An Improved Method for Successive Data Schism and Interpolation

AHMED N. H. ALNUAIMY, MAHAMOD ISMAIL, MOHD A. M. ALI and KASMIRAN JUMARI
Department of Electrical, Electronic & Systems Engineering
Faculty of Engineering & Built Environment
National University of Malaysia, UKM
43600 UKM Bangi Selangor Darul Ehsan
MALAYSIA
ahmed_alnuaimy@yahoo.com

Abstract: - This paper presents an improved type of data schism and interpolation called Alnuaimy-Mahamod Interpolation based on the mean and variance to calculate the entire point between two known points. A new mathematical method is developed for interpolation from a given set of data points in a plane and for fitting a smooth curve to the points. This method is devised in such a way that the resultant curve will pass through the given points and will appear smooth and natural. Data schism and interpolation describes the behaviors of the calculated points which behave as bacteria reproduction. This type of interpolation based on calculation of the mean and the variance of two adjacent points and modify of these calculate values by named factors to be used to calculate the entire points. Our type of interpolation can be working as linear midpoint interpolation, linear interpolation and smooth curve fitting.

Key-Words: - Interpolation, midpoint interpolation, linear interpolation and smooth curve fitting.

1 Introduction

Computation and measurement can be performed to determine a relation between two variables. The result is given as a set of discrete data points in a plane. Knowing that the relation can be represented by a smooth curve, then a curve fitting used to set of data points so that it will pass through all the points. Manual drawing is the most primitive method for this purpose and results in a reasonable curve if it is done by a well-trained scientist or engineer. But, since it is very tedious and time consuming a computer is used to draw the curve. The computer must then be provided with necessary instructions for mathematically interpolating additional points between the given data points.

There are several mathematical methods of interpolating a single-valued function from a given set of values [1], but their application to curve fitting sometimes results in a curve that is very different from one drawn manually. The common difficulty is that the resultant curve sometimes shows unnatural wiggles. This seems inevitable if we make any assumption concerning the functional form for the whole set of given data points other than the continuity and the smoothness of the curve.

The problem of constructing a continuously defined function from given discrete data is unavoidable whenever one wishes to manipulate the data in a way that requires information not included explicitly in the data. In this age of ever-increasing digitization in the storage, processing, analysis, and communication of information, it is not difficult to find examples of applications where this problem occurs. The relatively easiest and in many applications often most desired approach to solve the problem is interpolation, where an approximating function is constructed in such a way as to agree perfectly with the usually unknown original function at the given measurement points. In view of its increasing relevance, it is only natural that the subject of interpolation is receiving more and more attention these days. However, in times where all efforts are directed toward the future, the past may easily be forgotten. It is no sinecure, scanning the literature, to get a clear picture of the development of the subject through the ages. This is quite unfortunate, since it implies a risk of researchers going over grounds covered earlier by others. History has shown many examples of this and several new examples will be revealed here. In this paper we will present a systematic overview of the developments in interpolation theory, from the earliest times to the present date and to put the most well-known techniques currently used in signal and image processing applications into historical perspective and give a precise description for Alnuaimy-Mahamod Interpolation.

2 Interpolation Overview

The word “interpolation” originates from the Latin verb *interpolare*, a contraction of “inter,” meaning "between," and "polare," meaning "to polish." That is to say, to smooth in between given pieces of information. It seems that the word was introduced in the English literature for the first time around 1612 and was then used in the sense of "to alter or enlarge [texts] by insertion of new matter" [2]. The original Latin word appears [3] to have been used first in a mathematical sense by Wallis in his 1655 book on infinitesimal arithmetic [4].

In antiquity, astronomy was all about time keeping and making predictions concerning astronomical events. This served important practical needs: farmers, e.g., would base their planting strategies on these predictions. To this end, it was of great importance to keep up lists—so-called ephemerides—of the positions of the sun, moon, and the known planets for regular time intervals. Obviously, these lists would contain gaps, due to either atmospherical conditions hampering observation or the fact that celestial bodies may not be visible during certain periods. From his study of ephemerides found on ancient astronomical cuneiform tablets originating from Uruk and Babylon in the Seleucid period (the last three centuries BC), the historian-mathematician Neugebauer [5, 6] concluded that interpolation was used in order to fill these gaps. Apart from linear interpolation, the tablets also revealed the use of more complex interpolation methods. Precise formulations of the latter methods have not survived, however.

By the beginning of the 20th century, the problem of interpolation by finite or divided differences had been studied by astronomers, mathematicians, statisticians, and actuaries,21 and most of the now well-known variants of Newton's original formulae had been worked out. This is not to say, however, that there are no more advanced developments to report on. Quite to the contrary, already in 1821, Cauchy [6] studied interpolation by means of a ratio of two polynomials and showed that the solution to this problem is unique, the Waring–Lagrange formula being the special case for the second polynomial equal to one.22 Generalizations for solving the problem of multivariate interpolation in the case of fairly arbitrary point configurations began to appear in the second half of the 19th century, in the works of Borchardt and Kronecker [8].

A generalization of a different nature was published in 1878 by Hermite [9], who studied and solved the problem of finding a polynomial of which also the first few derivatives assume prespecified values at given points, where the order of the highest derivative may differ from point to point. In a paper [10] published in 1906, Birkhoff studied the even more general problem: given any set of points, find a polynomial function that satisfies prespecified criteria concerning its value and/or the value of any of its derivatives for each individual point.23 Hermite and Birkhoff type of interpolation problems—and their multivariate versions, not necessarily on Cartesian grids—have received much attention in the past decades. A more detailed treatment is outside the scope of this paper, however, and the reader is referred to relevant books and reviews [8].

Another important development from the late 1800s is the rise of approximation theory. For a long time, one of the main reasons for the use of polynomials had been the fact that they are simply easy to manipulate, e.g., to differentiate or integrate. In 1885, Weierstrass [11] also *justified* their use for approximation by establishing the so-called *approximation theorem*, which states that every continuous function on a closed interval can be approximated uniformly to any prescribed accuracy by a polynomial.24 The theorem does not provide any means of obtaining such a polynomial, however, and it soon became clear that it does not necessarily apply if the polynomial is forced to agree with the function at given points within the interval, i.e., in the case of an interpolating polynomial.

Examples of meromorphic functions for which the Waring–Lagrange interpolator does not converge uniformly were given by Méray [12], [13] and later Runge [14]—especially the latter has become well known and can be found in most modern books on the topic. A more general result is due to Faber [15], who, in 1914, showed that for any prescribed triangular system of interpolation points there exists a continuous function for which the corresponding Waring–Lagrange interpolation process carried out on these points does not converge uniformly to this function. Although it has later been proven possible to construct interpolating polynomials that *do* converge properly for all continuous functions, e.g., by using the Hermite type of interpolation scheme proposed by Fejér [16] in 1916, these findings clearly revealed the "inflexibility" of algebraic polynomials and their limited applicability to interpolation.

The early 1970s was the time when digital image processing really started to develop. One of the first applications reported in the literature was the geometrical rectification of digital images obtained from the first Earth Resources Technology Satellite launched by the United States National Aeronautics and Space Administration in 1972. The need for

more accurate interpolations than obtained by standard linear interpolation in this application led to the development of a still very popular technique known as *cubic Interpolation*. Meanwhile, research on cubic convolution interpolation continued. In 1981, Keys [17] published an important study that provided new approximation-theoretic insights into this technique.
The use of splines interpolation for digital-image interpolation was investigated by an important paper providing a detailed analysis was published in 1978 by Hou and Andrews [18].

3 Alnuaimy-Mahamod Interpolation

Our method is based on the mean and the variance of two known values to calculate the entire values. The mean and the variance functions will be considered as functions for the unknown values. The calculated values of the mean and variance should be modified by β and ξ factors respectively. Using of these new values to calculate the entire samples. These two factors will be described precisely later.

3.1 Formulation

The calculation method can be explained as follows:

1) Calculating of the mean and the variance of the two known values which have been used to calculate the entire values in between:

$$Mean\,(x_{p,p+1}) = \frac{1}{N_p}\sum_{i=1}^{N_p} x_i \qquad (1)$$

$$\sigma^2(x_{p,p+1}) = \frac{1}{N_p}\sum_{i=1}^{N_p}\left(x_i - \bar{x}_{p,p+1}\right)^2 \qquad (2)$$

where N_p is the number of values used in the equation.

2) Multiplying the mean and the variance by factors β and ξ respectively.

3) Calculating the values of the two entire values between the two known values using the following equation:

$$x_{e1,2} = \frac{c_1}{2} \pm \sqrt{c_2 + \left(\frac{c_1}{2}\right)^2} \qquad (3)$$

where x_e is the interpolated values while c_1 and c_2 are:

$$c_1 = N_p \times \beta\bar{x} - (x_p + x_{p+1}) \qquad (4)$$

$$c_2 = \frac{N_p \times (\beta\bar{x})^2 + N_p \times \xi\sigma^2 - (x_p + x_{p+1} + c_1)}{2} \qquad (5)$$

4) Using the new values to calculate others in between following the same method in order to get required number of the interpolated values.

3.2 β and ξ factors

β and ξ factors are the modification factors for the mean and variance respectively. These factors ca be considered as control factors; as these factors control the calculated values to be closer or further from the known values.

3.2.1 β factor

β factor controls the calculated values of the interpolation as those can be upper, lower or on the line (in between) of the used values. When β equal to 1 the calculated values will be on the line of the used values for interpolation. As β increased the interpolated values gets higher to the used sample while as it is decreased the interpolated values gets down to the used values as shown in Fig.1.

β factor may cause error in the interpolated values specially when the original used values are real values so it can cause generation of an complex values. Therefore, the default value for β factor is 1. But in some cases we can change this value if we need it.

2.2.2 ξ factor

ξ factor controls the calculated values of the interpolation as it can be closer to used known values or further but in between (on the line of the known values). When ξ equal to 0.5 the calculated samples will be just only a mid point between to the used values (close to each other). As ξ increase the interpolated values getting closer to the used values, but when ξ equal to 1 the value will be much closer or equal to the used values as β equals to 1 (i.e. equals to the used value) as shown in Fig.2.

The default used value for ξ factor is 0.556 which can give good accurate interpolated values.

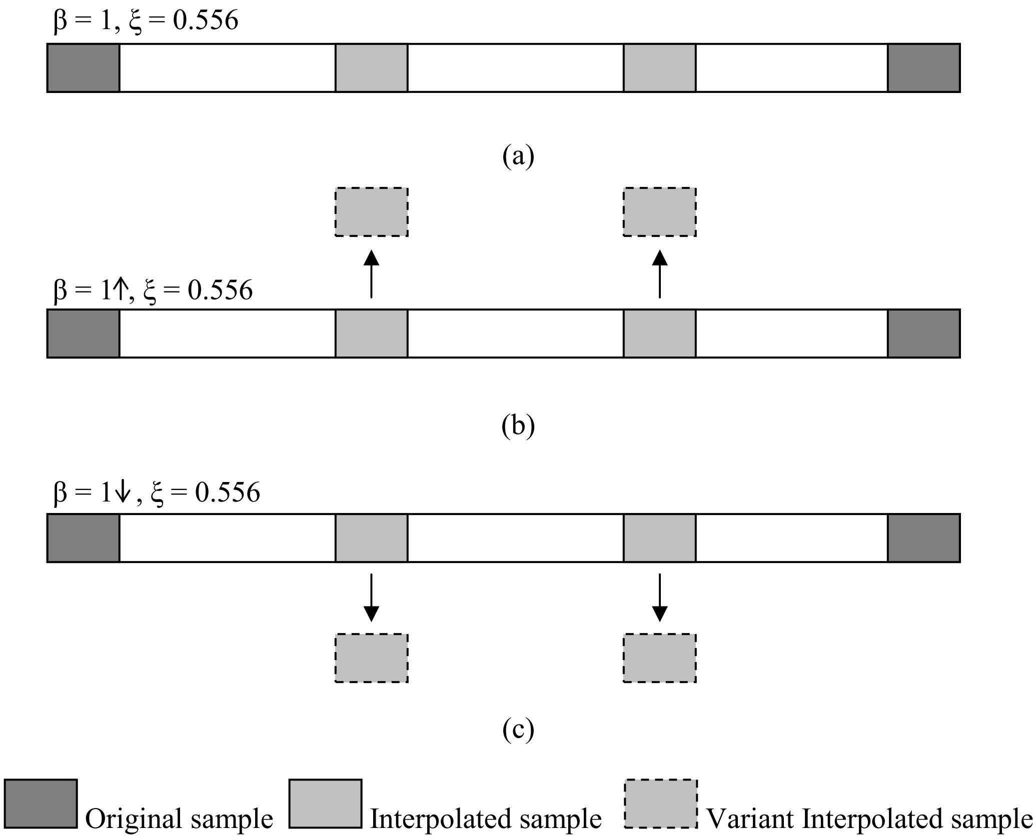


Fig.1 β factor over default ξ value

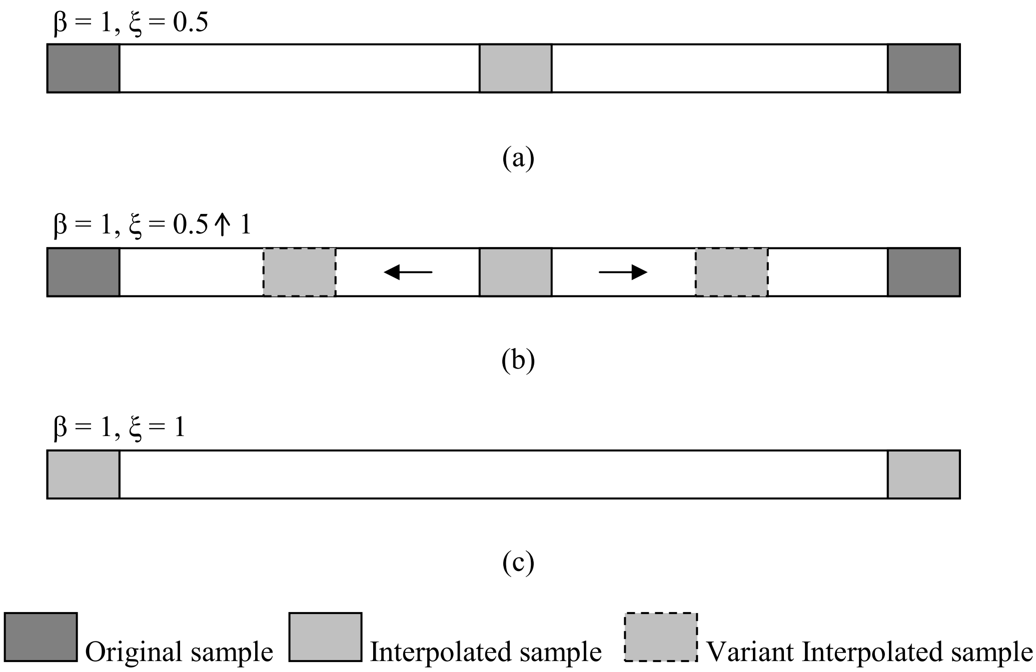


Fig.2 ξ factor over default β value

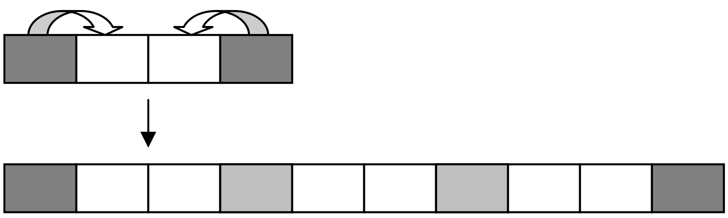
3.3 Samples Interpolation

The subdivision of the symbols should be submitted to some role, where the interpolation should be made between two adjacent samples as two another samples will be created which lay in between the original used samples. The calculated samples will be used to calculate another points lay in between.

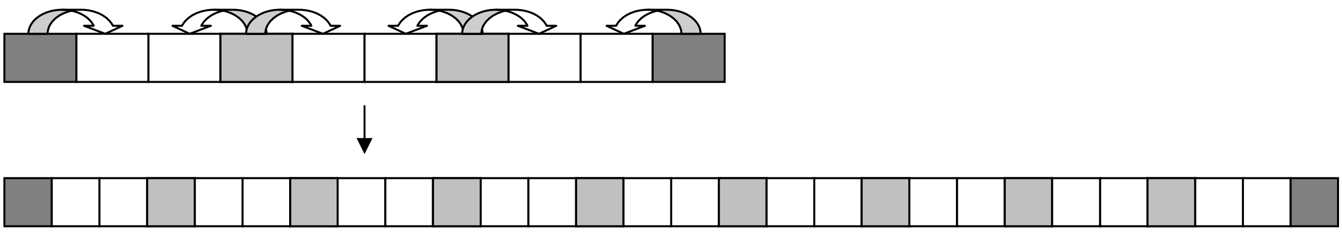
Preliminary the possible number of the calculated point can be calculated according to the following equation:

$$N = N_p + 2\times(N_p - 1) \tag{6}$$

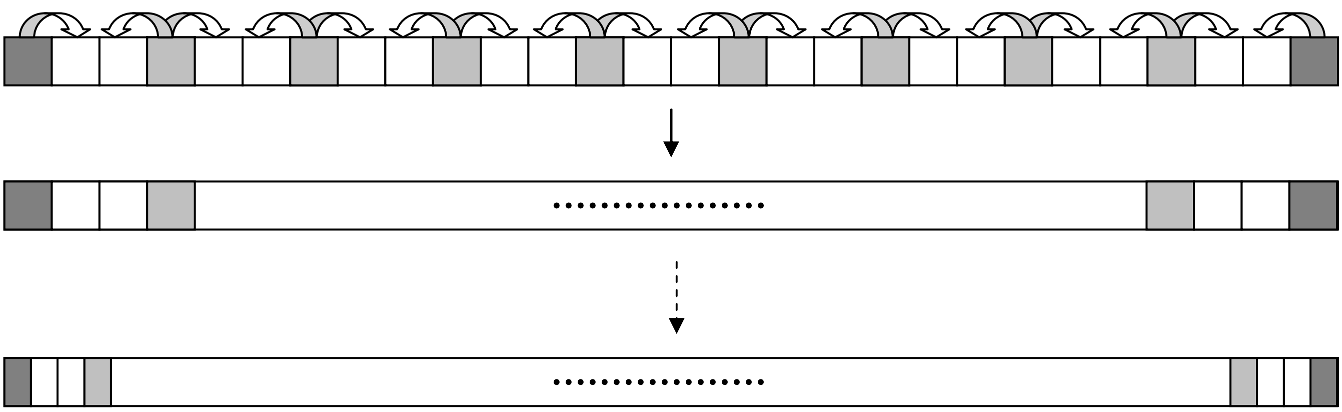
where N is the total number of the point after calculation and N_p is the number of known points. N_p will be updated to be equal to N each time we repeat the interpolation until we get the required number of the interpolated samples, as shown in Fig. 3.



Step 1: Calculating of two samples lay between the original ones.



Step 2: Using the created samples with the original ones to calculate another samples between each two adjacent samples.

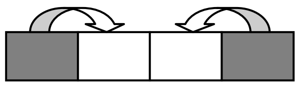


Step 3: Repeat of the Step 2 each time we need to create another samples.

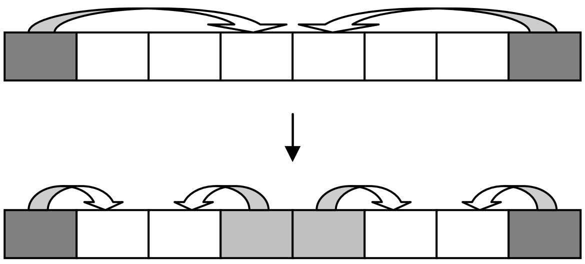
Fig.3 Interpolation subdivision

In order to get other numbers of the interpolated samples there should be other subdivisions for the entire interpolated samples as shown in Fig. 4(a-d) and 5(a-c).

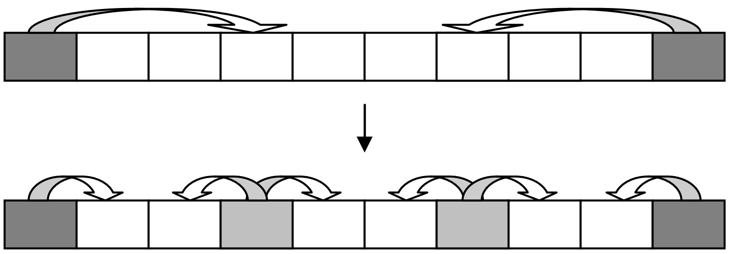
These distributions should be controlled by ξ factor, where ξ factor can control the interpolated samples values to be close to each other or close to the original samples.



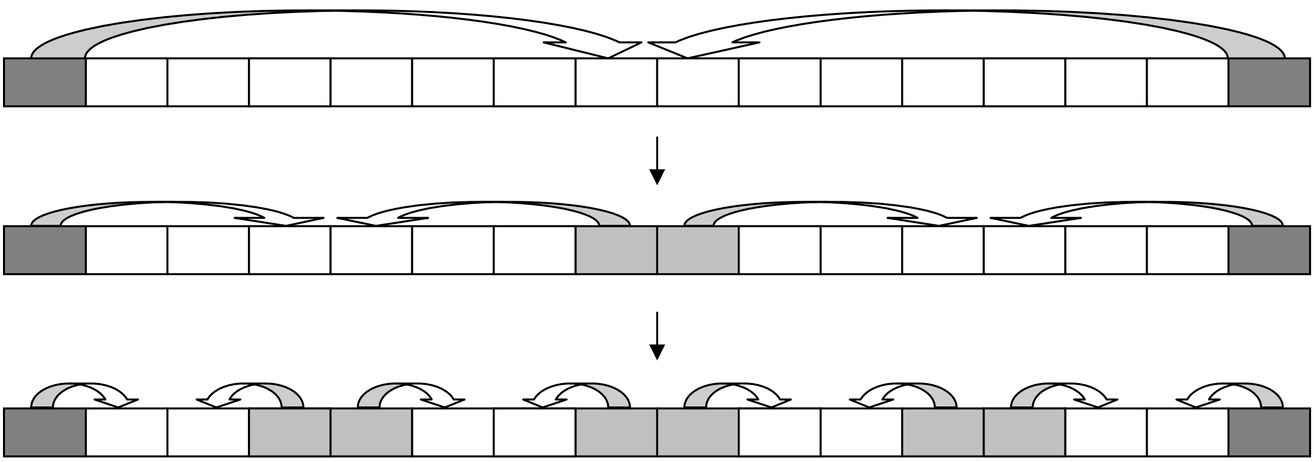
(a) 3 Steps – Single subdivision



(b) 7 Steps – 2 subdivisions

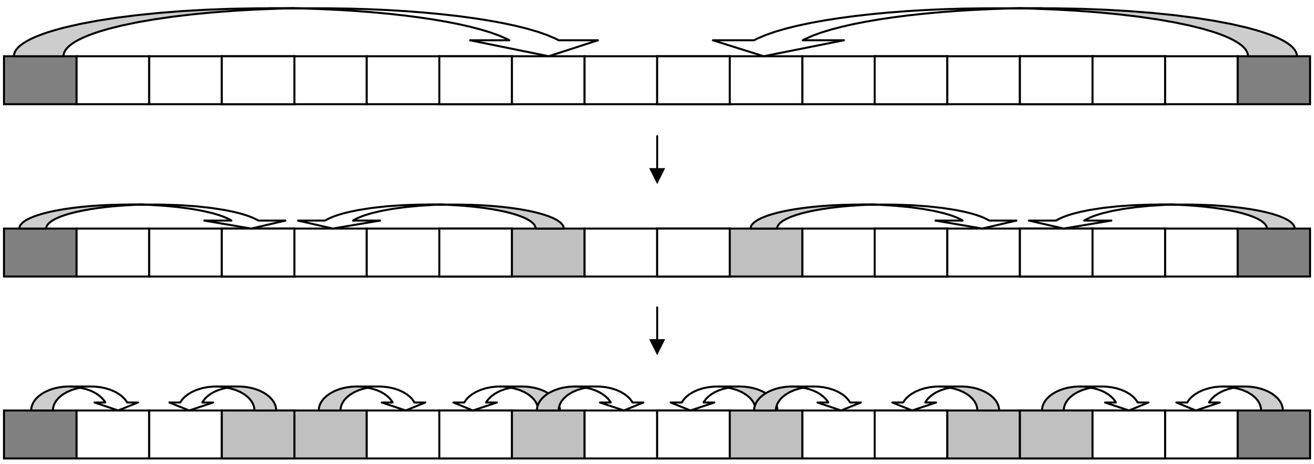


(c) 9 Steps – 3 subdivisions

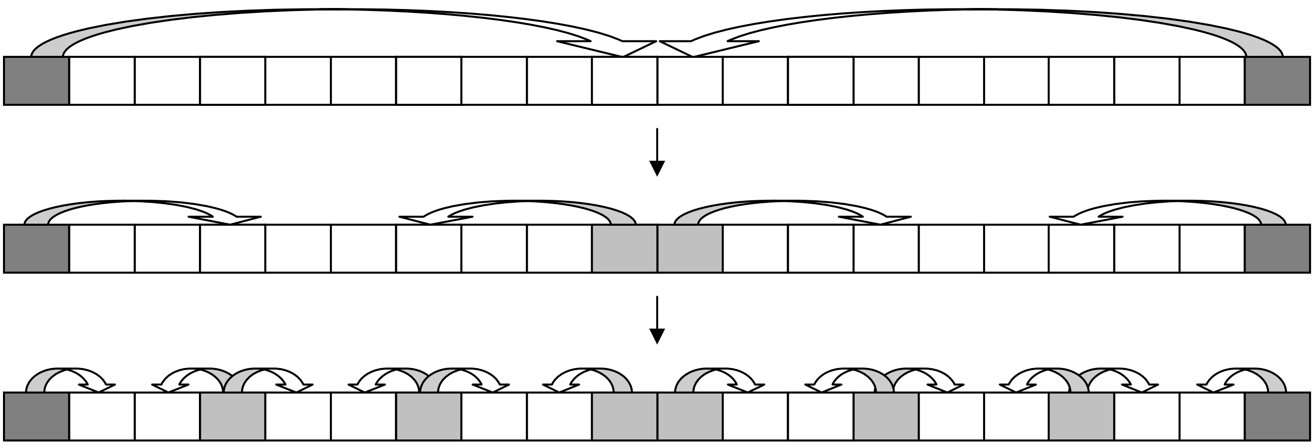


(d) 15 Steps – 2 subdivision – 4 subdivision

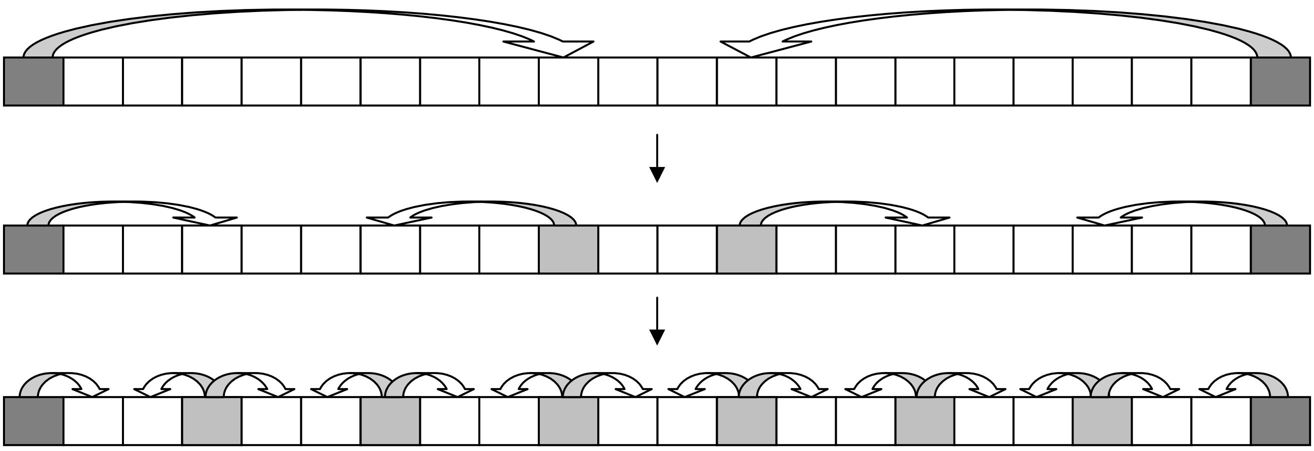
Fig. 4 Interpolation Subdivisions



(a) 17 Steps – 3 subdivision – 5 subdivision



(b) 19 Steps – 2 subdivision – 6 subdivision



(c) 21 Steps – 3 subdivision – 7 subdivision

Fig. 5 Other Interpolation Subdivisions

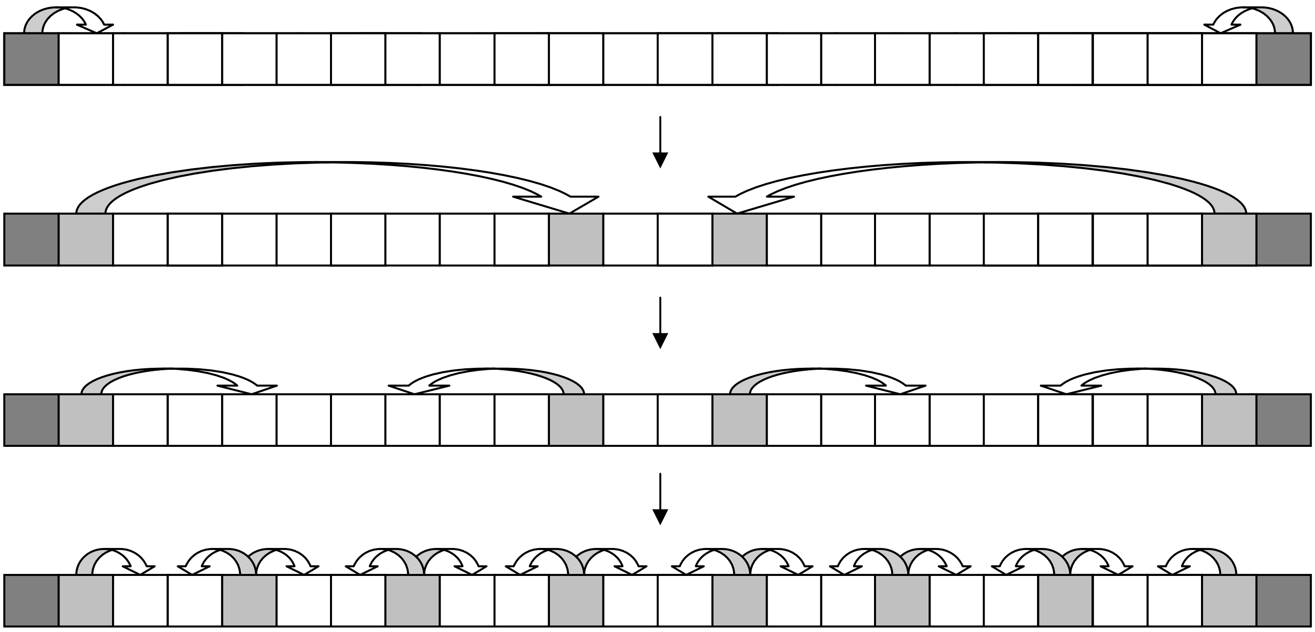


Fig. 6 Interpolation distribution based on variant values of ξ factors

Also we can get other distributions for the interpolated samples which are not uniformly distributed as they are controlled by variant values of ξ factors as can be seen above in Fig. 6.

3.4 Example

Let us suppose that we have a number of points on a line of a sinc function's curve which we want to interpolate other points between those to give more reality to the curve. Fig. 7 shows number of points on the curve in the *x-y* coordinates.

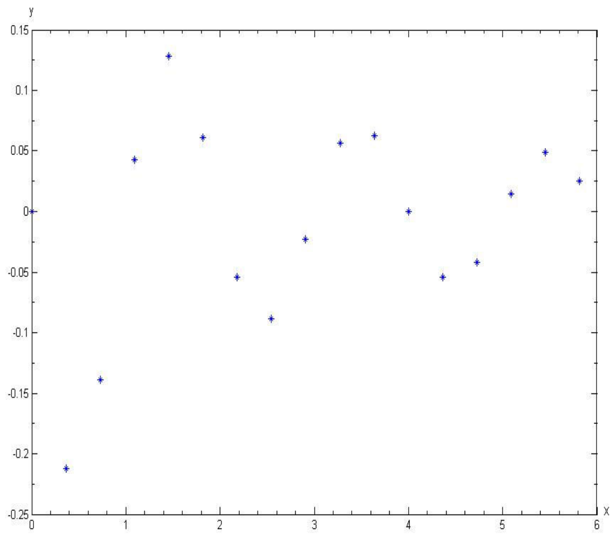


Fig. 7 points of a sinc curve

Now we will test of the work of Alnuaimy-Mahamod interpolation using of the default values of β and ξ factors (1 and 0.556 respectively). First we will choose 3rd and 4th point to be x_1 and x_2 for Alnuaimy-Mahamod interpolation, as their amplitudes are -0.1393 and 0.04289 respectively, and then apply the steps in section 3.1 to calculate x_{e1} and x_{e2} as following:

$$\bar{x} = \frac{-0.1393 + 0.04289}{2} = \text{-0.0482}$$

$$\sigma^2(x) = \frac{(-0.1393 - (-0.0482))^2 + (0.04289 - (-0.0482))^2}{2}$$

$$c_1 = N_p \times \beta\,\bar{x} - (x_1 + x_2) = -0.0964$$

$$c_2 = \frac{N_p \times (\beta\,\bar{x})^2 + N_p \times \xi\,\sigma^2 - (x_1 + x_2 + c_1)}{2}$$

$$= \text{-0.0014}$$

$$x_{e1,2} = \frac{c_1}{2} \pm \sqrt{c_2 + \left(\frac{c_1}{2}\right)^2}$$

$$= -0.07867, -0.01771$$

Now we will put these two points on the *x*-*y* coordinate between the used points as we divided the *x*-axis distance between the used points into three parts as shown in Fig. 8.

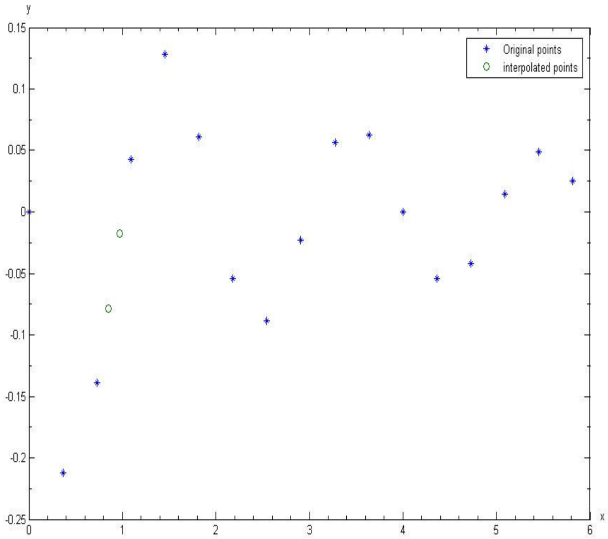


Fig. 8 Interpolation between 3rd and 4th point of the sinc curve

By continuing the interpolation for the rest points, we can get more points for the curve as shown in Fig. 9.

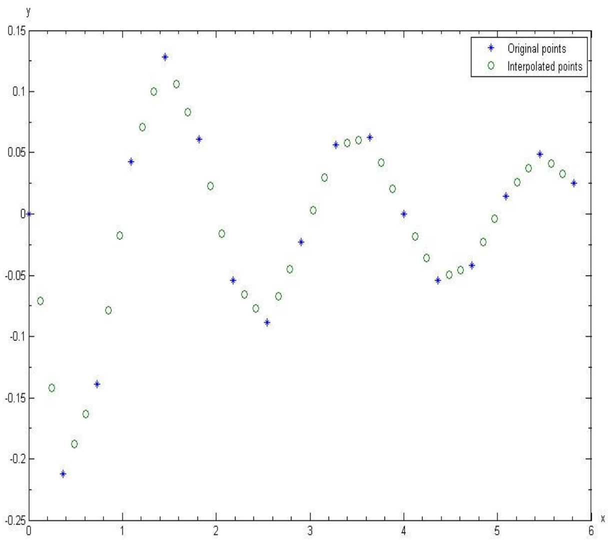


Fig. 9 Interpolation between all points of the sinc curve

Also we can continue to find other points using the original and the interpolated points over Alnuaimy-Mahamod interpolation to give more accuracy for the curve.

4 Result and Discussion

In this section a computer result for different parameters (β and ξ factors) of Alnuaimy-Mahamod interpolation over a number of samples of sinc function curve will be given.

Table 1 gives the accuracy of Alnuaimy-Mahamod interpolation as it measure the Root Mean Square Error (RMSE) for interpolated curve and the original one for different values of β and ξ factors.

Fig. 10 shows the performance of Alnuaimy-Mahamod interpolation for the default values of β and ξ factors (1 and 0.556 respectively). It is obvious that the interpolation work as linear interpolation.

Using default value of β factor and different values of ξ factor for interpolation to test the controlling of ξ factor on the interpolation as shown in Fig. 11. It can be seen that when ξ factor increase the curve become smoother but the accuracy of the curve reduce.

In Fig. 12 the effect of β factor variation over default value of ξ factor can be seen. It is obvious that increasing and decreasing the value of β factor will increase the concavity and the convexity of the curve.

Table 1: RMSE for different values of Alnuaimy-Mahamod interpolation's Parameters

β factor	*ξ factor*	*RMSE*
1	0.556	1.940e-02
1	0.6	2.210e-02
1	0.65	2.843e-02
1	0.7	3.520e-02
1.015	0.556	1.473e-02
1.02	0.556	1.414e-02
1.03	0.556	1.527e-02

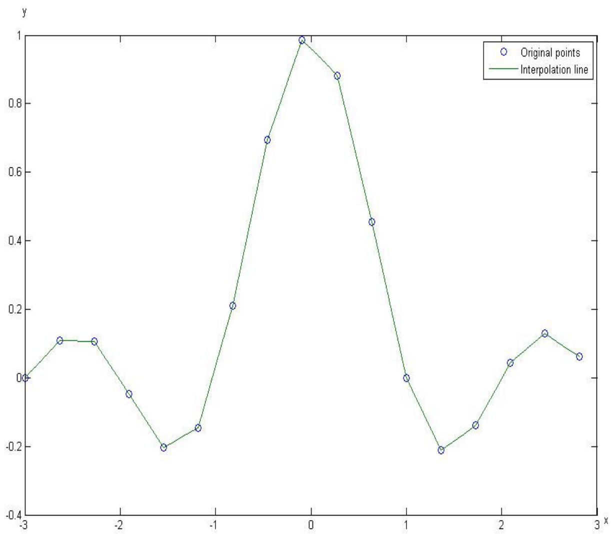
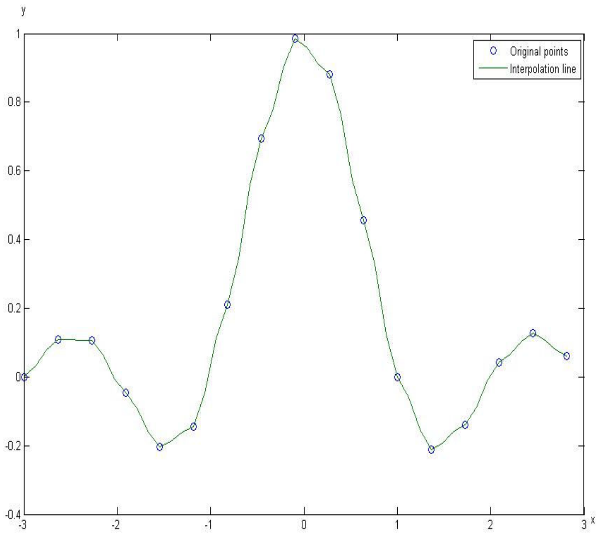
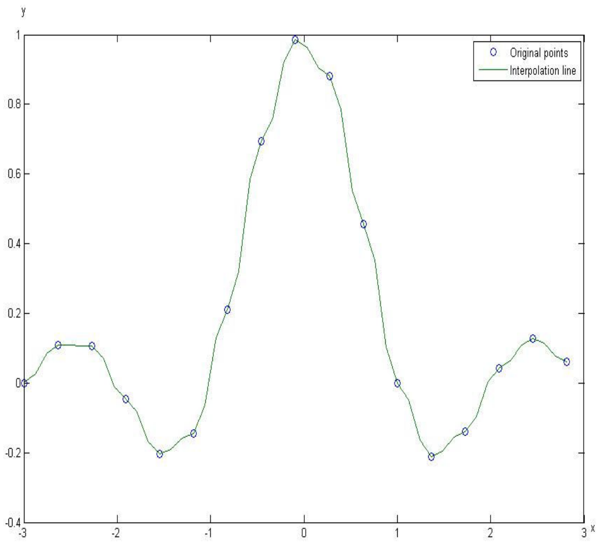


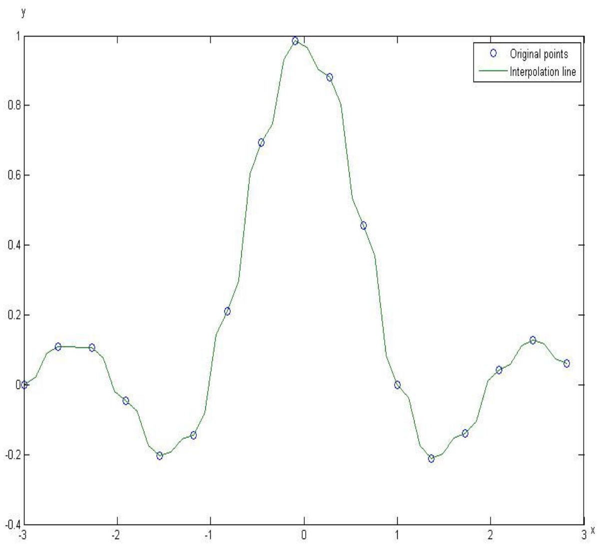
Fig. 10 Default value of β and ξ factors for Alnuaimy-Mahamod Interpolation



(a) β=1 and ξ=0.6

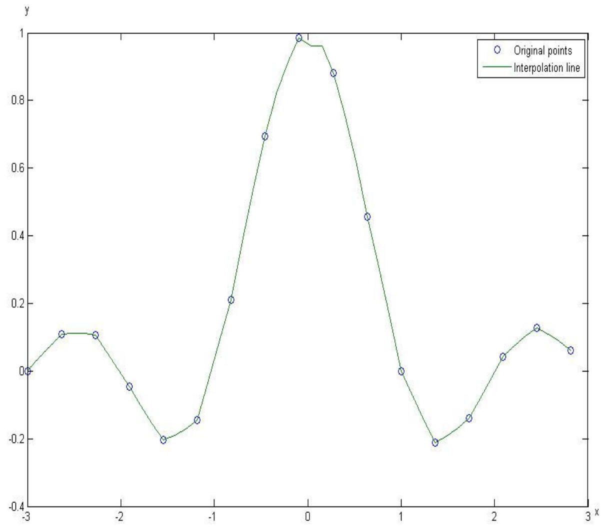


(b) β=1 and ξ=0.65

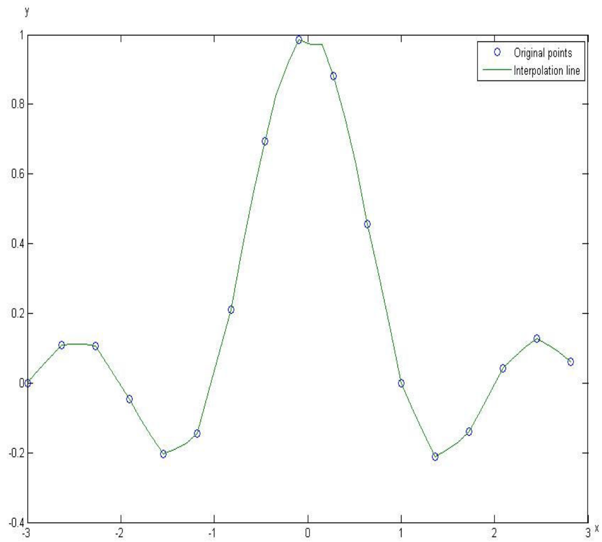


(c) β=1 and ξ=0.7

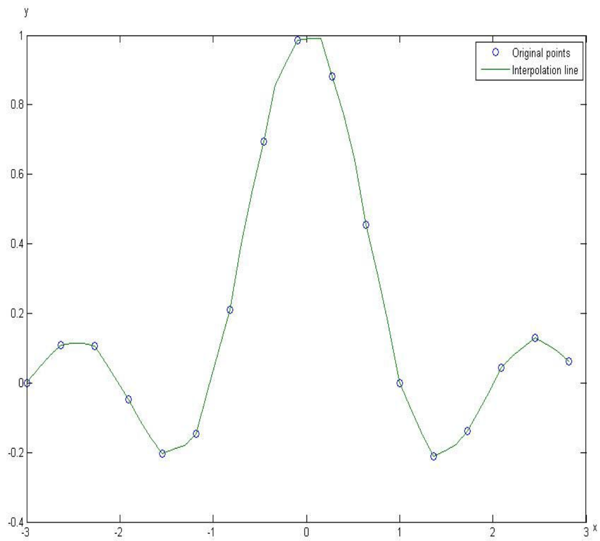
Fig. 11 Fixed value of β and different values of ξ for Alnuaimy-Mahamod Interpolation



(a) β=1.015 and ξ=0.556



(b) β=1.02 and ξ=0.556



(c) β=1.03 and ξ=0.556

Fig. 12 Different values of β and fixed value of ξ for Alnuaimy-Mahamod Interpolation

5 Conclusion

Alnuaimy-Mahamod interpolation can give more accuracy and reality for the interpolated curve. It can also work similar to other types of interpolation like linear interpolation, mid point interpolation and another type of interpolation. However, the interpolation is distinguishable from others since it can be controlled by two factors which are β and ξ. Hence, the interpolation becomes more flexible to the curvature of the line.

References:

[1] J. KARUP, *On a new mechanical method of graduation* In Transactions of the Second International Actuarial Congress. C. and E. Layton, London, 1899, pp. 78-109.

[2] J. Simpson and E.Weiner, Eds., *The Oxford English Dictionary*, 2nd ed. Oxford, U.K.: Oxford Univ. Press, 1989.

[3] J. Bauschinger, *Interpolation*, in Encyklopädie der Mathematischen Wissenschaften, W. F. Meyer, Ed. Leipzig, Germany: B. G.Teubner, 1900–1904, pp. 799–820.

[4] J. Wallis, *Arithmetica Infinitorum. Hildesheim*, Germany: Olms Verlag, 1972.

[5] O. Neugebauer, *Astronomical Cuneiform Texts. Babylonian Ephemerides of the Seleucid Period for the Motion of the Sun, the Moon and the Planets*, London, U.K.: Lund Humphries, 1955.

[6] O. Neugebauer, *A History of Ancient Mathematical Astronomy*, Berlin, Germany: Springer-Verlag, 1975.

[7] A.-L. Cauchy, *Cours d'Analyze de l'École Royale Polytechnique:Part I: Analyze Algébrique*, Paris, France: Imprimerie Royale, 1821, pt. I

[8] M. Gasca and T. Sauer, *On the history of multivariate polynomial interpolation*, J. Comput. Appl. Math., vol. 122, no. 1–2, pp. 23–35, 2000.

[9] C. Hermite, *Sur la formule d'interpolation de Lagrange*, Journal für die Reine und Angewandte Mathematik, vol. 84, no. 1, pp. 70–79, 1878.

[10] G. D. Birkhoff, *General mean value and remainder theorems with applications to mechanical differentiation and quadrature*, Trans. Amer. Math. Soc., vol. 7, no. 1, pp. 107–136, 1906.

[11] K. Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen reeller Argumente*, in Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, pp. 633/789–639/805.

[12] Ch. Méray, *Observations sur la légitimité de l'interpolation*, Annales Scientifiques de l'École Normale Supérieure, ser. 3, vol. 1, pp. 165–176, 1884.

[13] Ch. Méray, *Nouveaux exemples d'interpolations illusoires*, Bulletin des Sciences Mathématiques., ser. 2, vol. 20, pp. 266–270, 1896.

[14] C. Runge, *Über emperische Functionen und die Interpolation zwischen äquidistanten Ordinaten*, Zeitschrift für Mathematik und Physik, vol. 46, pp. 224–243, 1901.

[15] G. Faber, *Über die interpolatorische Darstellung stetiger Functionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23, pp. 192–210, 1914.

[16] L. Fejér, *Ueber interpolation*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse, pp. 66–91, 1916.

[17] R. G. Keys, *Cubic convolution interpolation for digital image processing*, IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-29, pp. 1153–1160, Dec. 1981.

[18] H. S. Hou and H. C. Andrews, *Cubic splines for image interpolation and digital filtering*, IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-26, pp. 508–517, Dec. 1978.