Truncation Approximations For Two Term Recursive Universal Form Of Matrix Ordinary Differential Equations

SEVDA ÜSKÜPLÜ
İstanbul Technical University
Informatics Institute
Ayazağa Campus, 34469, Maslak, İstanbul
TÜRKİYE
sevda@be.itu.edu.tr

METİN DEMİRALP
İstanbul Technical University
Informatics Institute
Ayazağa Campus, 34469, Maslak, İstanbul
TÜRKİYE
demiralp@be.itu.edu.tr

Abstract: This work focuses on the construction of an error bound formula for the series solution to Okubo Form of a set linear ordinary differential equation. Okubo Form is obtained using space extension concept which introduces new unknowns into the equation under consideration at the expense of a dimension growth. This is applied to the linear matrix ordinary differential equations in this work.

1 Introduction

In this work, we focus on the following form of matrix differential equations since all linear ordinary differential equations (ODE), in matrix or vector form, can be rewritten in that form

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t), \tag{1}$$

where dot stands for the differentiation with respect to the independent variable t, $\mathbf{A}(t)$ is a given square matrix function of t, and $\mathbf{X}(t)$ represents the unknown square matrix function of t which is same type with $\mathbf{A}(t)$. The matrix $\mathbf{X}(t)$ does not have to be same type of the matrix $\mathbf{A}(t)$. As long as the dimensions of these two matrices are compatible, any rectangular type can be assumed for $\mathbf{X}(t)$. However, we prefer the same type since the solutions for all the other problems with different type unknowns, under appropriate initial conditions, can be expressed as the product of an appropriate rectangular initial constant matrix with the solution of the square unknown case for unit matrix initial value. Those matrix solutions are called propagators in many branches of science and engineering since they characterize the evolution of a system in time. On the other hand, it is not always possible to impose unit matrix initial condition to (1). In the case where the coefficient matrix $\mathbf{A}(t)$ and therefore the differential equation has a singularity at the initial t value the behaviour of the solution may not be a constant matrix. It may have certain singularities enforcing us to write an appropriate initial t dependent matrix multiplying a power series which is analytic in some region around the initial t value. Even in that case, the leading term of the analytic factor of the solution can be initialized by the unit matrix. All these discussions urge us not to specify the initial form of the solution and to leave its specification after the Okubo Form is constructed.

The matrix ODE given in (1) can be converted to a certain universal form via a special technique which puts the ODE structure to a desired form at the expense of increasing the number of unknowns. This method we call "Space Extension Approach" takes us to the linear ODEs with matrix coefficients in the form which were first investigated by Okubo[1, 2, 3]. The structure of these resulting equations after space extension gives the possibility of constructing **two consecutive term recursions** to get the coefficients of the series solution in matrix algebraic analytical expressions. The details of the space extension approach and the conversion of the ODEs into Okubo Form can be found in related papers[4, 5].

The rest of the paper is organized as follows. The second section is devoted to obtain an error bound for the coefficient matrix $\mathbf{A}$(t) given in (1). In the third section the construction of the error bounds for the series solutions of Okubo Form is given. The fourth section is about the truncations as approximants. The fifth section finalizes the paper by presenting the concluding remarks.

2 Error Bound For The Coefficient Matrix

The bound construction for the coefficient matrix $\mathbf{A}(t)$ is not so difficult issue as long as its power series exists and converges in a nonempty disk centered at the t value, where the initial condition is imposed, in the complex plane of t. By using certain complex analytical theorems it is always possible to construct majorant series which can be summed up to a rational function converging inside the convergence domain of the function to be majorized. However, the result may be too pessimistic in many circumstances. Hence we may avoid to construct a bound for the coefficient matrix function $\mathbf{A}(t)$. Instead we convert the original matrix differential equation to Okubo form and then try to majorize its certain elements. To this end we are going to use the technique we call **space extension** in the solution of (1) where the matrices are all $n \times n$ type. Its main idea is basically to change the independent variable by powering. We define first a new independent variable as follows

$$y \equiv t^{m+1} \tag{2}$$

where m denotes any chosen positive integer, and then, we write

$$\mathbf{X}(t) \equiv \mathbf{X}_0(y) + t\mathbf{X}_1(y) + \cdots + t^m \mathbf{X}_m(y) \tag{3}$$

which holds inside the analyticity domain of the solution matrix $\mathbf{X}(t)$ around $t = 0$. If the initial condition given point is not 0 then we can deal with the powers of the differences between t and that initial point. We intend not to get into these types of details here. The right hand side component matrices in (3) are in fact the subseries of the series expansion for the solution matrix $\mathbf{X}(t)$ in powers of t. In this connection, $\mathbf{X}_0(t)$ contains the terms whose t dependence can be expressed via positive integer powers of t^{m+1}. Similarly, $\mathbf{X}_k(t)$ contains the terms whose t dependence can be expressed via positive integer powers of t^{m+1} multiplied by t^k where k is an integer varying between 0 and m inclusive.

If the series expansion of the matrix $\mathbf{A}(t)$ in nonnegative powers of t is truncated at the mth term and (3) is used in the resulting equation then we obtain an equation which contains t^ks for integer values of k between 0 and m inclusive after reexpressing any t^k term with greater than m values of k as the product of certain integer power of y with an appropriate term t^k where k is again between 0 and m inclusive. Since the homogenized form of this equation contains $m+1$ unknown matrices and a linear combination of $m+1$ terms each of which is proportional to a t^k type term with k values between 0 and m inclusive, the coefficient matrices of this linear combination can be set equal to zero. By reorganizing the resulting terms and putting into an extended matrix algebraic form we obtain the following Okubo form

$$\dot{\mathbf{Z}}(y) = \left[\frac{1}{y} \boldsymbol{\mathcal{A}}_0^{(m)} + \boldsymbol{\mathcal{A}}_1^{(m)} \right] \mathbf{Z}(y), \tag{4}$$

where

$$\mathbf{Z}(y)^T = \left[\mathbf{X}_0(y)^T \ ... \ \mathbf{X}_m(y)^T \right]. \tag{5}$$

The explicit structures of $\boldsymbol{\mathcal{A}}_0$ and $\boldsymbol{\mathcal{A}}_1$ matrices are

$$\boldsymbol{\mathcal{A}}_0^{(m)} \equiv \frac{1}{m+1} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_0 & -\mathbf{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{A_{m-1}} & \cdots & \mathbf{A}_0 & -m\mathbf{I} \end{bmatrix}, \tag{6}$$

$$\boldsymbol{\mathcal{A}}_1^{(m)} \equiv \begin{bmatrix} \mathbf{A}_m & \mathbf{A}_{m-1} & \cdots & \mathbf{A}_0 \\ \mathbf{0} & \mathbf{A}_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{A}_{m-1} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_m \end{bmatrix}. \tag{7}$$

Now we can write the following equality to investigate the norm properties of last two matrices

$$\left\| \boldsymbol{\mathcal{A}}_0^{(m)} \right\|_F^2 = \frac{1}{(m+1)^2} \left(\sum_{i=1}^{m} i \left\| \mathbf{A}_{m-i} \right\|_F^2 + \frac{m(m+1)(2m+1)}{6} \left\| \mathbf{I} \right\|_F^2 \right) \tag{8}$$

$$\left\| \boldsymbol{\mathcal{A}}_1^{(m)} \right\|_F^2 = \sum_{i=0}^{m} (i+1) \left\| \mathbf{A}_i \right\|_F^2 \tag{9}$$

where the subscript F denotes Frobenius matrix norm [6].

The analyticity of the coefficient matrix function $\mathbf{A}(t)$ enables us to write

$$\left\| \mathbf{A}_i \right\|_F^2 < \frac{B^2}{r^{2i}}, \qquad i = 0, 1, 2, ... \tag{10}$$

where r stands for the radius of the disc centered at the origin of the independent variable t's complex plane. It is assumed to be greater than T, the length of the interval in which t lies. Or, in other words, we choose the variation interval of t inside the convergence disk of $\mathbf{A}(t)$ in the complex plane of t. The parameter B above stands for the bound to the $\mathbf{A}(t)$'s norm on the circle centered at the origin with the radius r. It may depend on r. We do not need the explicit values of r and B for the moment since the present analysis is at a quite abstract level just now.

The utilization of (10) in (9) results in the following inequality

$$\left\| \boldsymbol{\mathcal{A}}_1^{(m)} \right\|_F^2 < B^2 \left[\frac{r^4}{(r^2-1)^2} + \frac{(m+1)-(m+2)r^2}{r^{2m}(r^2-1)^2} \right] \tag{11}$$

which means that the Frobenius norm of the constant matrix $\boldsymbol{\mathcal{A}}_1^{(m)}$ remains finite for all m values as long as the radius of analyticity is greater than 1. The same thing can not be said for the matrix $\boldsymbol{\mathcal{A}}_0^{(m)}$. It is unbounded with respect to m.

3 Error Bound Construction for the Series Solutions

It is possible to obtain a solution for the Okubo Form by expanding its solution into a power series around $y = 0$. There is a regular singular point at $y = 0$. So it is convenient to use the general term as y^{n+r} with undetermined coefficients. This means that we can write the following equalities

$$\begin{aligned} \mathbf{Z}(y) &= \sum_{n=0}^{\infty} y^{n+r}\mathbf{V}_n, \\ \dot{\mathbf{Z}}(y) &= \sum_{n=0}^{\infty}(n+r)y^{n+r-1}\mathbf{V}_n. \end{aligned} \tag{12}$$

If (12) is substituted into (4) the following structure is obtained

$$\sum_{n=0}^{\infty}(n+r)y^{n+r-1}\mathbf{V}_n = \sum_{n=0}^{\infty} y^{n+r-1}\boldsymbol{\mathcal{A}}_0^{(m)}\mathbf{V}_n + \sum_{n=0}^{\infty} y^{n+r}\boldsymbol{\mathcal{A}}_1^{(m)}\mathbf{V}_n. \tag{13}$$

This can be rewritten in a single series form as follows after certain alterations are performed in sums

$$\left(r\mathbf{I} - \boldsymbol{\mathcal{A}}_0^{(m)}\right) y^{r-1}\mathbf{V}_0 + \sum_{n=1}^{\infty}\Big[\,(n+r)\mathbf{V}_n - \boldsymbol{\mathcal{A}}_0^{(m)}\mathbf{V}_n - \boldsymbol{\mathcal{A}}_1^{(m)}\mathbf{V}_{n-1}\Big]\, y^{n+r-1} = 0. \tag{14}$$

The coefficients of y^{r-1} and y^{n+r-1} should be individually set equal to zero for the satisfaction of the last equation. This gives the following recursion

$$\begin{aligned} \left[(n+r)\mathbf{I} - \boldsymbol{\mathcal{A}}_0^{(m)}\right]\mathbf{V}_n &= \boldsymbol{\mathcal{A}}_1^{(m)}\mathbf{V}_{n-1} \\ n \geq 0,\ \mathbf{V}_{-1} &\equiv \mathbf{0}. \end{aligned} \tag{15}$$

where $\boldsymbol{\mathcal{A}}_0^{(m)}$ is not invertable due to its singular structure (it has a multiple zero eigenvalue).

The first equation in (15) takes the following form when $n = 0$

$$\left[\boldsymbol{\mathcal{A}}_0^{(m)} - r\mathbf{I}\right]\mathbf{V}_0 = \mathbf{0} \tag{16}$$

which means that r can be equal to one of $\boldsymbol{\mathcal{A}}_0^{(m)}$'s eigenvalues. The spectrum of $\boldsymbol{\mathcal{A}}_0^{(m)}$ is composed of m different values, 0, $1/m$, ..., $(m-1)/m$, each of which has a multiplicity equal to the dimension of the matrix $\boldsymbol{\mathcal{A}}(t)$. Only the zero eigenvalue is acceptable since matrix $\mathbf{Z}(y)$ was assumed to be analytic in y. Therefore we need to take $r = 0$. The eigenvectors corresponding to the zero eigenvalues are composed of m blocks each of which is a vector whose number of elements is same as the dimension of $\boldsymbol{\mathcal{A}}(t)$. All blocks, except the first one which is completely arbitrary, vanish. This is reflected to the matrix $\mathbf{V}_0$ as a square matrix arbitrariness with the same dimension of $\boldsymbol{\mathcal{A}}(t)$. In other words, the matrix $\mathbf{V}_0$ which is an m element vector of square matrices whose dimensions are same as the dimension of $\boldsymbol{\mathcal{A}}(t)$, can have a nonvanishing matrix block at its first element only. This means that the solution will have the arbitrariness we expect from the original matrix equation given in (1).

Now by taking $r = 0$ we can write the following recursion without specifying the explicit structure of $\mathbf{V}_0$

$$\mathbf{B}_0(n)\mathbf{V}_n = \mathbf{B}_1\mathbf{V}_{n-1} \tag{17}$$

where

$$\mathbf{B}_0(n) \equiv \left[n\mathbf{I} - \boldsymbol{\mathcal{A}}_0^{(m)}\right],\ \ \mathbf{B}_1 \equiv \boldsymbol{\mathcal{A}}_1^{(m)}\ \ n \geq 0. \tag{18}$$

The matrix $\mathbf{B}_0(n)$ here is invertible for all positive integer values of n. This urges us to write

$$\mathbf{V}_n = \mathbf{B}_0(n)^{-1}\mathbf{B}_1\mathbf{V}_{n-1}, \quad n = 1, 2, ... \tag{19}$$

which dictates us that all $\mathbf{V}_n$ matrices can be determined uniquely except a common rightmost factor which is equal to $\mathbf{V}_0$.

Now we can proceed to construct a majorant function for the norm of $\mathbf{Z}(y)$. We can write the following inequality

$$\begin{aligned} \left\|\mathbf{B}_0(n)^{-1}\right\| &\leq \frac{1}{n}\sum_{k=0}^{\infty}\frac{1}{n^k}\left\|\boldsymbol{\mathcal{A}}_0^{(m)}\right\|^k \\ &= \left[n\mathbf{I} - \left\|\boldsymbol{\mathcal{A}}_0^{(m)}\right\|\right]^{-1}, \\ &\qquad n > N \end{aligned} \tag{20}$$

where N stands for the least integer upper bound to the $\boldsymbol{\mathcal{A}}_0$'s norm and we have used the spectral norm to make the unit matrix norm 1.

(20) and (19) imply

$$\|\mathbf{V}_n\| \leq \frac{1}{n - \left\|\boldsymbol{\mathcal{A}}_0^{(m)}\right\|}\|\mathbf{B}_1\|\,\|\mathbf{V}_{n-1}\| \tag{21}$$

$$= \frac{\Gamma\left(N+1-\left\|\boldsymbol{\mathcal{A}}_0^{(m)}\right\|\right)}{\Gamma\left(n+1-\left\|\boldsymbol{\mathcal{A}}_0^{(m)}\right\|\right)}\|\mathbf{B}_1\|^{n-N}\|\mathbf{V}_N\|\,,$$
$$n = N, N+1, ... \qquad (22)$$

which means

$$\sum_{k=N}^{\infty}|y|^k\|\mathbf{V}_k\| \le \left(\sum_{k=N}^{\infty}|y|^k\frac{\Gamma\left(N+1-\left\|\boldsymbol{\mathcal{A}}_0^{(m)}\right\|\right)}{\Gamma\left(k+1-\left\|\boldsymbol{\mathcal{A}}_0^{(m)}\right\|\right)}\|\mathbf{B}_1\|^{k-N}\right)\|\mathbf{V}_N\|\,. \qquad (23)$$

This inequality reveals the fact that series solution of the Okubo form convergences for all finite values of its independent variable y as long as the dimensions of the matrix algebraic entities remain finite. This urges us to truncate this series solution by retaining a finite number of its first terms and discarding the remaining ones.

4 Truncations as Approximants and Error Estimations

Now we can define the following approximants for $\mathbf{Z}(y)$

$$\mathbf{Z}_k(t) \equiv \sum_{i=0}^{k} y^i\mathbf{V}_i, \qquad k = 0, 1, 2, ... \qquad (24)$$

which allows us to define the following approximants for $\mathbf{X}(t)$

$$\mathbf{X}_k(t) \equiv \mathbf{t}^T\mathbf{Z}_k(t), \qquad k = 0, 1, 2, ... \qquad (25)$$

where

$$\mathbf{t}^T \equiv \left[\, 1\; t\; t^2\; ...\; t^{m-1}\,\right]. \qquad (26)$$

Let us consider the sum of the discarded terms when we define $\mathbf{Z}_k(t)$ and write

$$E_k(y) \equiv \left\|\sum_{i=k+1}^{\infty} y^i\mathbf{V}_i\right\|, \qquad k = 0, 1, 2, ... \qquad (27)$$

as the norm of the error term. We can write the following inequalities by keeping the limitation $k > N-1$ in mind

$$\begin{aligned} E_k(y) &< \left(\sum_{i=0}^{\infty}\frac{\Gamma(k+2-N)}{\Gamma(k+2-N+i)}\|\mathbf{B}_1\|^i|y|^i\right) |y|^{k+1}\|\mathbf{V}_{k+1}\| \\ &= \frac{(k-N+1)!\,\|\mathbf{V}_{k+1}\|}{\|\mathbf{B}_1\|^{k+1-N}}|y|^N \left(e^{\|\mathbf{B}_1\||y|} - \sum_{i=0}^{k-N}\frac{\|\mathbf{B}_1\|^i|y|^i}{i!}\right), \end{aligned} \qquad (28)$$

where we have ignored the difference between the least integer upper bound to norm of $\boldsymbol{\mathcal{A}}_0^{(m)}$ and $\boldsymbol{\mathcal{A}}_0^{(m)}$'s norm without destroying the inequality. Although the resulting inequality becomes more pessimistic it helps us to express the infinite sum with a finite number of well known terms. In this analysis the truncation order k is assumed to be chosen greater than $N-1$ to be able to use the previous section's inequalities. The last error bound formula is constructed under this assumption.

5 Concluding Remarks

We have analysed the convergence of the Okubo form solutions to matrix ordinary differential equations in this paper. To this end we have used certain bounds to the matrices of the Okubo form for the construction of a majorant function to Okubo form solution. The truncation of the series solution to Okubo form at finite number of terms enabled us to use them as approximations. We have also constructed a bound to the error arising when we use these truncations. The error bound constructed here is quite pessimistic. We could construct much tighter error bounds by removing the requirement for k values to be greater than $N-1$ and/or by taking specific features of the matrices appearing in the analysis given here into consideration. However, we have not done so here because our main purpose has been just to get an idea about the errors, not to catch better explicit formulation (this may be considered as a separate future work). Hence we are not limited with the k values greater than $N-1$ in fact. We are going to use all k values especially the lowest ones for simplicity in our future applications.

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