to simulate the human observers' performance when compression factors are low. This mean opinion score (MOS) is to address the problem of unstable $R^2_{p=2}$ values for low compression factors. The six standard test images were compressed at compression factors 10, 20, 30 and 40. At each compression factor level, 20 human observers are requested to match the compressed images to its reference images provided.

Table 6. Accuracy rate (%) for matching the compressed image to its reference image.

Reference Image	Q=10	Q=20	Q=30	Q=40
Lena	25	20	95	100
Baboon	30	15	95	100
Airplane	70	85	100	100
Bridge	80	100	100	100
Boat	70	100	90	100
Peppers	65	90	95	100

Table 6 shows the results of this experiment where the figures indicate the percentage of the 20 human observers matched the compressed image to its reference image correctly. For example, only 25% of the human observers matched the Lena image correctly at compression factor $Q = 10$. While at compression factor 20, 20% of the human observers are able to match the Lena compressed image to its reference image. This is followed by 95% and 100% accuracy for compression factor 30 and 40, respectively. In general, the accuracy rate of matching the compressed image to its reference image increased when the compression factor increased. Human observers are still able to recognize the compressed image perfectly at compression factor, $Q \geq 40$. For the compression factor below 30, more human observers matched the compressed image to wrong reference image. This finding explains the reason why the quality values obtained from any quality metric may not be reliable for compression factor below 30.

3 Conclusion

This study proposed a solution to the practical image processing problem of how to investigate the relationship between two images or two sets of features extracted from these images. One example is to evaluate the quality of a compressed image by comparing its reference image, where both images contain noise. Since there are many image features such as image luminance, contrast and entropy can be extracted from an image, hence the multidimensional $\boldsymbol{Y}$ and $\boldsymbol{X}$ are used to represent the reference image and compressed image, respectively. Nevertheless, these image features should not be assessed separately, but a single value indicating the overall image quality is required (Keelan, 2002). This constraint leads us to consider a single slope in the proposed multidimensional unreplicated linear functional relationship model, which is an analogous to the unreplicated linear functional relationship model.

Estimation of parameters has been obtained by maximum likelihood estimation assuming there is a known ratio of error variances. Since the closed-form expression for each estimate is available, estimates can be obtained analytically. Due to the complexity of the model, we have only showed the estimates $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ are approximate unbiased using Taylor approximations. This result can be easily extended to other estimates. Further to this, the asymptotic properties of $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ can be obtained from Fisher's information matrix. Simulation study has been carried out to verify the results where the performance of the estimators remains good for the λ not more than 20. Lastly, the R_p^2 for two-dimensional MULFR model was applied to evaluate the quality of JPEG compressed image. The numerical examples indicate that R_p^2 is a good measure for comparing two images satisfied the three criteria stated by Avcibas et al. (2002), which are prediction monotonic, predication consistency and prediction accuracy. Furthermore, R_p^2 is a sensitive measure at low JPEG compression factors, which is closer to human's judgment.

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Appendix

Table 1. *Mean and standard deviation* $s(.)$ *of estimates of the desired true values* $\lambda=1, \beta=1, \boldsymbol{\alpha}=[0,0]$

n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0629	0.3869	0.9946	4.04E-03	6.50E-06	0.5205	0.6183	0.9997	1.58E-04
50	1.0102	0.1480	0.9987	1.01E-03	8.43E-07	0.1642	0.1665	0.9998	3.08E-05
100	1.0063	0.1029	0.9990	7.95E-04	4.65E-07	0.1302	0.1318	0.9998	2.58E-05
250	1.0023	0.0641	0.9993	5.47E-04	1.97E-07	0.0833	0.0901	0.9998	1.67E-05
1000	1.0005	0.0318	0.9997	2.60E-04	4.63E-08	0.0427	0.0447	0.9998	8.37E-06
4000	1.0001	0.0158	1.0000	2.16E-04	1.15E-08	0.0002	0.0002	0.9998	4.13E-06

Table 2. *Mean and standard deviation* $s(.)$ *of estimates of true values* $\lambda=1, \beta=0.5$ *and varying* $\boldsymbol{\alpha}$

$\lambda=1, \beta=0.5, \boldsymbol{\alpha}=[0,0]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0613	0.3871	0.4997	3.87E-03	4.23E-06	0.0535	0.0605	0.9992	3.75E-04
50	1.0111	0.1462	0.4998	1.50E-03	9.12E-07	0.0310	0.0304	0.9993	1.51E-04
100	1.0047	0.1016	0.5000	1.10E-03	4.45E-07	0.0001	0.0008	0.9993	1.04E-04
250	1.0022	0.0638	0.4999	6.50E-04	1.79E-07	0.0200	0.0237	0.9993	6.28E-05
1000	0.9999	0.0315	0.4999	3.36E-04	4.60E-08	0.0096	0.0091	0.9993	3.14E-05
4000	1.0004	0.0157	0.4999	1.68E-04	1.13E-08	0.0117	0.0124	0.9993	1.61E-05
$\lambda=1, \beta=0.5, \boldsymbol{\alpha}=[-0.5,-1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0625	0.3907	0.5000	4.01E-03	4.09E-06	-0.5018	-1.0015	0.9992	3.85E-04
50	1.0123	0.1470	0.4998	1.46E-03	9.78E-07	-0.4628	-0.9745	0.9992	1.54E-04
100	1.0039	0.1020	0.5000	9.94E-04	4.84E-07	-0.4979	-0.9965	0.9993	9.84E-05
250	1.0011	0.0641	0.4996	6.70E-04	1.89E-07	-0.4336	-0.9241	0.9992	6.97E-05
1000	1.0008	0.0314	0.4997	3.46E-04	4.63E-08	-0.4433	-0.9397	0.9993	3.24E-05
4000	1.0002	0.0159	0.4997	1.68E-04	1.17E-08	-0.4570	-0.9506	0.9993	1.65E-05
$\lambda=1, \beta=0.5, \boldsymbol{\alpha}=[-1,1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0620	0.3946	0.5000	3.07E-03	6.68E-06	-1.0015	0.9973	0.9992	3.98E-04
50	1.0125	0.1466	0.4996	1.65E-03	1.08E-06	-0.9241	1.0449	0.9992	1.54E-04
100	1.0032	0.1017	0.4996	1.03E-03	4.65E-07	-0.9060	1.0491	0.9993	1.01E-04
250	1.0012	0.0634	0.4997	6.78E-04	1.92E-07	-0.9360	1.0351	0.9992	6.71E-05
1000	1.0009	0.0318	0.4998	3.35E-04	4.50E-08	-0.9633	1.0211	0.9993	3.33E-05
4000	1.0004	0.0158	0.4998	1.66E-04	1.15E-08	-0.9606	1.0218	0.9993	1.63E-05
$\lambda=1, \beta=0.5, \boldsymbol{\alpha}=[0.5,1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0671	0.3946	0.4974	3.28E-03	4.49E-06	-0.4863	1.3417	0.9992	3.77E-04
50	1.0133	0.1496	0.4998	1.48E-03	9.08E-07	-0.9587	1.0212	0.9993	1.33E-04
100	1.0059	0.1023	0.4999	9.81E-04	4.39E-07	-0.9765	1.0067	0.9993	9.72E-05
250	1.0030	0.0639	0.4999	6.27E-04	1.84E-07	-0.9800	1.0088	0.9993	6.30E-05
1000	1.0002	0.0313	0.4998	3.22E-04	4.71E-08	-0.9470	1.0283	0.9993	3.10E-05
4000	1.0002	0.0158	0.5000	1.70E-04	1.15E-08	0.5005	0.9997	0.9993	1.62E-05

$\lambda=1,\ \beta=0.5,\ \boldsymbol{\alpha}=[10,10]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0584	0.3832	0.5000	3.36E-03	4.88E-06	9.9931	9.9928	0.9993	3.51E-04
50	1.0085	0.1457	0.5001	1.55E-03	9.59E-07	9.9830	9.9870	0.9993	1.47E-04
100	1.0054	0.1017	0.5000	1.22E-03	4.40E-07	10.0014	10.0014	0.9992	1.13E-04
250	1.0018	0.0634	0.5000	6.67E-04	1.86E-07	9.9899	9.9941	0.9993	6.62E-05
1000	1.0007	0.0317	0.5000	3.37E-04	4.58E-08	9.9924	9.9909	0.9993	3.34E-05
4000	1.0004	0.0157	0.5000	1.68E-04	1.15E-08	9.9932	9.9932	0.9993	1.63E-05

Table 3. *Mean and standard deviation* $s(.)$ *of estimates of true values* $\lambda=1,\ \beta=1.5$ *and varying* $\boldsymbol{\alpha}$

$\lambda=1,\ \beta=1.5,\ \boldsymbol{\alpha}=[0,0]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0615	0.3810	1.5000	5.39E-03	4.23E-06	0.0065	0.0066	0.9999	3.97E-05
50	1.0095	0.1497	1.5000	2.66E-03	8.90E-07	-0.0118	-0.0086	0.9999	1.68E-05
100	1.0058	0.1008	1.5000	1.86E-03	4.14E-07	-0.0025	-0.0022	0.9999	1.14E-05
250	1.0015	0.0637	1.5001	1.04E-03	1.76E-07	-0.0158	-0.0161	0.9999	7.58E-06
1000	1.0005	0.0319	1.5001	5.29E-04	4.61E-08	-0.0114	-0.0104	0.9999	3.66E-06
4000	1.0001	0.0158	1.5001	2.75E-04	1.15E-08	-0.0096	-0.0104	0.9999	1.83E-06
$\lambda=1,\ \beta=1.5,\ \boldsymbol{\alpha}=[-0.5,-1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0669	0.3926	1.4999	4.45E-03	5.27E-06	-0.4804	-0.9938	0.9999	3.49E-05
50	1.0121	0.1472	1.5000	2.55E-03	8.81E-07	-0.4973	-0.9973	0.9999	1.96E-05
100	1.0056	0.1010	1.4999	1.83E-03	4.37E-07	-0.4803	-0.9691	0.9999	1.13E-05
250	1.0021	0.0633	1.5000	1.07E-03	1.70E-07	-0.4943	-0.9909	0.9999	7.25E-06
1000	1.0005	0.0315	1.4999	5.36E-04	4.52E-08	-0.4888	-0.9833	0.9999	3.55E-06
4000	1.0004	0.0157	1.4999	2.72E-04	1.13E-08	-0.4934	-0.9888	0.9999	1.82E-06
$\lambda=1,\ \beta=1.5,\ \boldsymbol{\alpha}=[-1,1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0656	0.3897	1.5001	4.84E-03	3.77E-06	-1.0059	0.9953	0.9999	2.63E-05
50	1.0129	0.1491	1.5000	2.44E-03	9.38E-07	-0.9964	1.0035	0.9999	1.62E-05
100	1.0048	0.1005	1.5000	1.56E-03	4.32E-07	-0.9988	0.9982	0.9999	1.03E-05
250	1.0018	0.0649	1.5000	1.15E-03	2.00E-07	-1.0027	0.9971	0.9999	7.84E-06
1000	1.0007	0.0320	1.5000	5.45E-04	4.44E-08	-0.9947	0.9993	0.9999	3.67E-06
4000	1.0004	0.0158	1.5000	2.69E-04	1.15E-08	-0.9982	0.9923	0.9999	1.81E-06
$\lambda=1,\ \beta=1.5,\ \boldsymbol{\alpha}=[0.5,1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0645	0.3862	1.5000	3.96E-03	3.32E-06	0.5045	1.0054	0.9999	2.61E-05
50	1.0105	0.1482	1.5000	2.26E-03	8.48E-07	0.5015	1.0022	0.9999	1.39E-05
100	1.0057	0.1012	1.5002	1.72E-03	4.82E-07	0.4644	0.9630	0.9999	1.28E-05
250	1.0013	0.0632	1.5001	1.14E-03	1.85E-07	0.4824	0.9807	0.9999	7.33E-06
1000	1.0009	0.0318	1.5001	5.31E-04	4.49E-08	0.4818	0.9834	0.9999	3.68E-06
4000	1.0002	0.0158	1.5001	2.75E-04	1.15E-08	0.4891	0.9899	0.9999	1.83E-06

$\lambda=1,\ \beta=1.5,\ \boldsymbol{\alpha}=[10,10]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0663	0.3929	1.5012	7.09E-03	4.76E-06	9.8193	9.7600	0.9999	3.89E-05
50	1.0098	0.1476	1.5000	2.58E-03	9.33E-07	9.9927	9.9982	0.9999	1.62E-05
100	1.0034	0.1004	1.5001	1.78E-03	4.76E-07	9.9807	9.9702	0.9999	1.12E-05
250	1.0019	0.0635	1.5002	1.13E-03	2.01E-07	9.9705	9.9753	0.9999	7.77E-06
1000	1.0003	0.0315	1.5001	5.42E-04	4.54E-08	9.9799	9.9847	0.9999	3.53E-06
4000	1.0004	0.0157	1.5001	2.73E-04	1.15E-08	9.9803	9.9803	0.9999	1.81E-06

Table 4. *Mean and standard deviation* $s(.)$ *of estimates of true values* $\lambda=1,\ \beta=10$ *and varying* $\boldsymbol{\alpha}$

$\lambda=1,\ \beta=10,\ \boldsymbol{\alpha}=[0,0]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0606	0.3920	9.9999	1.80E-02	3.88E-06	-0.0223	-0.0234	1.0000	6.58E-07
50	1.0100	0.1471	10.0002	1.57E-02	8.23E-07	-0.0174	-0.0179	1.0000	4.10E-07
100	1.0055	0.1016	10.0003	9.90E-03	3.88E-07	-0.0585	-0.0390	1.0000	2.77E-07
250	1.0028	0.0636	10.0014	5.90E-03	1.76E-07	-0.2228	-0.2450	1.0000	1.61E-07
1000	1.0008	0.0317	10.0007	2.97E-03	4.65E-08	-0.1483	-0.1148	1.0000	8.08E-08
4000	1.0001	0.0158	10.0009	1.52E-03	1.14E-08	-0.1516	-0.1462	1.0000	4.10E-08
$\lambda=1,\ \beta=10,\ \boldsymbol{\alpha}=[-0.5,-1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0662	0.3874	10.0003	3.56E-02	5.24E-06	-0.5201	-1.0094	1.0000	1.27E-06
50	1.0095	0.1465	10.0000	1.29E-02	8.58E-07	-0.4869	-0.9832	1.0000	3.48E-07
100	1.0054	0.1012	10.0007	8.91E-03	4.21E-07	-0.6219	-1.0894	1.0000	2.47E-07
250	1.0031	0.0640	10.0005	6.06E-03	1.88E-07	-0.5776	-1.0964	1.0000	1.65E-07
1000	1.0004	0.0317	10.0007	3.02E-03	4.52E-08	-0.6266	-1.1183	1.0000	8.14E-08
4000	1.0002	0.0158	10.0007	1.52E-03	1.13E-08	-0.6154	-1.1340	1.0000	4.06E-08
$\lambda=1,\ \beta=10,\ \boldsymbol{\alpha}=[-1,1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0609	0.3874	10.0003	4.90E-02	3.51E-06	-1.0572	0.9690	1.0000	1.20E-06
50	1.0106	0.1453	10.0005	1.52E-02	8.54E-07	-1.0889	0.9438	1.0000	3.65E-07
100	1.0053	0.1012	10.0013	1.00E-02	4.57E-07	-1.2634	0.8377	1.0000	2.65E-07
250	1.0019	0.0641	10.0017	6.23E-03	1.84E-07	-1.3168	0.7473	1.0000	1.66E-07
1000	1.0007	0.0316	10.0009	3.09E-03	4.54E-08	-1.1530	0.8598	1.0000	8.06E-08
4000	1.0002	0.0158	10.0010	1.51E-03	1.16E-08	-1.1662	0.8301	1.0000	4.04E-08
$\lambda=1,\ \beta=10,\ \boldsymbol{\alpha}=[0.5,1]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{\hat{Var}(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0562	0.3865	9.9999	3.72E-02	3.22E-06	0.5407	1.0394	1.0000	6.50E-07
50	1.0097	0.1469	10.0000	1.07E-02	8.22E-07	0.4918	0.9917	1.0000	3.51E-07
100	1.0045	0.1008	10.0000	9.55E-03	4.62E-07	0.4988	1.0004	1.0000	2.71E-07
250	1.0015	0.0640	10.0005	5.71E-03	1.94E-07	0.4259	0.8876	1.0000	1.58E-07
1000	1.0005	0.0316	10.0007	3.05E-03	4.70E-08	0.3743	0.8895	1.0000	8.18E-08
4000	1.0001	0.0158	10.0011	1.53E-03	1.16E-08	0.3142	0.8271	1.0000	4.15E-08

$\lambda=1, \beta=10, \boldsymbol{\alpha}=[10,10]$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.0630	0.3837	10.0003	3.64E-02	3.01E-06	10.0060	10.0118	1.0000	9.03E-07
50	1.0091	0.1460	10.0022	1.34E-02	8.48E-07	9.5371	9.7276	1.0000	3.52E-07
100	1.0039	0.1009	10.0000	9.60E-03	4.48E-07	10.0067	10.0051	1.0000	2.63E-07
250	1.0016	0.0632	10.0010	6.20E-03	1.90E-07	9.8176	9.8648	1.0000	1.76E-07
1000	1.0005	0.0318	10.0008	2.91E-03	4.53E-08	9.8482	9.8647	1.0000	7.85E-08
4000	1.0001	0.0159	10.0010	1.50E-03	1.15E-08	9.8388	9.8274	1.0000	4.06E-08

Table 5. *Mean and standard deviation* $s(.)$ *of estimates of true values* $\beta=1, \boldsymbol{\alpha}=[0,0]$ *and varying* λ

$\lambda=0.5$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	0.5307	0.1954	1.0001	4.92E-03	3.24E-06	-0.0096	-0.0088	0.9998	8.38E-05
50	0.5038	0.0735	1.0000	1.26E-03	6.35E-07	-0.0011	0.0025	0.9999	2.22E-05
100	0.5016	0.0508	1.0000	1.05E-03	2.77E-07	0.0055	0.0017	0.9999	1.60E-05
250	0.5009	0.0316	1.0000	6.64E-04	1.09E-07	0.0064	0.0032	0.9999	1.02E-05
1000	0.5003	0.0158	1.0000	3.49E-04	2.91E-08	-0.0017	-0.0010	0.9999	5.30E-06
4000	0.5000	0.0079	1.0000	1.70E-04	7.19E-09	-0.0010	-0.0007	0.9999	2.55E-06
$\lambda=1.5$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	1.5862	0.5675	0.9998	5.81E-03	8.45E-06	0.0188	0.0138	0.9996	1.99E-04
50	1.5129	0.2183	0.9999	2.54E-03	1.72E-06	0.0052	0.0097	0.9997	6.17E-05
100	1.5064	0.1532	1.0000	1.84E-03	6.62E-07	0.0013	-0.0002	0.9997	4.28E-05
250	1.5028	0.0953	0.9999	1.06E-03	3.03E-07	0.0052	0.0037	0.9997	2.64E-05
1000	1.5000	0.0480	0.9999	5.41E-04	7.28E-08	0.0096	0.0108	0.9997	1.34E-05
4000	1.4997	0.0239	0.9999	2.71E-04	1.86E-08	0.0043	0.0049	0.9997	6.64E-06
$\lambda=5$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	5.3069	1.9575	0.9987	1.30E-02	4.10E-05	0.1984	0.2054	0.9982	8.45E-04
50	5.0708	0.7366	0.9991	5.82E-03	1.31E-05	0.0706	0.0174	0.9977	4.67E-04
100	5.0299	0.5094	0.9998	4.65E-03	6.12E-06	-0.0512	-0.0667	0.9977	3.32E-04
250	5.0131	0.3190	0.9996	2.90E-03	2.40E-06	-0.0209	-0.0332	0.9978	1.98E-04
1000	4.9993	0.1597	0.9992	1.49E-03	6.02E-07	0.0333	0.0516	0.9978	1.00E-04
4000	5.0004	0.0792	0.9991	7.50E-04	1.47E-07	0.0691	0.0605	0.9977	5.19E-05
$\lambda=10$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	10.6549	3.8835	1.0039	2.52E-02	2.44E-04	-1.4585	-0.7262	0.9945	2.59E-03
50	10.1052	1.4766	0.9939	1.53E-02	4.34E-05	0.7222	0.5801	0.9905	1.92E-03
100	10.0566	1.0119	0.9942	9.90E-03	2.41E-05	0.6047	0.5176	0.9904	1.36E-03
250	10.0233	0.6387	0.9967	5.55E-03	8.90E-06	0.2138	0.1526	0.9912	7.78E-04
1000	10.0062	0.3180	0.9957	2.96E-03	2.19E-06	0.3404	0.3644	0.9911	3.94E-04
4000	9.9981	0.1572	0.9958	1.43E-03	5.67E-07	0.3430	0.3237	0.9909	2.02E-04

$\lambda = 20$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	21.2793	7.8167	0.9833	5.78E-02	5.05E-04	2.3245	2.6933	0.9760	1.14E-02
50	20.2105	2.9715	0.9907	2.70E-02	1.95E-04	0.2606	-0.6120	0.9661	6.87E-03
100	20.0936	2.0276	0.9775	1.81E-02	8.73E-05	2.0631	2.4795	0.9645	5.00E-03
250	20.0350	1.2739	0.9868	1.11E-02	3.62E-05	0.5793	0.6664	0.9659	3.04E-03
1000	20.0098	0.6333	0.9826	5.42E-03	8.52E-06	1.3173	1.5826	0.9659	1.52E-03
4000	20.0034	0.3156	0.9824	2.79E-03	2.14E-06	1.4379	1.5035	0.9650	7.84E-04
$\lambda = 30$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	31.9325	11.8250	0.9968	6.39E-02	1.25E-03	-1.6135	-1.3343	0.9394	2.85E-02
50	30.3112	4.3741	0.9639	3.48E-02	3.48E-04	2.5995	3.5091	0.9241	1.51E-02
100	30.2073	3.0410	0.9685	2.50E-02	1.84E-04	1.6028	2.2684	0.9304	9.86E-03
250	30.0776	1.9222	0.9588	1.67E-02	7.84E-05	3.0734	3.5746	0.9216	6.91E-03
1000	30.0211	0.9461	0.9583	8.06E-03	1.94E-05	3.4321	3.3594	0.9214	3.45E-03
4000	29.9989	0.4709	0.9618	4.04E-03	4.71E-06	2.9392	2.9980	0.9249	1.65E-03
$\lambda = 100$									
n	$\bar{\lambda}$	$s(\lambda)$	$\bar{\hat{\beta}}$	$s(\hat{\beta})$	$\overline{V\hat{a}r(\hat{\beta})}$	$\bar{\hat{\alpha}}_1$	$\bar{\hat{\alpha}}_2$	$\bar{R}_p^2$	$s(R_p^2)$
10	106.170	39.0276	0.6567	2.47E-01	9.49E-03	34.3571	40.7447	0.5027	1.86E-01
50	101.399	14.8294	0.6570	9.39E-02	2.22E-03	31.7548	20.8756	0.4674	8.81E-02
100	100.410	10.1099	0.5464	5.99E-02	9.68E-04	55.8788	46.0113	0.3982	6.52E-02
250	100.168	6.3306	0.5626	3.89E-02	4.12E-04	46.0424	41.2834	0.4056	4.12E-02
1000	100.067	3.1353	0.5714	1.90E-02	1.01E-04	43.0090	45.0621	0.4170	2.02E-02
4000	100.019	1.5917	0.5749	9.64E-03	2.55E-05	43.3167	42.5100	0.4192	1.03E-02