Fluctuationlessness Theorem and its Application to Boundary Value Problems of ODEs

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Abstract: A numerical method based on Fluctuationlessness Approximation, which was developed recently, is constructed for solving Boundary Value Problems of Ordinary Differential Equations on appropriately defined Hilbert Spaces. The numerical solution is written in the form of a Maclaurin series. The unknown coefficients of this series are determined by constructing an $(n-2)$ unknown containing linear system of equations. The eigenvalues of the independent variable's matrix representation are used in the construction of the matrices and the vectors of the linear system. The numerical solution obtained by Fluctuationlessness Approximation is then compared with the Maclaurin coefficients of the analytical solution to observe the quality of the convergence. Some illustrative examples are presented in order to give an idea about the efficiency of the method explained here.

Key–Words: Boundary Value Problems, Eigenvalues, Fluctuationlessness Approximation, Hilbert Spaces.

1 Introduction

Ordinary Differential Equations (ODEs) are among the important subjects of the Applied Mathematics. They are mathematical models of the physical and engineering problems. Amongst these, Boundary Value Problems (BVP) occupy a large portion. They may be the model of the problem itself, or they may arise after using the method of separation of variables for Partial Differential Equations (PDEs). The Initial Value Problems (IVPs) have unique solutions at least for the case of linearity and for the many cases of nonlinear structures showing no bifurcative nature. BVPs may have a unique solution, no solution, or in certain cases, infinitely many solutions even in the case of nonlinearity, depending on the structures of the equations and the accompanying conditions. Despite some BVPs may have easily obtainable analytical solutions the others may involve equations that can not be solved analytically, or even they are solved, the solutions may be quite inconvenient for practical utilization. In the cases where a practically employable analytical solution can not be constructed the numerical solutions help us to find the very close values to the solution at certain fixed points in the domain of the independent variable. Certain numerical schemes like Shooting Method and Finite Difference Method have been developed in the past. However, as many numerical solutions have, these solutions may have some drawbacks such as needing a large number of nodes or producing unsatisfactory erroneous values.

Fluctuationlessness Theorem, which was proven recently, has an impressive feature that, even using few number of nodes, the numerical solution approximates the analytical solution satisfactorily. This theorem simply states that, in the absence of fluctuations, the matrix representation of a function operator, whose action is simply to multiply its operand by the function value, in any n–dimensional subspace of the Hilbert Space under consideration, is equal to the image of the independent variable's matrix representation in the same subspace under that function. There are three types of BVPs. If only a value to the normal derivative of the unknown function is given in the boundary conditions then it is a Neumann type boundary condition. If only a value to the unknown function is given in the boundary conditions then it is a Dirichlet type boundary condition. If the boundary has the form of a curve or surface that gives a value to the normal derivative of and itself of the function then it is a Cauchy type boundary condition or sometimes it is called "Mixed condition". The method we develop in this paper does not have any difference in the type of the boundary conditions of the problem. It is constructed to solve BVPs generally.

BVPs can also be defined for nonlinear ODEs, but we will restrict ourselves to a consideration of linear equations only in this work. These problems may also involve an unknown parameter which defines an eigenvalue problem of a linear ordinary differential operator if appropriate boundary conditions are imposed. Even if this is done the solution may not exist for all values of the unknown parameter. Instead, certain specific values for the parameter and the structures for the corresponding function are obtained as solution. What we get as solution is a set of specific parameter values which are called eigenvalues and corresponding functions which are called eigenfunctions. The set of eigenfunctions generally span a Hilbert space which is the domain of the linear ordinary differential operator mentioned above. We do not consider these types BVPs here in the perspective of our approach. They are left for some future works.

The rest of the paper is organized as follows. In the second section the Fluctuationlessness Theorem is explained briefly. The third section is devoted to the application of this theorem on the numerical solution of BVPs without unknown parameters. The fourth section involves numerical examples with illustrations. The fifth section involves the concluding remarks.

2 Fluctuationlessness Theorem

We take an n–dimensional subspace $\mathcal{H}_n$ of the Hilbert Space $\mathcal{H}$ spanned by the functions $u_1(x)$, $u_2(x)$,..., $u_n(x)$ which are analytic and therefore square integrable on the given finite interval $[a, b]$. These functions are first n elements of the basis function set for $\mathcal{H}$. We define the inner product of any two functions, f, g, from $\mathcal{H}$ as follows:

$$(f,g) \equiv \int_a^b dx w(x) f(x) g(x), \tag{1}$$

where $w(x)$ stands for a weight function. For an arbitrary function $g(x)$ in $\mathcal{H}_n$ we can write the following identity

$$g(x) \equiv P^{(n)} g(x). \tag{2}$$

Here $P^{(n)}$ is an operator which projects from $\mathcal{H}$ to the subspace $\mathcal{H}_n$. $P^{(n)}$ becomes the unit operator of $\mathcal{H}_n$ if it is considered from $\mathcal{H}_n$ to $\mathcal{H}_n$.

Now we can consider a new operator, $\widehat{x}$ which multiplies its operand by x and its domain is $\mathcal{H}$. The action of this operator on a function $g(x)$ from $\mathcal{H}_n$ can be expressed as follows:

$$\widehat{x} g(x) = \sum_{j=1}^{n} g_j \widehat{x} u_j(x) = \widehat{x} P^{(n)} g(x). \tag{3}$$

We can see from this equation that the function $\widehat{x} g(x)$ may not remain in the subspace $\mathcal{H}_n$ although $P^{(n)} g(x)$ is in this subspace. In such cases $\widehat{x}$ operator causes a space extension. In order to avoid from this situation, it is better to use $P^{(n)}\widehat{x}$ instead of $\widehat{x}$. So we can write

$$P^{(n)} \widehat{x} g(x) = P^{(n)} \widehat{x} P^{(n)} g(x). \tag{4}$$

The operator $\widehat{x}_{res}$ which is the restriction of $\widehat{x}$ from $\mathcal{H}_n$ to $\mathcal{H}_n$ can be explicitly given as follows:

$$\widehat{x}_{res} \equiv P^{(n)} \widehat{x} P^{(n)}. \tag{5}$$

Now we can write the following equation for the operator $\widehat{x}$.

$$\begin{aligned}
\widehat{x} \equiv & \left(P^{(n)} + \left[\widehat{I} - P^{(n)}\right]\right) \widehat{x} \left(P^{(n)} + \left[\widehat{I} - P^{(n)}\right]\right) \\
= & P^{(n)} \widehat{x} P^{(n)} + \left[\widehat{I} - P^{(n)}\right] \widehat{x} P^{(n)} \\
& + P^{(n)} \widehat{x} \left[\widehat{I} - P^{(n)}\right] \\
& + \left[\widehat{I} - P^{(n)}\right] \widehat{x} \left[\widehat{I} - P^{(n)}\right]
\end{aligned} \tag{6}$$

If we define $\widehat{x}_{fluc}$ as the last three additive terms of the right hand side in the above equality,

$$\begin{aligned}
\widehat{x}_{fluc} \equiv & \left[\widehat{I} - P^{(n)}\right] \widehat{x} P^{(n)} + P^{(n)} \widehat{x} \left[\widehat{I} - P^{(n)}\right] \\
& + \left[\widehat{I} - P^{(n)}\right] \widehat{x} \left[\widehat{I} - P^{(n)}\right],
\end{aligned} \tag{7}$$

then we can express the operator $\widehat{x}$ as the sum of $\widehat{x}_{res}$ and $\widehat{x}_{fluc}$. Here the operator $\left[\widehat{I} - P^{(n)}\right]$ approaches to $\widehat{0}$ operator as n goes to infinity. The approximations by ignoring the terms which contain this operator is called as *Fluctuationlessness Approximation*. Therefore the $\widehat{x}$ operator can now be expressed as

$$\widehat{x} \approx \widehat{x}_{res} \equiv P^{(n)} \widehat{x} P^{(n)} \tag{8}$$

in the fluctuationlessness limit. The matrix representation of this operator is $\mathbf{X}^{(n)}$, which has the general term $X_{ij}^{(n)}$ defined by the following expression

$$X_{ij}^{(n)} = \left(u_i, P^{(n)} \widehat{x} P^{(n)} u_j\right), \quad 1 \le i, j \le n. \tag{9}$$

Now we can define a function operator $\widehat{f}$ which multiplies its operand by a function $f(x)$ analytical on the interval $[a, b]$ as follows:

$$\widehat{f} \equiv f(\widehat{x}) \tag{10}$$

Hence $\widehat{f}$ is an algebraic operator in terms of $\widehat{x}$ and its fluctuationlessness approximation is given below.

$$\widehat{f} \approx f\left(\widehat{x}_{res}\right) \equiv f\left(P^{(n)} \widehat{x} P^{(n)}\right) \tag{11}$$

The matrix representation of the operator $\widehat{f}$'s restriction from $\mathcal{H}_n$ to $\mathcal{H}_n$ is written as follows

$$\mathbf{M}_f^{(n)} \approx f\left(\mathbf{X}^{(n)}\right), \tag{12}$$

where $\mathbf{M}_f^{(n)}$ stands for the matrix representation of the operator $\widehat{f}$'s restricted form mapping from $\mathcal{H}_n$ to $\mathcal{H}_n$. [1–5]

3 Numerical Solution of Boundary Value Problems in the Fluctuationlessness Theorem Perspective

We consider the following BVP composed of a second order linear and inhomogeneous ODE and two accompanying linear and inhomogeneous boundary conditions each of which is given at a different endpoint of the interval $[\,0,1\,]$

$$\begin{aligned} y'' + p(x)\,y' + q(x)\,y &= r(x)\,, && (13)\\ a_1 y(0) + a_2 y'(0) &= c, && (14)\\ b_1 y(1) + b_2 y'(1) &= d, && (15)\end{aligned}$$

where $0 < x < 1$. The ODE given by (13) has two linearly independent homogeneous solutions we denote by $\phi_1(x)$ and $\phi_2(x)$. These are assumed not to have any arbitrary constants. In other words, they are constructed by imposing specific values to the arbitrary constants of the general solution to the homogeneous form of (13). The inhomogeneous ODE in (13) has an additional particular solution $\phi_p(x)$ which identically vanishes when the right hand side becomes zero. Therefore the general solution of (13) can be written [6] as

$$y(x) = C_1\phi_1(x) + C_2\phi_2(x) + \phi_p(x) \tag{16}$$

The arbitrary constants C_1 and C_2 should take specific values to satisfy the boundary conditions given by (14) and (15) when they are imposed. If this happens then the following equations are obtained

$$\begin{aligned} A_{11}C_1 + A_{12}C_2 &= B_1,\\ A_{21}C_1 + A_{22}C_2 &= B_2, \end{aligned} \tag{17}$$

where

$$\begin{aligned} A_{11} &\equiv a_1\phi_1(0) + a_2\phi_1'(0)\\ A_{12} &\equiv a_1\phi_2(0) + a_2\phi_2'(0)\\ A_{21} &\equiv a_1\phi_1(1) + a_2\phi_1'(1)\\ A_{22} &\equiv a_1\phi_2(1) + a_2\phi_2'(1) \end{aligned} \tag{18}$$

$$\begin{aligned} B_1 &\equiv c - a_1\phi_p(0) - a_2\phi_p'(0)\\ B_2 &\equiv d - a_1\phi_p(1) - a_2\phi_p'(1) \end{aligned} \tag{19}$$

These mean that the existence and uniqueness related properties in the C_1 and C_2 constants are determined completely by the character of the equation set in (17). The equation set may or may not have solutions depending on the coefficient matrix composed of the elements A_{11}, A_{12}, A_{21}, and A_{11}, and the right hand side vector whose elements are B_1 and B_2. The solution can exist only when the right hand side vector of the equation set is orthogonal to the left nullspace of the coefficient matrix. Otherwise there is an incompatibility in the set of the equations. This means that the given boundary conditions are incompatible with the ordinary differential equation they accompany. In the case of compatibility there may be two possibilities: (1) The left and right nullspaces of the coefficient matrix are empty. This makes the solutions for C_1 and C_2 and therefore for whole BVP unique; (2) The rank of the coefficient matrix is just 1. This means that the left and right nullspaces are one dimensional and a one parameter arbitrariness appear in the solutions for C_1 and C_2 and therefore for the solution of the BVP. The case where the rank of the coefficient matrix corresponds to the situation where no boundary conditions are imposed.

We will focus on the BVPs where the boundary conditions are compatible to the ODE and produces unique solution. To find the solution numerically, we propose the following structure:

$$y(x) = f(x) = \sum_{k=0}^{n} f_k x^k \tag{20}$$

If we substitute this solution in the equations (13), (14) and (15), we obtain the following equations:

$$\begin{aligned} r(x) = {} & f_0 q(x) + f_1\left[\,q(x)\,x + p(x)\,\right]\\ & + f_2\left[\,q(x)\,x^2 + 2p(x)\,x + 2\,\right]\\ & + f_3\left[\,q(x)\,x^3 + 3p(x)\,x^2 + 6x\,\right]\\ & + \ldots\\ & + f_n\left[\begin{array}{l} q(x)\,x^n + np(x)\,x^{n-1}\\ +n(n-1)\,x^{n-2}\end{array}\right] \end{aligned} \tag{21}$$

$$c = a_1 f_0 + a_2 f_1 \tag{22}$$

$$\begin{aligned} d = {} & b_1 f_0 + (b_1 + b_2)\,f_1\\ & + \sum_{k=2}^{n}(b_1 + k b_2)\,f_k \end{aligned} \tag{23}$$

For the numerical solution we will apply Fluctuationlessness Theorem in the interval $[\,0,1\,]$. We write the following equation for (13) in terms of operators as follows:

$$\mathbf{M}_r^{(n)}\mathbf{e}_1^{(n)} = \left[\begin{array}{c} \mathbf{M}_{f''}^{(n)} + \mathbf{M}_p^{(n)}\mathbf{M}_{f'}^{(n)} \\ +\mathbf{M}_q^{(n)}\mathbf{M}_f^{(n)} \end{array} \right] \mathbf{e}_1^{(n)}, \quad (24)$$

where $\mathbf{M}_f^{(n)}$ stands for the matrix representation of the operator $\widehat{f}$ in the Hilbert Space $\mathcal{H}_n$. By Fluctuationlessness Theorem we know that

$$\mathbf{M}_f^{(n)} \approx f\left(\mathbf{X}^{(n)}\right). \quad (25)$$

The matrix $\mathbf{X}^{(n)}$ is symmetric and its spectral representation can be written as follows:

$$\mathbf{X}^{(n)} = \sum_{i=1}^{n} \xi_i \mathbf{x}_i \mathbf{x}_i^T \quad (26)$$

Here ξ_i is the i–th eigenvalue and $\mathbf{x}_i$ is its eigenvector with unit norm. Substituting (25) in (24) we obtain the following result.

$$\sum_{i=1}^{n} \left[\begin{array}{c} f''(\xi_i) + p(\xi_i) f'(\xi_i) \\ +q(\xi_i) f(\xi_i) - r(\xi_i) \end{array} \right] \times \left(\mathbf{x}_i^T \mathbf{e}_1^{(n)}\right) \mathbf{x}_i = 0 \quad (27)$$

Since the eigenvectors are linearly independent, this can be satisfied only when the coefficients of $\mathbf{x}_i$s at the left hand side are set equal to zero. So we can write the following equations:

$$f''(\xi_i) + p(\xi_i) f'(\xi_i) + q(\xi_i) f(\xi_i) - r(\xi_i) = 0 \quad (28)$$

To find the unknown constants, f_is, $(2 \leq i \leq n)$ in (21) we construct a set of vectors and matrices as follows

$$\begin{aligned}
[\mathbf{K}_1]_{ij} &\equiv \begin{cases} q(\xi_{i+1})\xi_{i+1}, & i = j \\ 0, & i \neq j \end{cases}, \\
[\mathbf{K}_2]_{ij} &\equiv \begin{cases} p(\xi_{i+1}), & i = j \\ 0, & i \neq j \end{cases}, \\
[\mathbf{K}_3]_{ij} &\equiv \begin{cases} i+1, & i = j \\ 0, & i \neq j \end{cases}, \\
[\mathbf{K}_4]_{ij} &\equiv \begin{cases} i(i+1), & i = j \\ 0, & i \neq j \end{cases}, \\
\mathbf{V}_1 &\equiv \begin{bmatrix} \xi_2 & \xi_2^2 & \dots & \xi_2^{n-1} \\ \xi_3 & \xi_3^2 & \dots & \xi_3^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_n & \xi_n^2 & \dots & \xi_n^{n-1} \end{bmatrix}, \\
\mathbf{V}_2 &\equiv \begin{bmatrix} 1 & \xi_2 & \xi_2^2 & \dots & \xi_2^{n-2} \\ 1 & \xi_3 & \xi_3^2 & \dots & \xi_3^{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \xi_n & \xi_n^2 & \dots & \xi_n^{n-2} \end{bmatrix}, \\
a_i &\equiv \left[\begin{array}{l} f_0 \cdot q(\xi_{i+1}) + \\ f_1 \cdot (p(\xi_{i+1}) + q(\xi_{i+1})\xi_{i+1}) \end{array} \right], \\
\mathbf{r}^T &\equiv \left[\begin{array}{ccc} r(\xi_2) & \dots & r(\xi_n) \end{array} \right], \\
\mathbf{f}^T &\equiv \left[\begin{array}{ccc} f_2 & \dots & f_n \end{array} \right],
\end{aligned} \quad (29)$$

where the matrices $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$, $\mathbf{K}_4$, and the vector $\mathbf{a}$ are given through elementwise definitions.

We can write the equation (21) in terms of these matrices and vectors as follows:

$$\mathbf{r} = (\mathbf{K}_1\mathbf{V}_1 + \mathbf{K}_2\mathbf{V}_1\mathbf{K}_3 + \mathbf{V}_2\mathbf{K}_4)\,\mathbf{f} + \mathbf{a} \quad (30)$$

Since the eigenvalues are distinct, the matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ are invertible. Therefore we can find $\mathbf{f}$ by using the following formula:

$$\mathbf{f} = (\mathbf{K}_1\mathbf{V}_1 + \mathbf{K}_2\mathbf{V}_1\mathbf{K}_3 + \mathbf{V}_2\mathbf{K}_4)^{-1}(\mathbf{r} - \mathbf{a}) \quad (31)$$

When we substitute the values of $\mathbf{f}$ in the equation (16) we obtain the following equation.

$$f(x) = f_0 + f_1 x + \sum_{k=2}^{n} f_k x^k \quad (32)$$

Here all the values except f_0 and f_1 are known. To find f_0 and f_1, the equations (22), (23) and (32) can be solved together. Hence we obtain the numerical solution. By comparing the coefficients of this solution with the Maclaurin coefficients of the analytical solution, we can see the quality of the approximation.

4 Numerical Implementation

In this section we are going to give certain numerical examples. For the implementations some linear second order Boundary Value Problems are chosen, their exact and numerical solutions are presented in figures comparatively. The mesh points we used for calculations are chosen as the eigenvalues of the independent variable's matrix representation $\mathbf{X}^{(n)}$. As can be shown by using Rayleigh ratios and integrals in the definitions of the elements of this matrix, these eigenvalues lie inside the interval $[0, 1]$. We used Mathematica 5.2 for calculations. We obtain the results on 10 grid points for the solutions of the differential equations. Our first problem is defined as follows:

$$y''(x) = 6x, \quad y(0) = 0, \quad y(1) = 1. \quad (33)$$

The analytical solution for this equation is easily found to be $y(x) = x^3$. The coefficients of the numerical solution are given as

$$\begin{aligned} f(x) &= 0.0x - 2.07 \times 10^{-16}x^2 + 1.0x^3 \\ &\quad +1.0 \times 10^{-16}x^4 + 8.5 \times 10^{-17}x^5. \end{aligned} \quad (34)$$

Since the analytical solution is a polynomial, its Maclaurin series equals to itself. Hence we can observe that a high quality approximation, an almost exact match is achieved successfully. A comparative figure of the solutions is displayed below:

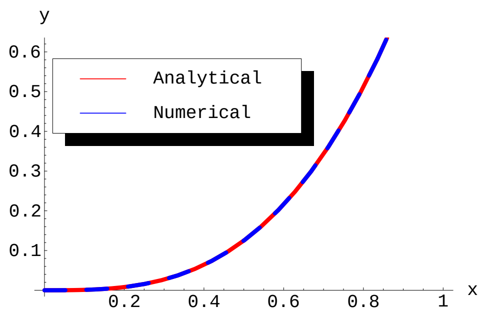

Figure 1: $y(x) = x^3$

The second problem we consider here is given as

$$\begin{aligned} y''(x) - \pi^2 y(x) = 0, \\ y(0) = 1, \quad y(1) = 0. \end{aligned} \tag{35}$$

The analytical solution for this equation is

$$y(x) = \frac{e^{\pi(2-x)} - e^{\pi x}}{e^{2\pi} - 1}. \tag{36}$$

Numerical solution obtained by the Fluctuationlessness Theorem is found to be as follows:

$$\begin{aligned} f(x) &= 1 - 3.1533481x + 4.9348015x^2 \\ &\quad -5.18702x^3 + 4.0583x^4 \\ &\quad -2.557x^5 + 1.32x^6 - 0.58x^7 \\ &\quad +0.20x^8 - 0.05x^9 + 0.006x^{10} \end{aligned} \tag{37}$$

Maclaurin coefficients of the analytical solution are given through the following equality:

$$\begin{aligned} y(x) &= 1 - 3.15334809x + 4.934802x^2 \\ &\quad -5.18704x^3 + 4.0587x^4 - 2.559x^5 \\ &\quad +1.33x^6 - 0.60x^7 + 0.23x^8 \\ &\quad -0.08x^9 + 0.02x^{10} + O\left(x^{11}\right) \end{aligned} \tag{38}$$

The analytic and approximate solutions are depicted together in Figure 2.

The last example we study in this section is given via the following equation

$$\begin{aligned} y''(x) + 2y(x) = 0, \\ y(0) = 1, \quad y(1) + y'(1) = 0. \end{aligned} \tag{39}$$

The analytical solution of this problem is

$$y(x) = \frac{\sin\left(\sqrt{2}x\right)}{\sin\left(\sqrt{2}\right) + \sqrt{2}\cos\left(\sqrt{2}\right)} - \frac{x}{2} \tag{40}$$

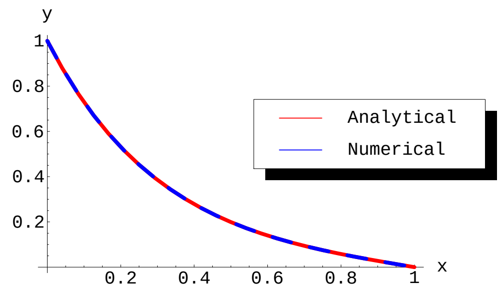

Figure 2: $y(x) = \frac{e^{\pi(2-x)} - e^{\pi x}}{e^{2\pi} - 1}$

The numerical solution is obtained as

$$\begin{aligned} f(x) &= 0.670412404x - 2.3 \times 10^{-10}x^2 \\ &\quad -0.39013745x^3 - 1.12 \times 10^{-7}x^4 \\ &\quad +0.039014x^5 - 2.8 \times 10^{-6}x^6 \\ &\quad -0.001851x^7 - 9.6 \times 10^{-6}x^8 \\ &\quad +6 \times 10^{-6}x^9 - 4.2 \times 10^{-6}x^{10}. \end{aligned} \tag{41}$$

Maclaurin coefficients of the analytical solution can be taken from the coefficients of the following equality

$$\begin{aligned} y(x) &= 0.670412405x - 0.390137468x^3 \\ &\quad +0.039013x^5 - 0.001857x^7 \\ &\quad +5.1 \times 10^{-6}x^9 + O\left(x^{11}\right). \end{aligned} \tag{42}$$

The comparison of the exact and approximate solutions can be seen in Figure 3.

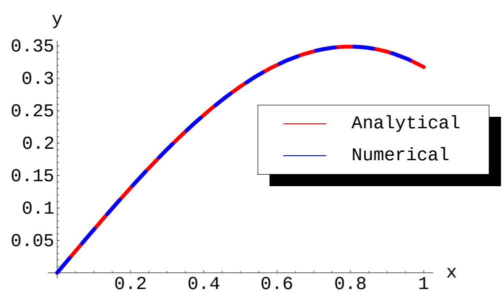

Figure 3: $y(x) = \frac{\sin(\sqrt{2}x)}{\sin(\sqrt{2}) + \sqrt{2}\cos(\sqrt{2})} - \frac{x}{2}$

5 Conclusion

In this work, we developed a new method based on the Fluctuationlessness Theorem for approximating the solution of ODE Boundary Value Problems over the interval $[0,1]$ numerically. The coefficients in the numerical solution are obtained very close to the

Maclaurin coefficients of the exact solution. The results obtained at the grid points have the precision about 10^{-9}. The method described here is a new study for the Boundary Value Problems of ordinary differential equations. For the future work our goal is to solve numerically Boundary Value Problems containing an unknown parameter and are in fact eigenvalue problems.

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