Numerical Integration of Bivariate Functions over a Non Rectangular Area by Using Fluctuationlessness Theorem

ERCAN GÜRVİT
Marmara University
Department of Mathematics
Göztepe, 34722, İstanbul
TÜRKİYE (Turkey)
ercan@be.itu.edu.tr

N.A. BAYKARA
Marmara University
Department of Mathematics
Göztepe, 34722, İstanbul
TÜRKİYE (Turkey)
baki@be.itu.edu.tr

METİN DEMİRALP
İstanbul Technical University
Informatics Institute
Maslak, 34469, İstanbul
TÜRKİYE (Turkey)
demiralp@be.itu.edu.tr

Abstract: "Fluctuation free matrix representation approximation Method" developed by M. Demiralp can be used in approximating the multiple remainder terms of the integral of the Multivariate Taylor expansion. This provides us with a new numerical integration method for multivariate functions. However in this paper instead of dealing with a single formula which takes care of the multiple remainder terms, a new approach is undertaken. At every step of a multivariate integration only one variable is taken care of. Thus an iterative procedure which speeds up the computation rate is obtained.

Key–Words: Multivariate Functions, Fluctuationlessness Theorem, Numerical Integration, Explicit Remainder Term, Taylor Expansion

1 Introduction

In the first part of this work we give a brief explanation of the Fluctuationlessness Theorem [1–7], then we make the necessary generalizations on the formal structure of the Taylor expansion and define a weight function and a set of orthonormal polynomials. Finally by making use of all ingredients we form the first approximation function for the inner integral. Repeating the same procedure the outer integral is approximated. Then certain numerical results obtained by the experiments are tabulated. At the final stage remarks concerning the generalization of the method from bivariate to multivariate functions are made.

2 Fluctuationlessness Theorem

The Fluctuationlessness approximation is based on a theorem which was conjectured and proven by M. Demiralp. This theorem states that the matrix representation of an algebraic operator which multiplies its argument by a scalar univariate function, is identical to the image of the independant variable's matrix representation over the same subspace via the same basis set, under that univariate function, when the fluctuation terms are ignored.

$\mathcal{M}_f$ standing for the matrix representation of the function f, we can write down the following approximation

$$\mathcal{M}_f = (u, fu^T) \approx (u, tu^T) \tag{1}$$

The function $f = f(t)$ is defined over the interval $[a, b]$ including $t = 0$. In case where 0 is not included a point inside the interval can be used as the expansion point. $u_i(t)$'s constitute orthogonal basis functions of the Hilbert space from which our functions are chosen. We define $u = (u_1(t), \ldots, u_n(t))$. The inner product of two functions under a weight function $w(t)$ is defined as follows

$$(g, h) = \int_a^b dt w(t) g(t) h(t) \tag{2}$$

Now, if we expand f as

$$f(t) = \sum_{i=0}^{\infty} f_i t^i \tag{3}$$

and replace this expression in 1 we obtain

$$\mathcal{M}_f = \left(u, \sum_{i=0}^{\infty} f_i t^i u^T \right) \tag{4}$$

This can than be written as

$$\mathcal{M}_f = \sum_{i=1}^{\infty} f_i (u, t^i u^T) \tag{5}$$

Here let us take two operators from Hilbert space, namely $\widehat{\mathcal{L}_1}$ and $\widehat{\mathcal{L}_2}$, and write the following inner product

$$(u, \widehat{\mathcal{L}_1}\widehat{\mathcal{L}_2} u^T) = (u, \widehat{\mathcal{L}_1}[\widehat{P}^{(n)} + (\widehat{I} - \widehat{P}^{(n)})]\widehat{\mathcal{L}_2} u^T) \tag{6}$$

The following definitions being made

$$\widehat{I}g = \sum_{i=1}^{\infty}(u_i g)u_i$$
$$\widehat{P}^{(n)}g = \sum_{i=1}^{n}(u_i g)u_i \quad (7)$$

we can rewrite 6 as

$$\begin{aligned}(u,\widehat{\mathcal{L}_1}\widehat{\mathcal{L}_2}u^T) &= (u,\widehat{\mathcal{L}_1}\widehat{P}^{(n)}\widehat{\mathcal{L}_2}u^T)\\ &+ (u,\widehat{\mathcal{L}_1}(\widehat{I}-\widehat{P}^{(n)})]\widehat{\mathcal{L}_2}u^T) \quad (8)\end{aligned}$$

It's clear that when n tends to infinity $\widehat{P}^{(n)}$ tends to $\widehat{I}$, from which we deduce that $\widehat{I}-\widehat{P}^{(n)}$ tends to 0. In this situation we are left with the fluctuationlessness term only, which allows us to write the approximation below

$$\begin{aligned}(u,\widehat{\mathcal{L}_1}\widehat{\mathcal{L}_2}u^T) &\approx (u,\widehat{\mathcal{L}_1}\widehat{P}^{(n)}\widehat{\mathcal{L}_2}u^T)\\ &= \left(u,\widehat{\mathcal{L}_1}\sum_{i=1}^{n}(u_i,\widehat{\mathcal{L}_2}u^T)u_i\right)\\ &= \sum_{i=1}^{n}\left(u,\widehat{\mathcal{L}_1}(u_i,\widehat{\mathcal{L}_2}u^T)u_i\right)\\ &= \sum_{i=1}^{n}(u,\widehat{\mathcal{L}_1}u_i)(u_i,\widehat{\mathcal{L}_2}u^T)\\ &= (u,\widehat{\mathcal{L}_1}u^T)(u,\widehat{\mathcal{L}_2}u^T) \quad (9)\end{aligned}$$

Thus the approximation can also be written as

$$(u,\widehat{\mathcal{L}_1}\widehat{\mathcal{L}_2}u^T) \approx (u,\widehat{\mathcal{L}_1}u^T)(u,\widehat{\mathcal{L}_2}u^T) \quad (10)$$

Generalizing this we obtain the expression

$$(u,\widehat{\mathcal{L}_1}\dots\widehat{\mathcal{L}_N}u^T) \approx (u,\widehat{\mathcal{L}_1}u^T)\dots(u,\widehat{\mathcal{L}_N}u^T) \quad (11)$$

Going back to $\widehat{t}$

$$\begin{aligned}\mathcal{M}_f &= \sum_{i=0}^{\infty}f_i(u,\widehat{t}^i u^T)\\ &\approx \sum_{i=0}^{\infty}f_i(u,\widehat{t}u^T)^i\\ &= f(u,\widehat{t}u^T) \quad (12)\end{aligned}$$

The argument being the matrix representation of the variable t, we can write the above approximation as

$$\mathcal{M}_f \approx f(\mathbf{T}) \quad (13)$$

where $\mathbf{T}$ is an $n\times n$ symmetric matrix.

3 The Method

In search of a numerical approximation of the definite double integral over a non-rectangular area we start with the following special integral over a rectangular geometry

$$I = \int_{a_1}^{b_1}dx_1\int_{a_2}^{b_2}dx_2 f(x_1,x_2) \quad (14)$$

Taking first into consideration the inner one

$$\begin{aligned}I_2 &= \int_{a_2}^{b_2}dx_2 f(x_1,x_2) = f(x_1,a_2)(b_2-a_2)\\ &+ \int_{a_2}^{b_2}dx_2(b_2-x_2)\frac{\partial}{\partial x_2}f(x_1,x_2) \quad (15)\end{aligned}$$

Generalizing this structure we obtain

$$\begin{aligned}I_2 &= \int_{a_2}^{b_2}dx_2 f(x_1,x_2)\\ &= \sum_{i=0}^{k_1}\frac{(b_2-a_2)^{i+1}}{(i+1)!}\frac{\partial^i}{\partial x_2^i}f(x_1,a_2)\\ &+ \int_{a_2}^{b_2}dx_2\frac{(b_2-x_2)^{k_1+1}}{(k_1+1)!}\frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}}f(x_1,x_2)\end{aligned} \quad (16)$$

We change the integration interval to $[0,1]$ by making a change of variable

$$\begin{aligned}\int_{a_2}^{b_2}dx_2 f(x_1,x_2) &= \sum_{i=0}^{k_1}\frac{(b_2-a_2)^{i+1}}{(i+1)!}\frac{\partial^i}{\partial x_2^i}f(x_1,a_2)\\ &+ \frac{(b_2-a_2)^{k_1+2}}{(k_1+2)!}\\ &\times \int_0^1 dt(k_1+2)(1-t)^{k_1+1}\\ &\times \frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}}f(x_1,(b_2-a_2)t+a_2)\end{aligned} \quad (17)$$

We replace the variable t by its operator counterpart $\widehat{t}$ and the definite integral by its inner product representation

$$\begin{aligned}\int_{a_2}^{b_2}dx_2 f(x_1,x_2) &= \sum_{i=0}^{k_1}\frac{(b_2-a_2)^{i+1}}{(i+1)!}\frac{\partial^i}{\partial x_2^i}f(x_1,a_2)\\ &+ \frac{(b_2-a_2)^{k_1+2}}{(k_1+2)!}\end{aligned}$$

$$\times(u_1, \frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}} f(x_1, (b_2-a_2)\widehat{t}+a_2)u_1) \qquad (18)$$

where

$$w_k(t) \equiv (k_1+2)(1-t)^{k_1+1}, \qquad k_1 = 1, 2, ... \quad (19)$$

is a weight function.

To utilize the Fluctuationlessness Theorem in approximating the integral, we need to construct a basis set spanning the Hilbert space under consideration. The elements of this basis set that we call $u_m(t)$ can be choosen as

$$u_m(t) = \frac{\sqrt{2m+k_1}}{\sqrt{k_1+2}} P_{m-1}^{(k_1+1,0)}(t), \qquad m = 1, 2, ... \quad (20)$$

with $P_{m-1}^{(k_1+1,0)}(t)$ standing for the relevant Jacobi polynomials. We can now write an approximation for the integral over the unit interval

$$\int_0^1 dt w_{k_1+1}(t) \frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}} f(x_1, (b_2-a_2)t+a_2)$$
$$\approx \mathbf{e}_1^T \frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}} f(x_1, (b_2-a_2)\mathbf{T}^{(n_1)} + a_2\mathbf{I}_{n_1})\mathbf{e}_1 \quad (21)$$

where $\mathbf{T}^{(n)}$ stands for the $(n \times n)$ matrix representation of t and $\mathbf{I}$ represents the *n*-dimensional unit matrix. $\mathbf{e}_1$ is the *n*-th dimensional unit vector with the only non-zero component being the first one. Now we embed it in its place

$$\int_{a_2}^{b_2} dx_2 f(x_1, x_2) \approx \sum_{i=0}^{k_1} \frac{(b_2-a_2)^{i+1}}{(i+1)!} \frac{\partial^i}{\partial x_2^i} f(x_1, a_2) +$$
$$\frac{(b_2-a_2)^{k_1+2}}{(k_1+2)!} \mathbf{e}_1^T \frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}} f(x_1, (b_2-a_2)\mathbf{T}^{(n_1)}$$
$$+a_2\mathbf{I}_{n_1})\mathbf{e}_1 \qquad (22)$$

For representation purposes we name the right-hand side of 22

$$\phi_{k_1,n_1}(a_2, b_2) = \sum_{i=0}^{k_1} \frac{(b_2-a_2)^{i+1}}{(i+1)!} \frac{\partial^i}{\partial x_2^i} f(x_1, a_2)$$
$$+\frac{(b_2-a_2)^{k_1+2}}{(k_1+2)!} \mathbf{e}_1^T \frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}} f(x_1, (b_2-a_2)\mathbf{T}^{(n_1)}$$
$$+a_2\mathbf{I}_{n_1})\mathbf{e}_1 \quad (23)$$

Thus, we can write the approximation in a simpler form.

$$\int_{a_2}^{b_2} dx_2 f(x_1, x_2) = \phi_{k_1,n_1}(a_2, b_2),$$
$$k_1 = 2, 3, ..., \qquad n_1 = 1, 2, ... \qquad (24)$$

Now, in order to obtain a scalar equivalent of the expression above we will proceed with the eigenvalues and the eigenvectors of the $\mathbf{T}^{(n_1)}$ matrix.

$$\mathbf{T}^{(n_1)}\mathbf{t}_i = \tau_i \mathbf{t}_i, \qquad i = 1, 2, ..., n_1 \qquad (25)$$

Here none of the eigenvalues is multiple and the eigenvectors are normalized in the Frobenius sense. By the continuation we write down the spectral decomposition of $\mathbf{T}^{(n_1)}$

$$\mathbf{T}^{(n_1)} = \sum_{i=1}^{n_1} \tau_i \mathbf{t}_i \mathbf{t}_i^T \qquad (26)$$

Consequently

$$\frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}} f(x_1, (b_2-a_2)\mathbf{T}^{(n_1)} + a_2\mathbf{I}_{n_1})$$
$$= \sum_{i=1}^{n_1} \frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}} f(x_1, (b_2-a_2)\tau_i + a_2)\mathbf{t}_i\mathbf{t}_i^T \quad (27)$$

Finally we obtain a function of a single variable.

$$\begin{aligned} g(x_1) &= \phi_{k_1,n_1}(a_2, b_2) \\ &= \sum_{i=0}^{k_1} \frac{(b_2-a_2)^{i+1}}{(i+1)!} \frac{\partial^i}{\partial x_2^i} f(x_1, a_2) \\ &+ \frac{(b_2-a_2)^{k_1+2}}{(k_1+2)!} \sum_{i=1}^{n_1} \frac{\partial^{k_1+1}}{\partial x_2^{k_1+1}} f(x_1, \\ &\quad (b_2-a_2)\tau_i + a_2) \left(\mathbf{e}_1^T \mathbf{t}_i\right)^2, \\ &\quad k_1 = 2, 3, ..., \qquad n_1 = 1, 2, ... \quad (28) \end{aligned}$$

Now that we have reduced the inner integral into a function of single variable, we can proceed as in the inner integral and integrate the outer one. But we replace x_1 by x for simplicity.

$$\begin{aligned} \int_{a_1}^{b_1} dx g(x) &= g(a_1)(b_1-a_1) \\ &+ \int_{a_1}^{b_1} dx (b_1-x) g^{(1)}(x) \qquad (29) \end{aligned}$$

Then we generalize

$$\begin{aligned} \int_{a_1}^{b_1} dx g(x) &= \sum_{i=0}^{k_2} \frac{(b_1-a_1)^{i+1}}{(i+1)!} g^{(i)}(a_1) \\ &+ \int_{a_1}^{b_1} dx \frac{(b_1-x)^{k_2+1}}{(k_2+1)!} g^{(k_2+1)}(x) \qquad (30) \end{aligned}$$

and make the necessary change of variable to obtain the new integration limits.

$$\begin{aligned}\int_{a_1}^{b_1} dx g(x) &= \sum_{i=0}^{k_2} \frac{(b_1-a_1)^{i+1}}{(i+1)!} g^{(i)}(a_1) \\ &+ \frac{(b_1-a_1)^{k_2+2}}{(k_2+2)!} \\ &\times \int_0^1 dt (k_2+2)(1-t)^{k_2+1} \\ &\times g^{(k_2+1)}((b_1-a_1)t+a_1) \end{aligned} \quad (31)$$

Proceeding in the same manner as in 18 we can write

$$\begin{aligned}\int_{a_1}^{b_1} dx g(x) &= \sum_{i=0}^{k_2} \frac{(b_1-a_1)^{i+1}}{(i+1)!} g^{(i)}(a_1) \\ &+ \frac{(b_1-a_1)^{k_2+2}}{(k_2+2)!} \\ &\times \left(v_1, g^{(k_2+1)}((b_1-a_1)\widehat{t}+a_1)v_1\right) \end{aligned} \quad (32)$$

Here again our weight function is

$$\overline{w}_{k_2+1}(t) \equiv (k_2+2)(1-t)^{k_2+1}, \qquad k_2 = 1,2,... \quad (33)$$

and the orthonormal basis functions are

$$v_m(t) = \frac{\sqrt{2m+k_2}}{\sqrt{k_2+2}} P_{m-1}^{(k_2+1,0)}(t), \qquad m = 1,2,... \quad (34)$$

with $P_{m-1}^{(k_2+1,0)}(t)$ standing for the Jacobi polynomials. We can again write an approximation to the integral over the unit interval

$$\begin{aligned}&\int_0^1 dt \overline{w}_{k_2+1}(t) g^{(k_2+1)}((b_1-a_1)t+a_1) \\ &\approx \mathbf{e}_1^T g^{(k_2+1)}((b_1-a_1)\overline{\mathbf{T}}^{(n_2)} + a_1 \mathbf{I}_{n_2})\mathbf{e}_1 \end{aligned} \quad (35)$$

and obtain the approximation

$$\begin{aligned}\int_{a_1}^{b_1} dx g(x) &\approx \sum_{i=0}^{k_2} \frac{(b_1-a_1)^{i+1}}{(i+1)!} g^{(i)}(a_1) \\ &+ \frac{(b_1-a_1)^{k_2+2}}{(k_2+2)!} \\ &\times \mathbf{e}_1^T g^{(k_2+1)}((b_1-a_1)\overline{\mathbf{T}}^{(n_2)} + a_1 \mathbf{I}_{n_2})\mathbf{e}_1 \end{aligned} \quad (36)$$

We name the right-hand side of the last approximation for the ease of notation

$$\begin{aligned}\theta_{k_2,n_2}(a_1,a_2) &= \sum_{i=0}^{k_2} \frac{(b_1-a_1)^{i+1}}{(i+1)!} g^{(i)}(a_1) \\ &+ \frac{(b_1-a_1)^{k_2+2}}{(k_2+2)!} \\ &\times \mathbf{e}_1^T g^{(k_2+1)}((b_1-a_1)\overline{\mathbf{T}}^{(n_2)} \\ &+ a_1 \mathbf{I}_{n_2})\mathbf{e}_1 \end{aligned} \quad (37)$$

and the approximation can simply be written as follows

$$\begin{aligned}&\int_{a_1}^{b_1} dx g(x) = \theta_{k_2,n_2}(a_1,b_1), \\ &k_2 = 2,3,..., \qquad n_2 = 1,2,... \end{aligned} \quad (38)$$

Finally, to get rid of the matrix entities, we start by calculating the eigenvalues and the eigenvectors of the $\mathbf{T}^{(n_2)}$ matrix

$$\overline{\mathbf{T}}^{(n_2)} \overline{\mathbf{t}}_i = \xi_i \overline{\mathbf{t}}_i, \qquad i = 1,2,...,n_2 \quad (39)$$

where again none of the eigenvalues is multiple and the eigenvectors are normalized in the Frobenius sense and then by continuation we write down the spectral decomposition of $\mathbf{T}^{(n_2)}$

$$\overline{\mathbf{T}}^{(n_2)} = \sum_{i=1}^{n_2} \xi_i \overline{\mathbf{t}}_i \overline{\mathbf{t}}_i^T \quad (40)$$

Hence

$$\begin{aligned}&g^{(k_2+1)}((b_1-a_1)\overline{\mathbf{T}}^{(n_2)} + a_1 \mathbf{I}_{n_2}) \\ &= \sum_{i=1}^{n_2} g^{(k_2+1)}((b_1-a_1)\xi_i + a_1) \overline{\mathbf{t}}_i \overline{\mathbf{t}}_i^T \end{aligned} \quad (41)$$

Therefore we get the final form of the approximation to the bivariate integration over a non-rectangular area.

$$\begin{aligned}\int_{a_1}^{b_1} dx g(x) &\approx \sum_{i=0}^{k_2} \frac{(b_2-a_2)^{i+1}}{(i+1)!} \frac{\partial^i}{\partial x_2^i} f(x_1,a_2) \\ &+ \frac{(b_2-a_2)^{k_2+2}}{(k_2+2)!} \\ &\times \sum_{i=1}^{n_2} \frac{\partial^{k_2+1}}{\partial x_2^{k_2+1}} f(x_1,(b_2-a_2)\tau_i+a_2) \\ &\times \left(\mathbf{e}_1^T \mathbf{t}_i\right)^2 \\ &, \quad (k_2 = 2,3,..., n_2 = 1,2,...) \end{aligned} \quad (42)$$

4 An Easy Calculation of $\mathbf{T}^{(n)}$

Let us observe the structure of the matrix representation of $\widehat{t}$. In order to achieve this, we make use of the three term recursion formula for Jacobi polynomials $P_m^{(k+1,0)}(t)$ and obtain the following three term recursion between the orthonormal basis functions $u_m(t)$

$$\begin{aligned} tu_m(t) &= \alpha_m u_{m+1} + \beta_m u_m(t) + \alpha_{m-1} u_{m-1}(t), \\ & \quad m = 1, 2, 3, ... \end{aligned} \tag{43}$$

where

$$\begin{aligned} \alpha_m &\equiv \frac{m(m+k+1)}{2m+k+1} \frac{1}{\sqrt{2m+k}} \frac{1}{\sqrt{2m+k+2}} \\ \beta_m &\equiv \frac{2m(m-1)+(k+1)(2m-1)}{(2m+k+1)(2m+k-1)} \end{aligned} \tag{44}$$

Now, working on $\mathbf{T} = (u_i, tu_j)$ we can easily find out that $\mathbf{T}$ matrix can be written in terms of α_m's and β_m's as follows

$$\mathbf{T} = \begin{bmatrix} \beta_1 & \alpha_1 & 0 & ... \\ \alpha_1 & \beta_2 & \alpha_2 & ... \\ 0 & \alpha_2 & \beta_3 & ... \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{45}$$

$\mathbf{T}$ is in fact an infinite matrix since the operator $\widehat{t}$ is defined on an infinite Hilbert space. However, by definition we have to use its finite *n*-dimensional truncation which we symbolize as $\mathbf{T}^{(n)}$. The (i, j)-th component of $\mathbf{T}^{(n)}$ can be calculated by $T_{i,j}^{(n)} = \mathbf{e}_i^T \mathbf{T}^{(n)} \mathbf{e}_j$. This yields the following

$$\begin{aligned} T_{i,j}^{(n)} &= \beta_i \delta_{i,j} + \alpha_i \delta_{i,j-1} + \alpha_{i-1} \delta_{i,j+1}, \\ & \quad 1 \leq i, j \leq n \end{aligned} \tag{46}$$

Having obtained the components of $\mathbf{T}^{(n)}$ We can now easily calculate the eigenvalues and eigenvectors of $\mathbf{T}^{(n)}$ and embed them into our approximation.

5 Experimental Results

In this section we give results from the implementation of the algorithm on a Pentium IV 2.0 Ghz processor and all the calculations were held with Mathematica 7.0.0 [8]. The working precision was 22 and the results are tabulated with an accuracy of 10. The first function is choosen from a textbook example:$f[x, y] = Exp[-(x-2)^2/4 - (y-3)^2/4]Cos[2x + y - 7]$

Three methods are used in integrating this function on a rectangular area: the first one is our actual iterative algorithm, the second is the symbolically held iterative integration by Mathematica which in fact gives us the exact result to be compared with, and the third one is the built-in numerical integration iteratively held by Mathematica [9]. Lower integration limits are $x = 2, y = 3$. The results can be seen in Table 1 together with the timing (in seconds) at the right of each column showing values.

As it can easily be seen the timing for our algorithm is much better than the same iteration done symbolically. But in fact a more relevant comparison is the one made with the built-in Mathematica algorithm for numerical integration which gives us about two fold better results in timing for the same level of accuracy.

The second function is choosen to be:$f[x, y] = 2y.Sin[x] + Cos[x]^2$

Three methods are used in integrating this function on a non-rectangular area: the first one is our actual iterative algorithm, the second is the symbolically held iterative integration by Mathematica which in fact gives us the exact result to be compared with, and the third one is the built-in numerical integration iteratively held by Mathematica. Lower integration limits are $x = 0, y = Sin[x]$. The upper limit of integration of y is $y = Cos[x]$. The results can be seen in Table 2 together with the timing at the left of each column.

This example which involves a non-constant integration limit in one variable emphasizes the power of the method. It is obvious that the present method provides a serious gain in time when applied to an integration over a non-rectangular area.

As an important remark we have to emphasize that the n values which, in our two examples, are chosen to start with relatively lower values and step up with a value of 5, should in practice be chosen much more systematically by watching the change in the integration results and comparing them with a predetermined tolerance value.

6 Conclusion

It would be helpful to note that this iterative algorithm is a generalization of the definite bivariate integration over a rectangular area. This can also be generalized to a broader range of integrals, namely multivariate integrals. We can repeat the same iteration as many times as needed. Another computational advantage is that in this method the nested sums do not exist and hence they are additive and not multiplicative, which in fact decreases immensely the overall computation time.

n	Upper Limits	Exact Value	Time	Present Method	Time	Built-in Method	Time
10	$x = 3, y = 4$	0.0931409270	9.671	0.0931409270	0.235	0.0931409270	0.297
15	$x = 4, y = 5$	-0.7447287576	7.625	-0.7447287576	0.469	-0.7447287576	0.812
20	$x = 5, y = 6$	-0.6371152492	8.500	-0.6371152492	0.781	-0.6371152492	1.391
25	$x = 6, y = 7$	-0.6240687828	7.313	-0.6240687828	1.296	-0.6240687828	2.016

Table 1: *Comparison of exact integration values of* $f[x, y] := Exp[-(x-2)^2/4 - (y-3)^2/4]Cos[2x + y - 7]$ *with the values obtained from present work results and the built-in numerical integration method applied by Mathematica 7.00* ($k = 5$)

n	Upper Limits	Exact Value	Time	Present Method	Time	Built-in Method	Time
5	$x = \Pi/8$	0.3583552320	2.781	0.3583552320	0.000	0.3583552320	0.297
10	$x = \Pi/7$	0.3971733701	3.485	0.3971733701	0.031	0.3971733701	0.265
15	$x = \Pi/6$	0.4411857195	0.484	0.4411857195	0.031	0.4411857195	0.266
20	$x = \Pi/5$	0.4859411442	1.875	0.4859411442	0.046	0.4859411442	0.297
25	$x = \Pi/4$	0.5118446353	0.406	0.5118446353	0.047	0.5118446353	0.250

Table 2: *Comparison of exact integration values of* $f[x, y] := 2y.Sin[x] + Cos[x]^2$ *with the values obtained from present work results and the built-in numerical integration method applied by Mathematica 7.00* ($k = 5$)

Acknowledgements: The third author is grateful to Turkish Academy of Sciences for their support.

References:

[1] Ercan Gürvit, N. A. Baykara, and Metin Demiralp, Evaluation of Univariate Integrals via Fluctuationlessness Theorem, AIP Conf. Proc. 1048, 239 (2008), DOI:10.1063/1.2990901

[2] Ercan Gürvit, N. A. Baykara, and Metin Demiralp, Evaluation of Multivariate Integrals via Fluctuationlessness Theorem and Taylor's Remainder, AIP Conf. Proc., ICCMSE (2008), To appear

[3] M. Demiralp, A Fluctuation Expansion Method for the Evaluation of a Function's Expectation Value, Int. Conf. on Numer. Anal. and Appl. Math., Wiley, Rhodes, Greece, Sept. 16 – 20, 2005, pp. 711-714

[4] M. Demiralp, Fluctuation Expansion at the horizon as a New and Efficient Tool for Integration and ODE and PDE Solving, under review (2008).

[5] B. Tunga and M. Demiralp, Fluctuationlessness Approximation Based Multivariate Integration in Hybrid High Dimensional Model Representation, Int. Conf. on Numer. Anal. and Appl. Math., AIP, Kos, Greece, Sept. 15 – 20, 2008, pp. 562-565

[6] İ. Yaman and M. Demiralp, An Integration Free Universal Approximation Method Based on Orthogonal Polynomials, Int. Conf. on Numer. Anal. and Appl. Math., AIP, Kos, Greece, Sept. 15 – 20, 2008, pp. 601-604

[7] S. Üsküplü and M. Demiralp, Univariate Integration via Space Extension Based Fluctuation Approximation, Int. Conf. on Numer. Anal. and Appl. Math., AIP, Kos, Greece, Sept. 15 – 20, 2008, pp. 566-569

[8] A.S. Cvetkovic and G. Milovanovic, The Mathematica Package "OrthogonalPolynomials", FACTA UNIVERSITATIS (NIS), SER. Math. Inform. 19(2004), 17-36

[9] S. Wolfram, The Mathematica Book 5'th Edition, Published by Wolfram Media