

$$K_h^0 = \{v_h \mid v_h \in V_h^0, \text{ or } v_h(P_i) \geq \chi(P_i) \\ \text{ and } \beta_j \geq 0, 1 \leq i \leq m_2, 1 \leq j \leq m_1 \\ \text{ or } v_h(P_i) \geq \chi(P_i) \text{ and } v_h(M_{jk}) \geq \chi(M_{jk}), \\ 1 \leq i \leq m_2, 1 \leq j \leq m_1, 1 \leq k \leq 3\}$$

For the u_h, v_h of V_h^0 , we can define:

$$a_h(u_h, v_h) = \sum_{i=1}^{m_1} \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx dy, \quad f(v_h) \\ = \int_{\Omega} f v_h \, dx dy$$

The same as the previous section, we consider the discrete minimum problem: finding u_h^* in K_h^0 such as

$$a_h(u_h^*, v_h - u_h^*) \geq f(v_h - u_h^*), \forall v_h \in K_h^0 \quad (16)$$

In the process, we adopt a typical cycle of AFEM, a standard iterative algorithm of the completion is the following. Let Γ denotes the set of triangles to be refined. More precisely to get a refined triangulation from the current triangulation, we first solve the PDE to get the solution on the current triangulation. The error is estimated using the solution, and used to mark a set of triangles that are to be refined or coarsened. Triangles are refined or coarsened in such away to keep two most important properties of the triangulations: shape regularity and conformity.

Algorithm 2.2

STEP1 Initialization: given initial mesh Γ and $0 < tol, \Theta_r < 1, \Theta_c < 1$.

STEP2 Solve: compute discrete solution u_h .

STEP3 Estimate: compute local error estimator η_T and set $\eta^2 = \sum_{T \in \Gamma} \eta_T^2$.

STEP4 IF $\eta < tol$ THEN Return

ELSE Mark: find subsets $\Gamma_r, \Gamma_c \subset \Gamma$, such that

$\sum_{T \in \Gamma_r} \eta_T^2 < \Theta_r \eta^2, \sum_{T \in \Gamma_c} \eta_T^2 < \Theta_c \eta^2$, and η_T small enough for $T \in \Gamma_r$.

Refine / Coarsen: refine triangles $T \in \Gamma_r$ and coarsen triangles $T \in \Gamma_c$ generate a new mesh Γ

Go to STEP 2.

END IF

This approach is based on an observation that if a triangle is not compatible, then after a single division of the the neighbor opposite the peak, it will be. Of course, it may be possible that the neighboring triangle is also not compatible, so the algorithm recursively check the neighboring triangle until a compatible triangle is found. The recursion occurs before the division, so it always bisect a pair of compatible triangles (except near the boundary) and thus the conformity is ensured.

In the following, we discuss the implementation of the vertex bisection and apply it to the obstacle problem. For getting more exact triangulation which is refined from the current triangulation, First of all we must solve the PDE in the current triangulation, and get the answer. Error can be estimated with the current solution, and then, we can label a series of triangles, and these will be subdivided. When the triangles are subdivided, we need to maintain two important properties of triangulation: convergence and conformity.

3 Numerical Results

In this section, numerical examples are given for the obstacle-free problem and the obstacle problem for a membrane. It is seen that the contact region of the obstacle problem is approximated by implementing the AFEM algorithm on the computer.

3.1 The obstacle-free problem

In the following, we mainly show self adaptive finite element method with bisection algorithm through a numerical example. We consider the following elliptic partial

differential problem:

Suggest: $\Omega = \{|x|+|y|<10\} - \{0 \leq x \leq 1, y = 0\}$,

finding the solution of the poisson equation:

$$-\Delta u = f \text{ in } \Omega, u = u_D \text{ on } \Gamma_1, \frac{\partial u}{\partial n} = g \text{ on } \Gamma_2$$

here $f = 1, \Gamma_1 = \partial\Omega, \Gamma_2 = \emptyset$. The following

Fig.2 and Fig.3 are the solution graphics by a different number of iterations in MATLAB.

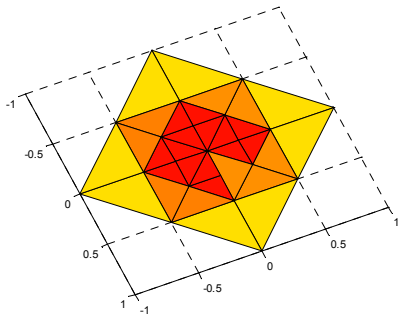
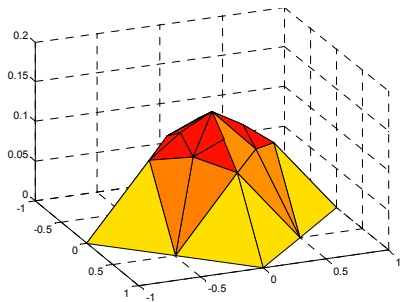


Fig. 2: Numerical solution of obstacle-free problem after 5 iterations

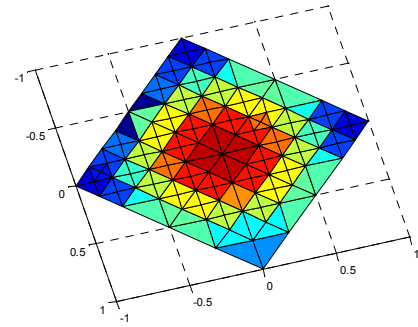
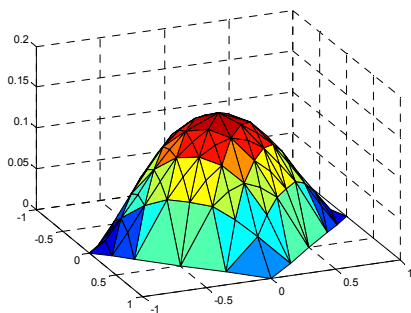


Fig. 3: Numerical solution of obstacle-free problem after 10 iterations

3.2 The obstacle problem

We propose its obstacle function is

$$z = \begin{cases} 7 - x^2 - y^2 & (\text{when } \sqrt{x^2 + y^2} < \frac{1}{8}) \\ 0 & (\text{others}) \end{cases}$$

We draw the figure of obstacle function in (see figure1) as follows

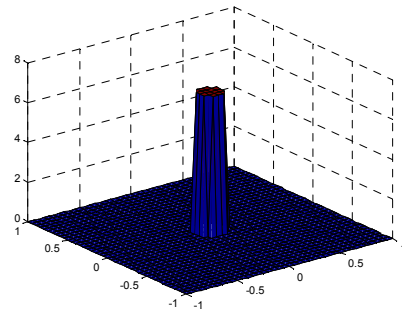
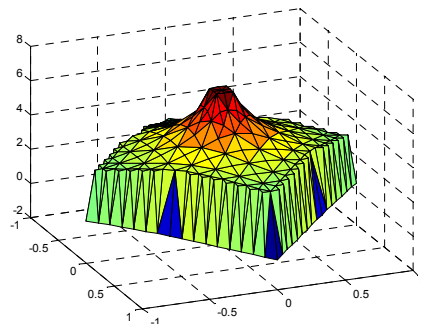


Fig. 4: obstacle function

In the following Fig.5 and Fig.6, the solution graphics of a different number of iterations are given.



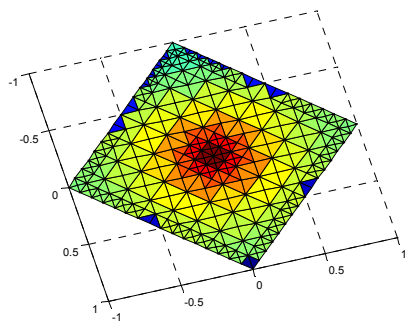


Fig. 5: Numerical solution of obstacle problem after 3 iterations

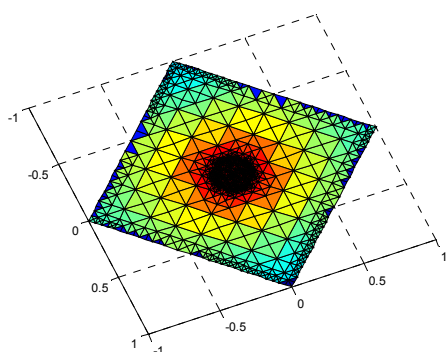
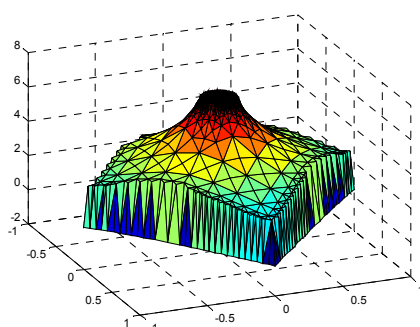


Fig. 6: Numerical solution of obstacle problem after 10 iterations

By the iteration number in the above example, we can see that only through a few iterations, numerical solution of obstacle problem can be obtained easily. It is concluded that the self adaptive finite element method is effective and very easy implement when applied to obstacle problem.

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