

The Numerical Solution of Obstacle Problem by Self Adaptive Finite Element Method

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Abstract: - In this paper, the bisection of the local mesh refinement in self adaptive finite element is applied to the obstacle problem of elliptic variational inequalities. We try to find the approximated region of the contact in the obstacle problem efficiently. Numerical examples are given for the obstacle problem.

Key-Words: - obstacle problem, variational inequalities, self adaptive finite element, bisection, triangulation refinement

1 Introduction

The history of the study about self adaptive mesh is more than 30 years, domestic and foreign researchers have proposed various algorithms. The self adaptive finite element method is a process which estimates calculation error according to the results of finite element method based on the existing grid and then re-partition the grid where the error is large and re-calculated. When the error is up to the required value, the self adaptive process stops. Therefore, effective error estimation and self adaptive mesh generation are the two key technologies of self adaptive finite element method.

The obstacle problem is one of the simplest unilateral problems, it arises when modelling a constrained membrane in the classical elasticity theory. Many important problems, such as the torsion of an elastic-plastic cylinder, the Stefan problem can be formulated by transformation to an obstacle problem. Several comprehensive monographs can be consulted for the theory and numerical solution of variational inequalities, e.g.[1]-[4]. Since obstacle problems are highly

nonlinear, it is difficult for the computation of approximate solutions. The approximate solution of obstacle problems is usually solved by variable projection method, for example, the relaxation method [2], multilevel projection method[5], multigrid method[6]-[7] and projection method[8] for nonlinear complementarity problems.

In most finite element methods that can be applied to the obstacle problem, the error estimates are acquired under regular and quasi-conforming subdivision [9], but research on anisotropic subdivision is less, which limits the application of the finite element method in engineering. This paper mainly discusses the numerical solution to elliptic partial differential equations and the bisection method is applied to the self adaptive finite element method which will be applied to the obstacle problem. The following is the variational principle of elliptic boundary value problems. First of all, consider the boundary value problem

$$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega \\ \mathcal{B}u = 0, & \text{on } \partial\Omega^* \end{cases} \quad (1)$$

In this equation, Ω is the bounded domain of R^d , the border is $\Gamma = \partial\Omega$, $\partial\Omega^*$ can be the part of the border of $\partial\Omega$ or the entire border. \mathcal{L} is the linear differential operators, \mathcal{B} is the boundary operator.

The problem (1) is about how to find the solution u in the sets of function that make (1) meaningful. In this paper, \mathcal{L} is the uniform operator of $2m$ orders, that is:

$$\begin{cases} \mathcal{L} = \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} (-1)^{|\beta|} \partial^\beta (a_{\alpha\beta}(x) \partial^\alpha), & \alpha, \beta \in R_+^d \\ \mathcal{L}_0 = (-1)^{m|\alpha|} \sum_{|\alpha|=|\beta|=m} \partial^\beta (a_{\alpha\beta}(x) \partial^\alpha), & (\text{principal part}) \end{cases}$$

(2)

And there is constant $\alpha_0 > 0$ such that:

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \xi^{\alpha+\beta} \geq \alpha_0 |\xi|^{2m} \quad \forall m \in R_+^1, \alpha \in R_+^d, \\ \forall a.e. x \in \Omega, \xi \in R^d, a_{\alpha\beta} \in L^\infty(\bar{\Omega})$$

\mathcal{B} is the homogeneous Dirichlet boundary operator, that is on the $\partial\Omega$,

$$u = 0, \frac{\partial u}{\partial \nu} = 0, \left(\frac{\partial}{\partial \nu}\right)^2 u = 0, \dots, \left(\frac{\partial}{\partial \nu}\right)^{m-1} u = 0$$

(3)

Here $\nu = (\nu_1, \dots, \nu_d)$ is the unit outer normal vector on $\partial\Omega$. As long as $\partial\Omega$ is full-smooth, (3) is meaningful. Especially when $m = 1$, (3) is $u = 0$. By the Sobolev space theory it's easy to know that conditions (3) can guarantee $u \in \mathcal{H}_0^m(\Omega)$.

If $u \in \mathcal{C}^{2m}(\Omega) \cap \mathcal{H}_0^m(\Omega)$ is the solution of boundary value problem defined by (2) and (3), in order to derive the variational form and bilinear form, for any $v \in \mathcal{C}_0^\infty(\Omega)$, there is

$$a(u, v) = (\mathcal{L}u, v)_0 = \sum_{\alpha, \beta} (-1)^{|\beta|} \int_{\Omega} v \partial^\beta a_{\alpha\beta}(x) \partial^\alpha u dx$$

Here $(\mathcal{L}u, v)_0$ is the inner product of $\mathcal{L}^2(\Omega)$.

By the Green formula: for any $u, v \in \mathcal{C}^1(\bar{\Omega})$

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = - \int_{\Omega} u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} u v \nu_i ds, \quad i = 1, 2, \dots, d$$

(4)

We could get the variational form of elliptic boundary value problems (2):

$$\begin{aligned} a(u, v) &= \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} a_{\alpha\beta}(x) (\partial^\alpha u(x)) (\partial^\beta v(x)) dx \\ &= \int_{\Omega} f(x) v(x) dx = (f, v)_0 = F(v) \quad \forall v \in \mathcal{C}_0^\infty(\Omega) \end{aligned}$$

Conversely, if $u \in \mathcal{C}^{2m}(\Omega)$ is the solution of problem (4) whose boundary condition is (3). By the Green formula (4), we can get

$$\int_{\Omega} (f - \mathcal{L}u) v dx = 0, \quad \forall v \in \mathcal{C}_0^\infty(\Omega)$$

That is $\mathcal{L}u = f$. In other words, the solution of variational problem that meet the boundary conditions (3) is the same as the original boundary value problem's. Therefore, for the purposes of classical solutions, boundary value problem and variational problem is equivalent, but classical solution u of the boundary conditions (3) satisfies $u \in \mathcal{C}^{2m}(\Omega) \cap \mathcal{H}_0^m(\Omega)$.

By the above-mentions the original solution $u \in \mathcal{H}_0^m(\Omega)$, also by the condition that $\mathcal{H}_0^m(\Omega)$ is the closure of $\mathcal{C}^{2m}(\Omega)$ under the norm of $\mathcal{H}^m(\Omega)$. Therefore, the boundary problem's variational form or weak form can be written to: Finding $u \in \mathcal{H}_0^m(\Omega)$ such that

$$a(u, v) = (f, v)_0 \quad \forall v \in \mathcal{H}_0^m(\Omega) \quad (5)$$

The solution u of (5) is in $\mathcal{H}_0^m(\Omega)$ and there

is not necessarily $u \in C^{2m}(\Omega)$, therefore, the solution of (5) is the weak solution for the boundary value problem (1).

The obstacle problem is a typical example of the elliptic variational inequality of the first kind. Consider the obstacle problem : Find $u \in K$ such that

$$E(u) = \inf_{v \in K} E(v) \tag{6}$$

Where

$$K = \{v \in H_0^1(\Omega) \mid v \geq \psi \text{ a.e. in } \Omega\} \tag{7}$$

$$E(v) = \int_{\Omega} \left(\frac{1}{2} |\nabla v|^2 - fv \right) dx \tag{8}$$

And the obstacle function ψ satisfies the condition $\psi \in H^1(\Omega) \cap C(\Omega)$ and $\psi \leq 0$ on Γ , the boundary of the domain Ω . Problems (6)-(8) describe the equilibrium position u of an elastic membrane constrained to lie above a given obstacle ψ under an external force $f(x) \in L^2(\Omega)$. It is well known that the solution of (6)-(8) is characterized by the following variational inequality: find $u \in K$, such that

$$\int_{\Omega} \nabla u \cdot \nabla (v-u) dx \geq \int_{\Omega} f(v-u) dx, \quad \forall v \in K \tag{9}$$

If the solution $u \in C^2(\Omega) \cap H_0^1(\Omega)$, then we have

$\Omega = \Omega_1 \cup \Omega_2$ - the coincidence set Ω_1 and its

complement Ω_2 ,

$$u = \psi, \text{ and } -\Delta u - f > 0, \text{ in } \Omega_1 \tag{10}$$

$$u > \psi, \text{ and } -\Delta u - f = 0, \text{ in } \Omega_2 \tag{11}$$

Notice that the region of contact

$$\Omega_1 = \{x \in \Omega \mid u(x) = \psi(x)\} \tag{12}$$

is an unknown a priori.

In this paper, we will apply the newest bisection of the local mesh refinement in self

adaptive finite element method to solve the obstacle problem (6)-(8). We first present the obstacle problem and its numerical approximation by finite element method. In section 2, we apply the self adaptive finite element method to the obstacle problem. In section 3, the self adaptive finite element method is applied to one example of the obstacle-free problem and the obstacle problem, the implementation is achieved in MATLAB.

2 The Numerical Algorithm

The finite element method is one of the most commonly used discretization methods for the numerical simulation of many practical models. Now we apply the finite element method to the obstacle problem. We present the discreted obstacle problem by the finite element method. Let $V_h \subset H_0^1(\Omega)$ be a linear finite element space. The discrete admissible set is

$$K_h = \{v_h \in V_h \mid v_h(x) \geq \psi(x), \text{ for any node } x\} \tag{13}$$

Then the approximation of problem (1)-(3) is to find

$u_h \in K_h$ such that

$$E(u_h) = \inf_{v_h \in K_h} E(v_h) \tag{14}$$

The error estimate in H^1 norm or L^∞ norm was proved in [2,10,11].

Theorem 2.1 Assume that the solution u of (6)-(8) and the obstacle function ψ are in the space $W^{2,\infty}$. Then, there exists a constant C independent of h and solution u_h of (8) satisfies

$$\|u - u_h\|_{0,\infty} \leq Ch^2 |\log(h)| (\|u\|_{2,\infty} + \|\psi\|_{2,\infty}) \tag{15}$$

To speed up computing numerical simulations, AFEM (adaptive finite element method) is introduced to reduce computational costs while keeping optimal accuracy.

Now we discuss the self adaptive finite element method and algorithm of this article. When we do the mechanical analysis of engineering

structures, we did not know the extent of stress concentration and its location. Mesh is refined in the areas of large stress gradient only by virtue of experience. But self adaptive finite element method re-partition the grid through the error analysis of finite element method so as to use the least degree of freedom to obtain the best results in the range of error allowed and avoid mesh density is too small where stress gradient is large, or small stress gradient is over the local mesh density. Here we only discuss the tag strategy of triangulation in the realization of the process.

2.1 Application of Bisection Method in the self adaptive finite element method

We first introduced the two important properties of triangulation. The triangulation \mathcal{T}_h of $\Omega \subset R^2$ (or in grid) is the sets which divided Ω into a series of triangles. Family Triangulation is conforming, if the intersection of two triangles τ and τ' in \mathcal{T}_h

is made up by the common vertex x_i or edges E or empty set (not intersect). Edge of a triangle is called non-conforming, if there is a vertex, and it falls on this edge, this vertex is called within the next hanging point. To the suspension grid points, it may need some specific base vector and matrix assembled complexly. For a conforming grid only needs a set of base vectors of finite element. In the following, we will have been used this property of triangulation.

If $\max_{\tau \in \mathcal{T}_h} \frac{\text{diam}(\tau)^2}{|\tau|} \leq \sigma$ that $\text{diam}(\tau)$ is the

diameter of τ , $|\tau|$ is the area of τ , we say family triangulation is regular. If σ is independent of k in the formula above, family triangulation is uniformly regular. Regular triangulation ensure that each corner of the triangular element are maintained to $0 - \pi$. It's important to the H^1 norm estimates of controlled error and the condition number of

stiffness matrix. After the marked triangle set is refined, it needs to design a criterion for the triangular element partitioned and marked so that the refined grid remains the conformity and regularity. Nowadays the most popular two methods are as follow:

2.1.1 The vertex bisection

We gives a group of triangulation \mathcal{T}_h in Ω , and label a vertex of τ as the highest point or the vertex for each triangular element $\tau \in \mathcal{T}$. Its opposite edge is called basic edge. This process is called as the Labeling Process of \mathcal{T} . The vertex bisection criterion includes:

- (1) A triangle can be divided into two new sub-triangle by connecting with the highest point and the mid-point of the basic side;
- (2) A new vertex is produced in the following way: the mid-point of basic side is the highest point of two sub-triangles. Once labels the triangulation of the initial group, to make the bisection process can be continuously carried on, the label of family triangulation inherited by criterion (2).

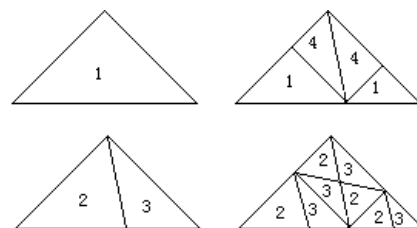


Fig. 1: Four kinds of similar type triangulations in the vertex bisection

In order to illustrate the effectiveness of the process of the bisection, given the following three important properties of the vertex bisection:

Rivara [13] pointed out that in the course of triangulation, it will only generate four kinds of similar type (see Fig.1), So the triangular element achieved by this bisection is uniform and regular.

After labeled triangles have been bisected, it may destroy the conformity of family

triangulation. In order to restore conformity, in the bisection process we should eliminate hanging points, this process can be described as perfection. The perfection process may generate more hanging points, so it needs to stop the process. This issue will be discussed in the following. The last property for the AFEM optimization problem is very important. It points out that compared to the marked units, the perfection process will not increase many units.

2.1.2 The longest edge bisection

The longest edge bisection is proposed by Rivara's study group [16,17]. In this method, the longest edge of the triangle is always used for second-class subdivision. Every time the maximum angle is divided in the longest edge bisection. Therefore, we can expect that this bisection can remain regularity.

In fact, Rosenberg and Stenger [20] proved that in the process of dividing the triangle the smallest angle is at least half of the smallest angle of the initial triangle. Rivara pointed out that this perfection process must be terminated. It can be seen that the longest edge bisection is a special case of the latest top bisection in which different labels are used.

2.1.3 Labeling process to reduce error

We suppose $\eta^2 = \sum_{\tau \in T} \eta_\tau^2$ is the cumulative error index of local error contribution η_τ in a triangular element τ . For the traditional labeling strategy, it marks family triangulation so that:

$$\eta_\tau \geq \theta \max_{\tau \in T_h} \eta_\tau, \quad \forall \theta \in (0,1)$$

This labeling strategy was first proposed by Babuska and Vogelius [4].

Here we used the volume- marked strategy raised by Dorfler. This strategy defines a tag set M so that

$$\sum_{\tau \in M} \eta_\tau^2 \geq \theta \eta^2, \quad \forall \theta \in (0,1)$$

The larger θ makes more triangles to be refined in a circle. Although the smaller θ can lead to grid optimization in the result, it can lead to more refined cycle. Generally, we choose $\theta = 0.2 - 0.5$. The advantage of the volume-marked strategy is: For some elliptic problems, it can prove that the approximation error of the fixed factor for each cycle is diminishing. Therefore, this partial refinement process is convergent. For many degrees of freedom, its best numerical approximation has been put forward.

2.1.4 Perfection

After the triangle has been marked by the bisection, the major problem become: How to maintain the conformity of the grid? We first consider the process of the two basic approaches in perfection, followed by a new strategy for edge marking.

A standard iterative algorithm of perfection is given as follows. Suggest, M is the triangle set which need refinement. Mitchell proposed a more efficient iterative algorithm, Kossaczky extend it to 3-D.

This approximation algorithm is based on the following steps: If a triangle is non-conforming, while we apply a single partition to the triangle opposite to its highest point, it will become conforming. Of course, its adjacent triangles may also be non-conforming. Therefore, in this algorithm it is repeatedly asked to check the adjacent triangle, until we find a conforming triangle. Because it always appears in the iteration before the bisection, there always appear a pair of conforming triangle when the second division is occurred (Except near its borders), and also to ensure conformity. Mitchell proved that if the initial triangulation is the label of conformity, this iteration would be terminated.

2.1.5 marking the edge to Ensure conformity

In order to achieve conformity, we will put forward the new approximation method in this part. Noted that in the output of the grid, new nodes are always those mid-points of some edges in the

inputted grid. Our labeling strategy is: If one side is marked, basic sides of all the triangles sharing this side will be marked.

It is only need slightly modified triangulation (τ), we will be able to achieve this tag by iteration.

Because in each iteration an edge must be marked, and the number of edges of the triangulation is limited, so the termination of this iterative algorithm is obvious.

2.2 The numerical solution of obstacle problem by the self adaptive finite element method

AFEMs are now widely used in the numerical solution of PDEs to achieve better accuracy with minimum degrees of freedom. A typical loop of AFEM through local refinement involves

solution \rightarrow *error estimation* \rightarrow
labeling triangular element \rightarrow *refinement*

More precisely to get a refined triangulation from the current triangulation, we first solve the PDE to get the solution on the current triangulation. The error is estimated using the solution, and used to mark a set of triangles that are to be refined. Triangles are refined in such away to keep two most important properties of the triangulations: shape regularity and conformity.

Recently, several convergence and optimality results have been obtained for adaptive finite element methods on elliptic PDEs [12]-[17] which justify the advantage of local refinement over uniform refinement of the triangulations. In most of those works, newest vertex bisection is used in the refine step. It has been shown that the mesh obtained by this dividing rule is conforming and uniformly shape regular. In addition the number of elements added in each step is under control which is crucial for the optimality of the local refinement. Therefore we mainly discuss vertex bisection in this report and include another popular bisection rule, longest edge bisection, as a variant of it.

From the paper [19,20], we can see that the

mesh is refined in the areas of large error by self adaptive finite element method. The self adaptive finite element method re-partition the grid through the error analysis of finite element method so as to use the least degree of freedom to obtain the best results in the range of error allowed and avoid mesh density is too small where stress gradient is large, or small stress gradient is over the local mesh density. In the paper [19,20], the author proposed bisection method in the self adaptive finite element method and prove its convergence. In the following, we will apply this method to the discreted obstacle method, which can be obtained by the numerical algorithm in the paper [21] or [22].

Suggest Γ_i ($i=1,2,\dots,m_1$) is one triangulation of Ω , Satisfying the regularity conditions, $h_i = \text{diam}(\Gamma_i)$. Denote ρ_i the diameter of inscribed sphere of Γ_i , $h = \min_{1 \leq i \leq m_1} h_i$. Let P_i ($1 \leq i \leq m_2$) is all of the nodes and G_i ($1 \leq i \leq m_1$) is all the units focus.

After giving the value α_i ($1 \leq i \leq m_3$) of P_i , the value β_i ($1 \leq i \leq m_1$) of G_i , we can only get a function of Ω by using the way we construct (9), the function is v_h , and its set is V_h .

In fact, if the mid-points of three edges are denoted separately as M_{i1}, M_{i2}, M_{i3} ($1 \leq i \leq m_1$), and $V^0 = H_0^1(\Omega)$, we can get the solution space like

$$K_h^0 = \{v_h \mid v_h \in V_h^0, \text{ or } v_h(P_i) \geq \chi(P_i) \\ \text{ and } \beta_j \geq 0, 1 \leq i \leq m_2, 1 \leq j \leq m_1 \\ \text{ or } v_h(P_i) \geq \chi(P_i) \text{ and } v_h(M_{jk}) \geq \chi(M_{jk}), \\ 1 \leq i \leq m_2, 1 \leq j \leq m_1, 1 \leq k \leq 3\}$$

For the u_h, v_h of V_h^0 , we can define:

$$a_h(u_h, v_h) = \sum_{i=1}^{m_1} \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx dy, \quad f(v_h) \\ = \int_{\Omega} f v_h \, dx dy$$

The same as the previous section, we consider the discrete minimum problem: finding u_h^* in K_h^0 such as

$$a_h(u_h^*, v_h - u_h^*) \geq f(v_h - u_h^*), \forall v_h \in K_h^0 \quad (16)$$

In the process, we adopt a typical cycle of AFEM, a standard iterative algorithm of the completion is the following. Let Γ denotes the set of triangles to be refined. More precisely to get a refined triangulation from the current triangulation, we first solve the PDE to get the solution on the current triangulation. The error is estimated using the solution, and used to mark a set of triangles that are to be refined or coarsened. Triangles are refined or coarsened in such away to keep two most important properties of the triangulations: shape regularity and conformity.

Algorithm 2.2

STEP1 Initialization: given initial mesh Γ and $0 < tol, \Theta_r < 1, \Theta_c < 1$.

STEP2 Solve: compute discrete solution u_h .

STEP3 Estimate: compute local error estimator η_T and set $\eta^2 = \sum_{T \in \Gamma} \eta_T^2$.

STEP4 IF $\eta < tol$ THEN Return

ELSE Mark: find subsets $\Gamma_r, \Gamma_c \subset \Gamma$, such that

$\sum_{T \in \Gamma_r} \eta_T^2 < \Theta_r \eta^2, \sum_{T \in \Gamma_c} \eta_T^2 < \Theta_c \eta^2$, and η_T small enough for $T \in \Gamma_r$.

Refine / Coarsen: refine triangles $T \in \Gamma_r$ and coarsen triangles $T \in \Gamma_c$ generate a new mesh Γ

Go to STEP 2.

END IF

This approach is based on an observation that if a triangle is not compatible, then after a single division of the the neighbor opposite the peak, it will be. Of course, it may be possible that the neighboring triangle is also not compatible, so the algorithm recursively check the neighboring triangle until a compatible triangle is found. The recursion occurs before the division, so it always bisect a pair of compatible triangles (except near the boundary) and thus the conformity is ensured.

In the following, we discuss the implementation of the vertex bisection and apply it to the obstacle problem. For getting more exact triangulation which is refined from the current triangulation, First of all we must solve the PDE in the current triangulation, and get the answer. Error can be estimated with the current solution, and then, we can label a series of triangles, and these will be subdivided. When the triangles are subdivided, we need to maintain two important properties of triangulation: convergence and conformity.

3 Numerical Results

In this section, numerical examples are given for the obstacle-free problem and the obstacle problem for a membrane. It is seen that the contact region of the obstacle problem is approximated by implementing the AFEM algorithm on the computer.

3.1 The obstacle-free problem

In the following, we mainly show self adaptive finite element method with bisection algorithm through a numerical example. We consider the following elliptic partial

differential problem:

Suggest: $\Omega = \{|x|+|y|<10\} - \{0 \leq x \leq 1, y = 0\}$,

finding the solution of the poisson equation:

$$-\Delta u = f \text{ in } \Omega, u = u_D \text{ on } \Gamma_1, \frac{\partial u}{\partial n} = g \text{ on } \Gamma_2$$

here $f = 1, \Gamma_1 = \partial\Omega, \Gamma_2 = \emptyset$. The following

Fig.2 and Fig.3 are the solution graphics by a different number of iterations in MATLAB.

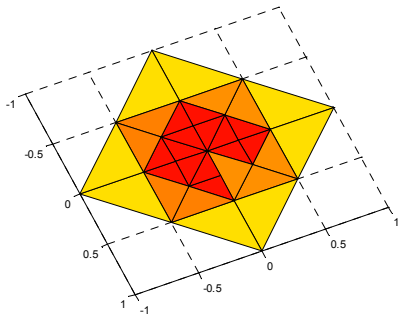
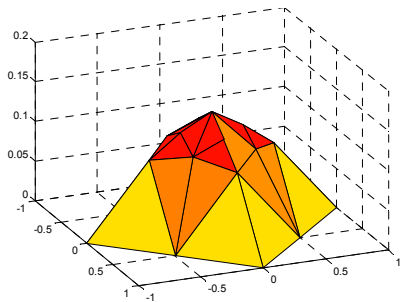


Fig. 2: Numerical solution of obstacle-free problem after 5 iterations

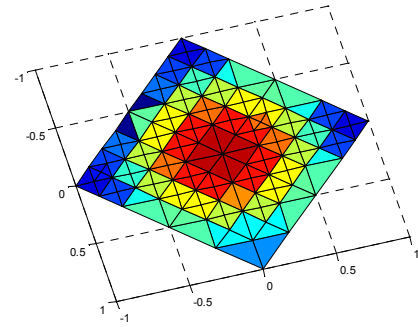
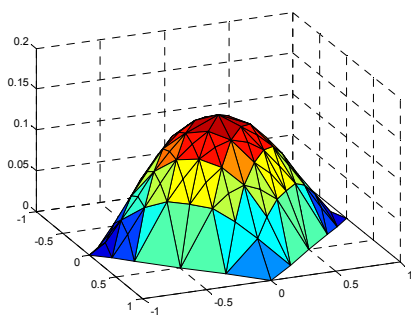


Fig. 3: Numerical solution of obstacle-free problem after 10 iterations

3.2 The obstacle problem

We propose its obstacle function is

$$z = \begin{cases} 7 - x^2 - y^2 & (\text{when } \sqrt{x^2 + y^2} < \frac{1}{8}) \\ 0 & (\text{others}) \end{cases}$$

We draw the figure of obstacle function in (see figure1) as follows

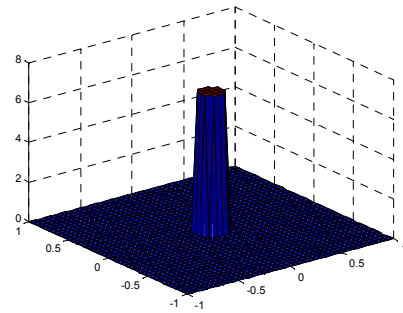
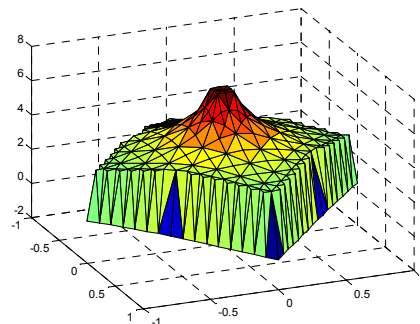


Fig. 4: obstacle function

In the following Fig.5 and Fig.6, the solution graphics of a different number of iterations are given.



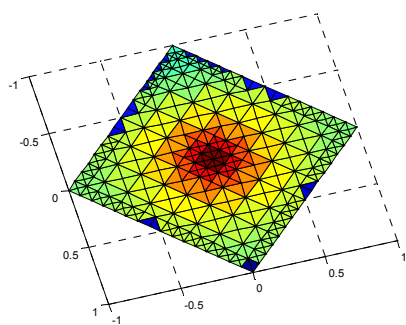


Fig. 5: Numerical solution of obstacle problem after 3 iterations

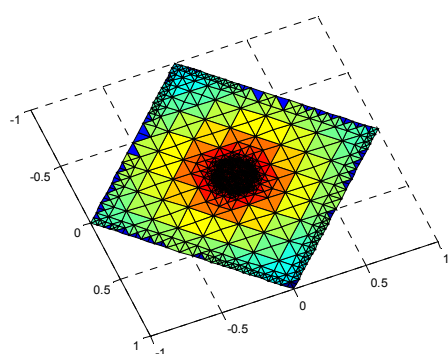
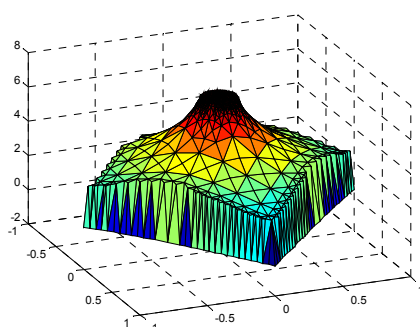


Fig. 6: Numerical solution of obstacle problem after 10 iterations

By the iteration number in the above example, we can see that only through a few iterations, numerical solution of obstacle problem can be obtained easily. It is concluded that the self adaptive finite element method is effective and very easy implement when applied to obstacle problem.

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