

# Applications of High Dimensional Model Representations to Computer Vision

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*Abstract:* A new and powerful method for matrix decomposition is developed in this work. It is similar to singular value decomposition and the main idea comes from the univariate approximation of a function, given on a planar grid's nodes, by two variable high dimensional model representation. The proposed method is less iteration dependent than the singular value decomposition and the components are determined via straightforward steps containing recursions. It seems to have more capabilities than the singular value decomposition as an alternative method. An illustrative application is also given.

*Key-Words:* Singular Value Decomposition, High Dimensional Model Representation, Matrix Decomposition

## 1 Introduction

One of the widely used approaches in computer vision is to consider the image's pixel map and use the singular value decomposition of the corresponding matrix for approximating and interpreting various features. Singular value decomposition constructs a set of basis matrices from a given matrix such that the basis matrices are mutually orthonormal with respect to Frobenius norm inducing inner product which is the trace of the product of its first argument's transpose with the second argument. Basis matrices are linearly combined to give the corresponding original matrix. The linear combination coefficients are the singular values of the related matrix and can take nonnegative values. If we consider the matrix corresponding to an image, and its singular value decomposition, then each basis matrix multiplied by the corresponding singular value can be interpreted as a subimage. The original image is the linear combination of subimages. A careful numerical experimentation on various images shows that many important characteristics of an image can be adequately approximated by linearly combining rather small percentage of the basis matrices with dominant singular values. For example, an image of  $400 \times 500$  pixels can have at most 400 nonzero singular values. If so then, say 40 or 60, subimages coming from the singular value decomposition may suffice for a sufficiently good visualization. Assume that 40 subimages will be used. The original image contains 2000000 pixels. As the worst case, we can assume that the pixels are all independent from each other. Each subimage has also 2000000 pixels however they are produced from an outer product which has 900 independent values. Hence, 40 subimages have a total of

36000 independent values. This means that an entity with 2000000 values can be, approximately but efficiently, characterized by a quite small number (36000) of values. This is very similar to the principal component analysis in statistics where covariance matrix is the focus instead of the pixel map. Truncated singular value decomposition approximation's quality comes from the specific nature of the images. Their basic characteristics are generally contained in the first few basis matrix of singular value decomposition. Numerical experimentations show that the dominance of their singular values in first few components (for example 40) is about ninety or more percents. Truncated singular value decomposition can also be used in data compression.

Singular value decomposition is closely related to matrix eigenvalue problems and requires iteration unless the image under consideration has a very specific structure. Iteration is undesired because it increases the total number of mathematical operations and therefore the method's cost, hence it is preferred to avoid. Here we try to develop a new method to resolve an image to subimages by avoiding iteration as much as possible. To this end we use the experiences gathered from high dimensional model representation (HDMR) of multivariate analysis. HDMR has been developed in last two decades. Although it has matured considerably there are still new extensions and applications to increase the method's power. HDMR, and especially its application on functions given not analytically but by a set of data is the source of inspiration for this work. The data given points are located at the nodes of a rectangular hyperprismatic regular grid without leaving any node empty. We are not go-

ing to use all the theory of HDMR comprehensively. Instead, we will take its univariance properties and conceptually use them in the construction of the new method described here.

Paper is organized as follows. Next section gives the mainlines of the singular value decomposition with a special emphasis on computational cost increasing aspects. The third section covers a sufficiently brief but detailed summary of HDMR. The fourth section is devoted to the presentation of the main conceptual and technical aspects of this work's new method. The fifth section presents an illustrative application of the new method on decomposing and reconstructing an image. The sixth section finalizes the paper via concluding remarks.

## 2 Singular Value Decomposition

Consider an  $m \times n$  type real (for simplicity) matrix  $\mathbf{A}$  which maps from the  $n$  element cartesian vectors' space  $\mathcal{K}_n$  to the  $m$  element cartesian vectors' space  $\mathcal{K}_m$ . If  $m$  would be equal to  $n$  then we would be able to define an eigenvalue problem on  $\mathbf{A}$  since it would transform from one element to the other in the same space. Without confining ourselves to very limited case of  $m = n$  we try to find a way such that certain specific vectors in  $\mathcal{K}_n$  are transformed to themselves by using only  $\mathbf{A}$  and certain related matrices. To this end we can consider the transpose of  $\mathbf{A}$ , denoted by  $\mathbf{A}^T$ , which maps from  $\mathcal{K}_m$  to  $\mathcal{K}_n$ . This urges us to consider  $\mathbf{A}^T\mathbf{A}$  which is an  $n \times n$  matrix and maps from  $\mathcal{K}_n$  to  $\mathcal{K}_n$ . Since  $\mathbf{A}^T\mathbf{A}$  is square we can consider its eigenpairs. Thus, acting  $\mathbf{A}^T\mathbf{A}$  on one of its eigenvectors scales that eigenvector. However this is in fact a composite operation. In this action, first,  $\mathbf{A}$  takes that eigenvector to some other corresponding vector, but not in  $\mathcal{K}_n$ , in  $\mathcal{K}_m$ . Then  $\mathbf{A}^T$  maps the corresponding vector to the vector which is original vector's scaled form in  $\mathcal{K}_n$ . A similar action is realized by  $\mathbf{A}\mathbf{A}^T$ , but this time, between  $\mathcal{K}_m$  and  $\mathcal{K}_m$ . Everything remains valid for this time, with a difference, the eigenvector is now in  $\mathcal{K}_m$ .

All these discussions motivate us to write

$$\begin{aligned} \mathbf{A}\mathbf{u} &= \sigma\mathbf{v}, \\ \mathbf{A}^T\mathbf{v} &= \sigma\mathbf{u}, \quad \mathbf{u} \in \mathcal{K}_n, \quad \mathbf{v} \in \mathcal{K}_m \end{aligned} \quad (1)$$

where  $\sigma$  is an arbitrary constant for this moment. These equations can be put into the following more amenable form

$$\begin{aligned} \mathbf{A}^T\mathbf{A}\mathbf{u} &= \sigma^2\mathbf{u}, \\ \mathbf{A}\mathbf{A}^T\mathbf{v} &= \sigma^2\mathbf{v}, \quad \mathbf{u} \in \mathcal{K}_n, \quad \mathbf{v} \in \mathcal{K}_m \end{aligned} \quad (2)$$

which define the eigenvalue problems of  $\mathbf{A}^T\mathbf{A}$  and  $\mathbf{A}\mathbf{A}^T$ . Since  $\mathbf{A}^T\mathbf{A}$  and  $\mathbf{A}\mathbf{A}^T$  are symmetric and non-negative definite their eigenvalues are also nonnegative and, because of the symmetry, the eigenvectors corresponding to different eigenvalues are mutually orthogonal, separately for each matrix.

Let us focus on the eigenvalues first and assume that  $m < n$ . Since the rank of  $\mathbf{A}$  can be at most  $m$  and the rank of the product of two matrices can never increase, one can prove that the rank of the matrix  $\mathbf{A}^T\mathbf{A}$  can be at most  $m$ . This implies that the matrix  $\mathbf{A}^T\mathbf{A}$  should have a zero eigenvalue whose multiplicity is at least  $(n - m)$ . Therefore the number of nonzero  $\sigma$  values are at most  $m$ . These discussions can be equivalently applied on the case where  $n < m$ .

The second important issue about the eigenvalues is the equivalence between the nonzero eigenvalues of  $\mathbf{A}^T\mathbf{A}$  and  $\mathbf{A}\mathbf{A}^T$ . This can be shown by using the characteristic polynomials of both matrices. To this end one can use the fact that the traces of the same integer powers of these matrices are identical. This means that the characteristic polynomial coefficients starting from the highest power should be same. Therefore the characteristic polynomials should be proportional by a factor of certain power of eigenvalue parameter (power is at least  $(n - m)$  when  $m < n$ ). Since the eigenvalues are the square of  $\sigma$  values there is a sign uncertainty in these values and general tendency is naturally to choose positive values.

If we index the normalized eigenvectors in descending corresponding eigenvalues (this is the general tendency in scientific community) then we can define the following orthogonal matrices

$$\mathbf{Q}_r \equiv [\mathbf{u}_1 \dots \mathbf{u}_n], \quad \mathbf{Q}_\ell \equiv [\mathbf{v}_1 \dots \mathbf{v}_m] \quad (3)$$

which enable us to write

$$\begin{aligned} \mathbf{Q}_r^T\mathbf{A}^T\mathbf{A}\mathbf{Q}_r &= \mathbf{Q}_r^T\mathbf{A}^T\mathbf{Q}_\ell\mathbf{Q}_\ell^T\mathbf{A}\mathbf{Q}_r \\ &= (\mathbf{Q}_\ell^T\mathbf{A}\mathbf{Q}_r)^T (\mathbf{Q}_\ell^T\mathbf{A}\mathbf{Q}_r) \\ &= \mathbf{\Lambda} \end{aligned} \quad (4)$$

where we have used the fact  $\mathbf{Q}_\ell\mathbf{Q}_\ell^T = \mathbf{I}_m$  ( $\mathbf{I}_m$  symbolizes  $m \times m$  unit matrix) coming from orthonormality and  $\mathbf{\Lambda}$  stands for the diagonal matrix whose diagonal elements are  $\mathbf{A}^T\mathbf{A}$ 's eigenvalues sorted in decreasing order. If we define an  $m \times n$  type matrix  $\mathbf{\Sigma}$  which has two horizontally located blocks, first, an  $m \times m$  diagonal block whose elements are the  $\mathbf{A}^T\mathbf{A}$ 's eigenvalues' positive square sorted in decreasing order, and second,  $m \times (n - m)$  type zero matrix then it is not hard to get the following relation

$$\mathbf{\Lambda} = \mathbf{\Sigma}^T\mathbf{\Sigma} \quad (5)$$

whose comparison with (4) reveals

$$\mathbf{Q}_\ell^T \mathbf{A} \mathbf{Q}_r = \Sigma \quad (6)$$

and therefore

$$\mathbf{A} = \mathbf{Q}_\ell \Sigma \mathbf{Q}_r^T \quad (7)$$

where we have used the orthonormalities of the matrices  $\mathbf{Q}_\ell$  and  $\mathbf{Q}_r$  once more. This is the singular value decomposition of the matrix  $\mathbf{A}$ .

As we have seen the singular value decomposition is based on the solution of an eigenvalue problem. The eigenvalues and eigenvectors can only be found by iterative methods unless the target matrix  $\mathbf{A}$  has a specific nature enabling us to work analytically. In this work we try to get rid of iteration as much as possible by constructing a new method.

Although we have given important aspects of the singular value decomposition method for matrices one can look at some papers [1–10] in the scientific literature for further readings.

### 3 High Dimensional Model Representation (HDMR)

HDMR has been under development for multivariate analysis since the 1990s. It is principally based on the divide-and-conquer philosophy and is used to approximate a multivariate function in terms of less variate functions. Its expression is presented below for a given multivariate function denoted by  $f(x_1, \dots, x_N)$

$$\begin{aligned} f(x_1, \dots, x_N) = & f_0 + \sum_{i=1}^N f_i(x_i) \\ & + \sum_{\substack{i_1, i_2=0 \\ i_1 < i_2}} f_{i_1 i_2}(x_{i_1}, x_{i_2}) \\ & + \dots + f_{12\dots N}(x_1, \dots, x_N) \quad (8) \end{aligned}$$

where the independent variables are assumed to be in a given rectangular hyperprism ( $x_i \in [a_i, b_i]$ , ( $i = 1, \dots, N$ )) and the “vanishing under any univariate integration” condition is imposed on the right hand side components except the constant term. That is, the integral of any nonconstant term at the right hand side, over its any independent variable and over the related interval, vanishes. This means orthogonality amongst the right hand side components in an appropriately defined Hilbert space of square integrable multivariate functions with respect to the HDMR’s integration geometry. The integration domain is restricted to orthogonal geometries only, to enable the separation of the multivariate integrations to the consecutive independent univariate integrations. The general

tendency is to take the hyperprismatic geometry mentioned above. However, some other orthogonal geometries like hyperspherical or hyperelliptic regions can also be equivalently used depending on the modelling.

It is also possible to use a weight function for the integration. However, the weight function is also restricted. It should be a product of univariate weight functions each of which depends on a different independent variable. Each univariate weight factor is normalized to have unit integral value, for simplicity. These restrictions on the HDMR geometry and weight function are imposed to get compatibility and uniqueness in the determination of HDMR components. Otherwise, the basic assumption of (8) can never be fulfilled.

The orthogonality conditions suffice to uniquely determine all HDMR components as long as the restrictions on the HDMR’s geometry and weight are fulfilled. Constant term is in fact the weighed average of the given function for the HDMR’s orthogonal geometry (the rectangular hyperprism defined by the cartesian product of the abovementioned intervals in this work). Univariate terms correspond to variances, or in other words, fluctuations in the direction defined by the relevant independent variable in the space of all independent variables. The bivariate terms are related to the correlations between two relevant independent variables. The other components can be interpreted in similar ways.

What we have mentioned above were the multivariate functions which are given in analytic structures. However, in many applications, the target function is not known analytically, instead, its values at certain, generally in finite number, points of the independent variables’ space. Hence, the integrals appearing in the structures of the HDMR components must pick up the given function values only. Although there may be different ways to this end, we use the linear combination of Dirac delta functions in the HDMR weights. Then all integrations turn out to be linear combinations over given function values.

Although it may seem to be rather trivial, the number of HDMR dimensions necessary for computer vision is 2 as long as the pixel map of the image is considered. There the pixel values are considered to be values of a color intensity function and the independent variables are taken as the horizontal and vertical positions of the pixels. This defines a planar rectangular grid, on the nodes of which the color intensities are given. Here we consider the case where a monochromatic image is given. This permits us to use just one information for each pixel position. The most general colored images then can be treated by considering the matrices composed of main color component values

separately as independent images. For example, the red, blue, or green color intensities can be individually considered as independent images. Then the original image can be considered as the overlapping of these images on the same frame.

If we focus on an  $m \times n$  type monochromatic image then we consider two independent variables  $x_1 \in [0, n + 1]$  (horizontal) and  $x_2 \in [0, m + 1]$  (vertical) and the color function  $f(x_1, x_2)$  then the corresponding HDMR can be written as follows

$$f(x_1, x_2) = f_0 + f_1(x_1) + f_2(x_2) + f_{12}(x_1, x_2) \quad (9)$$

where

$$f_0 = \int_0^{n+1} dx_1 W_1(x_1) \int_0^{m+1} dx_2 W_2(x_2) f(x_1, x_2) \quad (10)$$

$$f_1(x_1) = \int_0^{m+1} dx_2 W_2(x_2) f(x_1, x_2) - f_0 \quad (11)$$

$$f_2(x_2) = \int_0^{n+1} dx_1 W_1(x_1) f(x_1, x_2) - f_0 \quad (12)$$

$$f_{12}(x_1, x_2) = f(x_1, x_2) - f_0 - f_1(x_1) - f_2(x_2) \quad (13)$$

where the overall weight function  $W(x_1, x_2)$  is taken as the product of two univariate functions depending on  $x_1$  and  $x_2$  respectively, as it should be to avoid certain incompatibilities in the determination of the HDMR components (these incompatibilities can be removed by defining new type of HDMRs which are out of the scope of this work). Hence we write

$$W(x_1, x_2) \equiv W_1(x_1) W_2(x_2) \quad (14)$$

Since  $x_1$  and  $x_2$  are continuous variables, the discreteness in the function values should be provided by the structures of the integrals where the weights are the only agents to do so. Thus, we define the weights as follows

$$W_1(x_1) \equiv \frac{1}{n} \sum_{i_1=1}^n \delta_D(x_1 - i_1),$$

$$W_2(x_2) \equiv \frac{1}{m} \sum_{i_2=1}^m \delta_D(x_2 - i_2) \quad (15)$$

where  $\delta_D$  stands for Dirac delta function which replaces the function's argument by its support ( $i_1$  and

$i_2$  above) if the support is an interior point in the interval, otherwise the function's argument is replaced with the support and the resulting term is multiplied by half (this explains why we have taken the intervals as above, it was for making all independent variable values interior points of the interval and we could choose some other endpoints for the same purpose but it would not change the results as long as the smallest and/or largest independent variable values remain as interior points). The utilization of the weights given by (15) in (10), (11), (12), (13) produces the following results

$$f_0 = \frac{1}{mn} \sum_{i_1=1}^n \sum_{i_2=1}^m f(i_1, i_2) \quad (16)$$

$$f_1(i_1) = \frac{1}{m} \sum_{i_2=1}^m f(i_1, i_2) - f_0, \quad i_1 = 1, \dots, n \quad (17)$$

$$f_2(i_2) = \frac{1}{n} \sum_{i_1=1}^n f(i_1, i_2) - f_0, \quad i_2 = 1, \dots, m \quad (18)$$

$$f_{12}(i_1, i_2) = f(i_1, i_2) - f_0 - f_1(i_1) - f_2(i_2),$$

$$i_1 = 1, \dots, n, \quad i_2 = 1, \dots, m \quad (19)$$

which can be rewritten in the following matrix algebraic forms

$$f_0 = \frac{1}{mn} \mathbf{w}_\ell^T \mathbf{F} \mathbf{w}_r \quad (20)$$

$$\mathbf{f}_1 = \frac{1}{m} \mathbf{F} \mathbf{w}_r \quad (21)$$

$$\mathbf{f}_2 = \frac{1}{n} \mathbf{F}^T \mathbf{w}_\ell \quad (22)$$

$$\mathbf{F}_{12} = \mathbf{F} - f_0 \mathbf{w}_\ell \mathbf{w}_r^T - \mathbf{f}_1 \mathbf{w}_r^T - \mathbf{w}_\ell \mathbf{f}_2^T \quad (23)$$

where  $\mathbf{w}_\ell$  and  $\mathbf{w}_r$  are vectors with  $m$  and  $n$  elements of 1 value respectively and the element of the  $m \times n$  type matrix  $\mathbf{F}$  at the intersection of  $i_1$ -th row with  $i_2$ -th column is  $f(i_1, i_2)$ . The matrix  $\mathbf{F}_{12}$  corresponds to the bivariate component of the image's HDMR. (23) leads us to write the image's HDMR as the following matrix equality

$$\mathbf{F} = f_0 \mathbf{w}_\ell \mathbf{w}_r^T + \mathbf{f}_1 \mathbf{w}_r^T + \mathbf{w}_\ell \mathbf{f}_2^T + \mathbf{F}_{12} \quad (24)$$

from which we can get the following univariate approximation  $\mathbf{F}_u$  by excluding the bivariate matrix

$$\mathbf{F}_u = \left( \mathbf{f}_1 + \frac{f_0}{2} \mathbf{w}_\ell \right) \mathbf{w}_r^T + \mathbf{w}_\ell \left( \mathbf{f}_2^T + \frac{f_0}{2} \mathbf{w}_r^T \right) \quad (25)$$

This matrix is an approximation to the original image, and typically contains the most dominant information. In other words, its dominance in norm square may climb to ninety percents. However this does not mean that  $\mathbf{F}_u$  creates the display of the original image. Instead, it contains horizontal and vertical lines only. It is in fact the background. The picture details are hidden in small deviations from this matrix, that is, in bivariate terms. Hence, univariate HDMMR does not suffice for computer vision purposes. Bivariate term should be somehow taken into consideration. Same thing is valid also for singular value decomposition. Its first dominant component behaves very similar to HDMMR's univariate portion. However its construction is very complicated in comparison with HDMMR's univariate approximation whose construction is quite straightforward.

Although the information given here for HDMMR suffice for this work's purposes interested reader can find many HDMMR related papers in scientific literature. Some of them [11–28] are included in the reference list of this paper.

## 4 HDMMR Based Matrix Decomposition

Singular value decomposition of an  $m \times n$  ( $m < n$ ) type matrix  $\mathbf{A}$  can be written as a sum from (7) as follows

$$\mathbf{A} = \sum_{i=1}^m \sigma_i \mathbf{u}_i \mathbf{v}_i^T \quad (26)$$

where the outer products  $\mathbf{u}_i \mathbf{v}_j^T$  ( $i = 1, \dots, n; j = 1, \dots, m$ ) are mutually orthonormal with respect to Frobenius matrix norm. Hence they form a set of basis matrices. This urges us to seek a similar matrix decomposition which does not involve this form of one-rank outer products. As we have mentioned in the previous section univariate HDMMR truncation (25) can be a good candidate for the construction of the first basis matrix. Thus, we propose

$$\mathbf{M}_1 \equiv \mathbf{s}_1 \mathbf{w}_1^T + \mathbf{w}_2 \mathbf{s}_2^T \quad (27)$$

where the column vectors  $\mathbf{w}_1$  and  $\mathbf{w}_2$  have  $n$  and  $m$  elements respectively. The elements of  $\mathbf{w}_1$  are all same and have the value  $1/\sqrt{n}$ . Same thing is also valid for

$\mathbf{w}_2$  and the common values of the elements are  $1/\sqrt{m}$  this time. These values set the square norms of  $\mathbf{w}_1$  and  $\mathbf{w}_2$  equal to 1.  $\mathbf{s}_1$  and  $\mathbf{s}_2$  are determined to minimize the following functional

$$\Delta_1(\mathbf{s}_1, \mathbf{s}_2) \equiv \|\mathbf{A} - \mathbf{M}_1\|^2 \quad (28)$$

which measures the deviation of the first basis matrix  $\mathbf{M}_1$  from the original matrix  $\mathbf{A}$ .

If the gradient of the right hand side of (28) with respect to  $\mathbf{s}_1$  and  $\mathbf{s}_2$  is separately set equal to zero then the following equations are obtained

$$\mathbf{s}_1 + (\mathbf{w}_2 \mathbf{w}_1^T) \mathbf{s}_2 = \mathbf{A} \mathbf{w}_1 \quad (29)$$

$$\mathbf{s}_2 + (\mathbf{w}_1 \mathbf{w}_2^T) \mathbf{s}_1 = \mathbf{A}^T \mathbf{w}_2 \quad (30)$$

where we have used the facts  $\mathbf{w}_1^T \mathbf{w}_1 = 1$  and  $\mathbf{w}_2^T \mathbf{w}_2 = 1$ . The elimination of  $\mathbf{s}_2$  between these two equations results in

$$[\mathbf{I}_n - \mathbf{w}_2 \mathbf{w}_2^T] (\mathbf{s}_1 - \mathbf{A} \mathbf{w}_1) = \mathbf{0} \quad (31)$$

where the matrix enclosed between left and right brackets has a nonempty nullspace spanned by the vector  $\mathbf{w}_2$ . This means

$$\mathbf{s}_1 = \mathbf{A} \mathbf{w}_1 + \alpha_1 \mathbf{w}_2 \quad (32)$$

where  $\alpha_1$  is arbitrary at this moment. Similar discussions allow us to get the following result for  $\mathbf{s}_2$

$$\mathbf{s}_2 = \mathbf{A}^T \mathbf{w}_2 + \alpha_2 \mathbf{w}_1 \quad (33)$$

where  $\alpha_2$  is also arbitrary at this point. To determine  $\alpha_1$  and  $\alpha_2$  first we can obtain the following equality from either (29) or (30) via multiplying both sides by appropriate vectors

$$\mathbf{w}_2^T \mathbf{s}_1 + \mathbf{w}_1^T \mathbf{s}_2 = \mathbf{w}_2^T \mathbf{A} \mathbf{w}_1 \quad (34)$$

Similarly we can get the following equalities from (32) and (33)

$$\mathbf{w}_2^T \mathbf{s}_1 = \alpha_1 + \mathbf{w}_2^T \mathbf{A} \mathbf{w}_1 \quad (35)$$

$$\mathbf{w}_1^T \mathbf{s}_2 = \alpha_2 + \mathbf{w}_1^T \mathbf{A} \mathbf{w}_1 \quad (36)$$

The last three equations mean

$$\alpha_1 + \alpha_2 + \mathbf{w}_2^T \mathbf{A} \mathbf{w}_1 = 0 \quad (37)$$

where one of  $\alpha_1$  and  $\alpha_2$  can be chosen arbitrarily. To provide a symmetric structure for  $\mathbf{s}_1$  and  $\mathbf{s}_2$  we can take  $\alpha_1 = \alpha_2$  and obtain

$$\alpha_1 = \alpha_2 = -\frac{1}{2} \mathbf{w}_2^T \mathbf{A} \mathbf{w}_1 \quad (38)$$

The use of these formulae in the previous expressions of  $\mathbf{s}_1$  and  $\mathbf{s}_2$  and then the reorganization of the results enable us to write

$$\begin{aligned} \mathbf{s}_1 &= \left[ \mathbf{I}_m - \frac{1}{2} \mathbf{w}_2 \mathbf{w}_2^T \right] \mathbf{A} \mathbf{w}_1 \\ \mathbf{s}_2 &= \left[ \mathbf{I}_n - \frac{1}{2} \mathbf{w}_1 \mathbf{w}_1^T \right] \mathbf{A}^T \mathbf{w}_2 \end{aligned} \quad (39)$$

Now we define

$$\mathbf{A}_1 = \mathbf{A} - \mathbf{M}_1, \quad \Delta_1 = Tr(\mathbf{A}_1^T \mathbf{A}_1) \quad (40)$$

The matrix  $\mathbf{A}_1$  corresponds some portion of the bivariate term in HDMR. Hence we do not expect that the second basis matrix will have the univariance based structure of  $\mathbf{M}_1$ . However, we may try to maintain the two one-rank matrix structure. This can be done by changing the structures in  $\mathbf{w}_1$  and  $\mathbf{w}_2$  self-consistently and we can use the orthonormalized forms of the vectors  $\mathbf{s}_2$  and  $\mathbf{s}_1$  for the vectors  $\mathbf{w}_1$  and  $\mathbf{w}_2$  and symbolize the new vectors by  $\mathbf{w}_3$  and  $\mathbf{w}_4$  respectively. The new unknown vectors denoted by  $\mathbf{s}_3$  and  $\mathbf{s}_4$  are determined by minimizing the norm square of the new remainder matrix symbolized by  $\mathbf{A}_2$  and defined as  $\mathbf{A}_1 - \mathbf{s}_3 \mathbf{w}_3^T - \mathbf{s}_4 \mathbf{w}_4^T$ . We give only the results below since the intermediate algebra is same as before

$$\begin{aligned} \mathbf{s}_3 &= \left[ \mathbf{I}_m - \frac{1}{2} \mathbf{w}_4 \mathbf{w}_4^T \right] \mathbf{A}_1 \mathbf{w}_3 \\ \mathbf{s}_4 &= \left[ \mathbf{I}_n - \frac{1}{2} \mathbf{w}_3 \mathbf{w}_3^T \right] \mathbf{A}_1^T \mathbf{w}_4 \end{aligned} \quad (41)$$

The new functional to be minimized for this case is as follows

$$\Delta_2 = Tr(\mathbf{A}_2^T \mathbf{A}_2) \quad (42)$$

This has been the second step of the basis matrix construction. The results for the  $k$ -th optimization are

$$\begin{aligned} \mathbf{s}_{2k-1} &= \left[ \mathbf{I}_m - \frac{1}{2} \mathbf{w}_{2k} \mathbf{w}_{2k}^T \right] \mathbf{A}_{k-1} \mathbf{w}_{2k-1} \\ \mathbf{s}_{2k} &= \left[ \mathbf{I}_n - \frac{1}{2} \mathbf{w}_{2k-1} \mathbf{w}_{2k-1}^T \right] \mathbf{A}_{k-1}^T \mathbf{w}_{2k} \end{aligned} \quad (43)$$

and

$$\begin{aligned} \mathbf{A}_k &= \mathbf{A}_{k-1} - \mathbf{s}_{2k-1} \mathbf{w}_{2k-1}^T - \mathbf{s}_{2k} \mathbf{w}_{2k}^T, \\ \Delta_k &= Tr(\mathbf{A}_k^T \mathbf{A}_k) \end{aligned} \quad (44)$$

The vectors  $\mathbf{w}_{2k-1}$  and  $\mathbf{w}_{2k}$  for a specific  $k$  value may become the left and right singular vectors of  $\mathbf{A}$ . Then

the recursion in (43) returns same  $\mathbf{w}_{2k-1}$  and  $\mathbf{w}_{2k}$  vectors and enters an infinite loop. As long as this does not happen, the recursion continues to produce basis matrices. However there is a possibility of producing linearly dependent basis matrices. This must of course be avoided. This is an open question yet but certain modifications seem to be possible to get rid of these types of undesired behaviours. What we expect is to get  $\min(m, n)$  linearly independent basis matrices in exactly  $\min(m, n)$  recursive steps. If the original matrix  $\mathbf{A}$  has the vector  $\mathbf{w}_1$  or  $\mathbf{w}_2$  as one of its singular vectors the recursion stops at the very beginning if a control exists over infinite loops.

If the recursion continues until  $k$  becomes equal to  $\min(m, n)$  and all produced basis matrices are linearly independent then we get the whole basis matrix set whose  $k$ -th element is given by

$$\begin{aligned} \mathbf{M}_k &= \mathbf{u}_{2k-1} \mathbf{w}_{2k-1}^T + \mathbf{w}_{2k} \mathbf{u}_{2k}^T, \\ k &= 1, \dots, \min(m, n) \end{aligned} \quad (45)$$

The matrix  $\mathbf{A}$  should be sum of these matrices as follows

$$\mathbf{A} = \sum_{k=1}^{\min(m, n)} \mathbf{M}_k \quad (46)$$

This sum has summands with generally nonunit norms and the norm of each summand can be extracted as a linear combination coefficient by making the summand matrix normalized to 1 with respect to Frobenius norm. There is no orthogonality amongst the summands unless certain coincidences occur. Orthogonality is provided by using an appropriate method to this end. If this is done and the resulting matrices are denoted by  $\mathbf{N}_k$  then

$$(\mathbf{N}_i, \mathbf{N}_j) = \delta_{ij}, \quad i, j = 1, \dots, \min(m, n) \quad (47)$$

which permits us to write

$$\mathbf{A} = \sum_{k=1}^{\min(m, n)} \nu_k \mathbf{N}_k \quad (48)$$

where

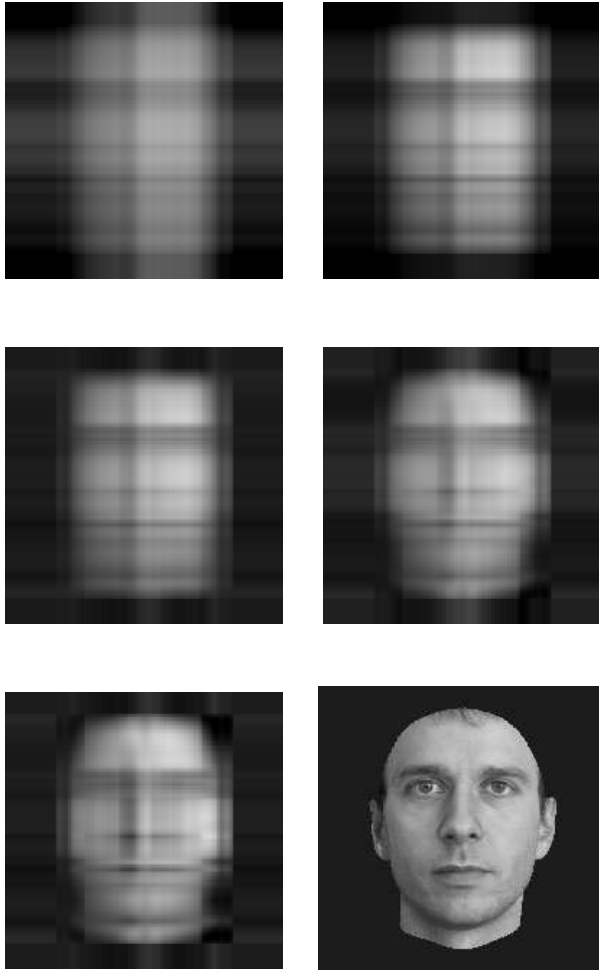
$$\nu_k \equiv (\mathbf{N}_k, \mathbf{A}), \quad k = 1, \dots, \min(m, n) \quad (49)$$

The  $\nu_k$  scalars are somehow corresponding to singular values and may match them in certain cases.

## 5 An Illustrative Application: Image Decomposition

We apply the presented method on an image in this section. First five frames from decomposition are cumulatively displayed below. That is, the first is the

display of the first subimage, the second is the display of the sum of first two subimages, and so on. The original image is also given here at the right bottommost location.



**Figure 1:** The comparison of the cumulative displays of first five subimages with the original image

As Figure 1 shows the first display is far from the original image and without knowing the original image it is almost impossible to recognize the picture although there is some hidden information in the frame. The quality increases as we proceed to the higher order frames as the comparison with the sixth display (original image) at the right bottommost shows.

The face images displayed here are taken from the MIT CBCL face database [29]. Credit is hereby given to the Massachusetts Institute of Technology and to the Center for Biological and Computational Learning for providing the database of facial images.

To get more clear idea about the quality increase we give the 10th, 25th, and, 40th cumulative displays together with the original display in Figure 2.



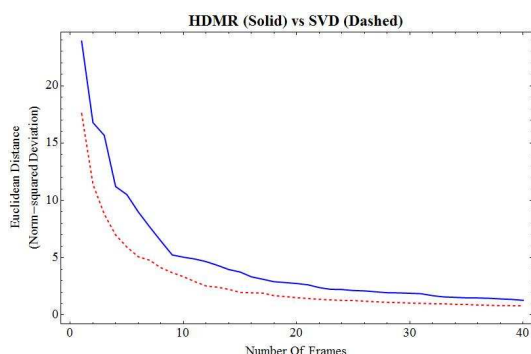
**Figure 2:** The comparison of the cumulative displays of 10th, 25th, 40th basis matrices with the original image

As seen from Figure 2 the 40th display is almost same as the original image although there are slight differences which can not be noticed unless sufficient attention is paid.

Before closing this section we need to make a comparison with the most widely used method, singular value decomposition. We could do this by using the displays of the cumulative matrices but the discrepancy between the present method and singular value decomposition is not visually distinguishable as the displays in Figure 3 show. Therefore we compare the norm squares of the differences between the cumulative display matrices and the original image with respect to the basis matrix number. The plots are given in Figure 4 where the dashed curve is for singular value decomposition and remains less than the solid curve which is produced from the present, HDMR based, method in all frame number values.



**Figure 3:** The comparison of the cumulative displays of 40th basis matrices for present method (left) and singular value decomposition



**Figure 4:** The deviations of the subimage norm squares for singular value decomposition and for the present method from the original image

All numerical implementations in this sections are realized by using Mathematica.

## 6 Concluding Remarks

In this work we have developed a new matrix decomposition method whose construction is inspired by the univariate truncation of two variable HDMR. The method is similar to the singular value decomposition of matrices, which uses single outer products as the basis matrices. Whereas, each individual basis matrix of the new method is taken as a sum of two outer products where there are four vectors, two of which are to be determined and the remaining ones being given. Unknown vectors are determined via the minimization of the deviation between the basis matrix and the matrix to be decomposed. The couple vector determined in this way is fed as the given vectors of the next step where the deviation of the previous step is used as the given matrix. This produces a recursive but not iterative decomposition method. The method does not guarantee the construction of a linearly independent vector in each step. However, the steps where no new vectors are generated to the outer products of the construction correspond to singular vectors of the target matrix and starts an infinite recursion which can be stopped by a check and the aim is achieved since singular vectors are also acceptable as what we desire to get.

The presented method, because of its noniterative nature, stands as a powerful candidate to take the place of singular value decomposition in many applications. We have applied it to computer vision and have found that a given monochromatic image can be dissolved to subimages with different dominancies such that first few of them becomes capable of sufficiently well describing the original image. It is quite promising to develop new tools for image processing and similar

areas.

This is the first step of the method's construction and there remains certain open questions we are intensely dealing with. For example, to provide orthogonality in the resulting basis set is our next and urgent step. The convergence properties should and will also be investigated appropriately. The linear algebraic issues are also important and the author believes that there appears to be important developments in the horizon.

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