The Study of Micro-Fluid Boundary Layer Theory

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Abstract: Similar to the study of $Prandtl$ system, by the well-known $Oleinik$ linear method, the paper gets existence, uniqueness of the solution for the following initial boundary problem in $\mathcal{D} = \{(t,\, x,\, y) \mid 0 < t < T,\, 0 < x < X,\, 0 < y < \infty\}$,

$$\begin{cases} u_t + uu_x + vu_y = U_t + UU_x + (\nu(y)u_y)_y, \\ u_x + v_y = 0, \\ u(0,\, x,\, y) = u_0(x,\, y),\, u(t,\, 0,\, y) = 0, \\ u(t,\, x,\, 0) = 0,\, v(t,\, x,\, 0) = v_0(t,\, x),\, \lim\limits_{y\to\infty} u(t,\, x,\, y) = U(t,\, x), \end{cases}$$

where T is sufficient small and $\nu(y)$ is a bounded function.

Key–Words: Micro-fluid boundary layer, Uniqueness, Existence, Classical solution

1 Introduction

As well known, $Prandtl$ proposed the conception of the boundary layer in 1904[1]. From then on, the interest for the theory of boundary layer has been steadily growing, due to the mathematical questions it pose, and its important practical applications. According to $Prandtl$ boundary layer theory, the flow about a solid body can be divided into two regions: a very thin layer in the neighborhood of the body (the boundary layer) where viscous friction plays an essential part, and the remaining region outside this layer where friction may be neglected (the outer flow). Thus, for fluids whose viscosity is small, its influence is perceptible only in a very thin region adjacent to the walls of a body in the flow; the said region, according to $Prandtl$, is called $the\ boundary\ layer$. This phenomenon is explained by the fact that the fluid sticks to the surface of a solid body and, this adhesion inhibits the motion of a thin layer of fluid adjacent to the surface. In this thin region the velocity of the flow past a body at rest undergoes a sharp increase: from zero at the surface to the values of the velocity in the outer flow, where the fluid may be regarded as frictionless. $Prandtl$ derived the system of equations for the first approximation of the flow velocity in the boundary layer. This system served as a basis for the development of the boundary layer theory, which has now become one of the fundamental parts of fluid dynamics. Assume that the motion of a fluid occupying a two-dimensional region is characterized by the velocity vector $V = (u,\, v)$, where $u,\, v$ are the projections of V onto the coordinate axes $x,\, y$, respectively, the $Prandtl$ system for a non-stationary boundary layer arising in an axially symmetric incompressible flow past a solid body has the form

$$\begin{cases} u_t + uu_x + vu_y = U_t + UU_x + \nu u_{yy}, \\ (ru)_x + (rv)_y = 0, \\ u(0,\, x,\, y) = u_0(x,\, y),\, u(t,\, 0,\, y) = 0, \\ u(t,\, x,\, 0) = 0,\, v(t,\, x,\, 0) = v_0(t,\, x), \\ \lim\limits_{y\to\infty} u(t,\, x,\, y) = U(t,\, x), \end{cases} \tag{1}$$

in a domain $\mathcal{D} = \{(t,\, x,\, y) \mid 0 < t < T,\, 0 < x < X,\, 0 < y < \infty\}$, where ν=const> 0 is the coefficient of kinematic viscosity; $U(t,\, x)$ is called the velocity at the outer edge of the boundary layer, $U(t,\, 0) = 0$, $U(t,\, x) > 0$ for $x > 0$; $r(x)$ is the distance from that point to the axis of a rotating body, $r(0) = 0$, $r(x) > 0$ for $x > 0$.

In recent decades, many scholars have been carrying out research in this field, achievements are abundant in literature on theoretical, numerical experimental aspects of the theory[2, 3]. However, some discrepancies were founded between theoretical and experimental results for several important practical problems. $Prandtl$ boundary theory does not consider both the influence of wall's properties on the characteristic of the boundary layer and the interaction of the actual solid wall with the flow of water. In the absence of chemical reactions and chemical absorption, the polar interaction plays the most important role. Under different kinds of solid surface, the absorption of water molecule's behavior are different from one to another. This fact also makes that the corresponding boundary layer's characteristics are different from one to another. When the interfacial adsorption is relatively stronger, at this time, experiments show that: the boundary layer system embodies the micro-fluid characteristic. To be specific, we should modify the $Prandtl$ system as[4]

$$\begin{cases} u_t + uu_x + vu_y = U_t + UU_x + (\nu(y)u_y)_y, \\ u_x + v_y = 0, \\ u(0,\, x,\, y) = u_0(x,\, y),\; u(t,\, 0,\, y) = 0, \\ u(t,\, x,\, 0) = 0,\; v(t,\, x,\, 0) = v_0(t,\, x), \\ \lim_{y\to\infty} u(t,\, x,\, y) = U(t,\, x), \end{cases} \tag{2}$$

where $(t,\, x,\, y) \in \mathcal{D}$, and $\nu(y) > 0$. In this paper, we get the following result.

Theorem 1 *Assume that U_x, U_t/U, U, v_0 are bounded functions having bounded derivatives with respect to t, x in $\mathcal{D} = \{(t,\, x,\, y) \mid 0 < t < T, 0 < x < X, 0 < y < \infty\}$, where T is sufficient small; $\nu(y) > 0$ is a bounded function of which the first, the second and the third order derivatives are bounded in $\mathcal{D}$; $\lim_{y\to\infty} u_0(x,y) = U(0,x)$, $u_0(x,0) = 0$; u_0/U, u_{0y}/U are continuous in $\overline{\mathcal{D}}$; $u_{0y} > 0$ for $y \geq 0, x > 0$,*

$$K_1(U(0,\, x) - u_0(x,\, y)) \\ \leq u_{0y}(x,\, y) \leq K_2(U(0,x) - u_0(x,y)),$$

with positive constants K_1 and K_2. Assume also that there exist bounded derivatives u_{0y}, u_{0yy}, u_{0yyy}, u_{0x}, u_{0xy}, and the ratios

$$\frac{u_{0yy}}{u_{0y}}, \frac{u_{0yyy}u_{0y} - u_{0yy}^2}{u_{0y}^2}$$

are bounded on the rectangle $\{0 \leq x \leq X,\, 0 \leq y < \infty\}$. Moreover, u_0, v_0 satisfy the following compatibility condition

$$v_0(0,\, x)u_{0y}(x,\, 0) \\ = U_t(0,\, x) + U(0,\, x)U_x(0,\, x) + (\nu(0)u_{0y})_y(x,\, 0), \tag{3}$$

and let

$$\left|\frac{u_{0yx} - u_{0x}u_{0yy}}{u_{0y}} + U_x\frac{u_0u_{0yy} - u_{0y}^2}{Uu_{0y}}\right| \\ \leq K_5(U - u_0(x,y)).$$

Then, problem (2) in $\mathcal{D}$ has a unique solution $(u,\, v)$ with the following properties: u/U, u_y/U are continuous and bounded in $\overline{\mathcal{D}}$; $u_y/U > 0$ for $y \geq 0$; $\lim_{y\to\infty} u_y/U = 0$, $u(t,\, x,\, 0) = 0$; v is continuous in y and bounded for bounded y; the weak derivatives u_t, u_x, v_y, u_{yt}, u_{yx}, u_{yy}, u_{yyy} are bounded measurable functions in $\mathcal{D}$; the first equation in (2) hold almost everywhere in $\mathcal{D}$; the functions u_t, u_x, v_y, u_{yy} are continuous with respect to y; moreover

$$\frac{u_{yy}}{u_y}, \frac{u_{yyy}u_y - u_{yy}^2}{u_y^2}, \tag{4}$$

are bounded and the following inequality hold:

$$C_1(U(t,\, x) - u(t,\, x,\, y)) \leq u_y(t,\, x,\, y) \\ \leq C_2(U(t,\, x) - u(t,\, x,\, y)), \tag{5}$$

$$\begin{array}{l} \exp(-C_2y) \leq 1 - \frac{u(t,x,y)}{U(t,x)} \leq \exp(-C_1y), \\ \left|\frac{u_{yx}u_y - u_xu_{yy}}{u_y} + U_x\frac{uu_{yy} - u_y^2}{Uu_y}\right| \leq C_3(U - u), \\ \left|\frac{u_{yt}u_y - u_tu_{yy}}{u_y} + U_t\frac{uu_{yy} - u_y^2}{Uu_y}\right| \leq C_4(U - u). \end{array} \tag{6}$$

Although our method is similar to $Oleinik$'s[5], however, ν is related to y, which makes the corresponding transform and the calculation become more complicated. Moreover, if we permit $\nu(y)$ or its derivatives to be a unbounded functions, then, $Oleinik$'s linear method will be invalid, in these circumstances, to get the same results seem very difficult. By the way, there are some important progresses to the existence of the global solutions of (1) in recent recent years, one can refer to [7-9] et.al.

2 Some Important Lemmas

Consider the following initial boundary problem

$$\begin{array}{c} u_t + uu_x + vu_y = U_t + UU_x + (\nu(y)u_y)_y \\ u_x + v_y = 0, \end{array} \tag{7}$$

where

$$(t,\, x,\, y) \in \mathcal{D} = \{(t,\, x,\, y) \mid: \\ 0 < t < T,\, 0 < x < X,\, 0 < y < \infty\},$$

and with the following initial boundary conditions

$$\begin{cases} u(0,\, x,\, y) = u_0(x,\, y),\; u(t,\, 0,\, y) = 0, \\ u(t,\, x,\, 0) = 0,\; v(t,\, x,\, 0) = v_0(t,\, x), \\ \lim_{y\to\infty} u(t,\, x,\, y) = U(t,\, x), \end{cases} \tag{8}$$

Definition 2 *A solution of problem (7)-(8) is a pair of functions $u(t,\, x,\, y)$, $v(t,\, x,\, y)$ with the following properties: $u(t,\, x,\, y)$ is continuous and bounded in $\overline{\mathcal{D}}$; $v(t,\, x,\, y)$ is continuous with respect to y in $\overline{\mathcal{D}}$ and bounded for bounded y; the weak derivatives u_t, u_x, u_y, u_{yy}, v_y are bounded measurable functions; equation (7) holds for u, v in $\mathcal{D}$, and conditions of (8) are satisfied.*

If we introduce the Crocco transformation

$$\tau = t,\, \xi = x,\, \eta = \frac{u(t,\, x,\, y)}{U(t,\, x)},$$

we obtain the following equation for $w(\tau,\, \xi,\, \eta) = u_y(t,\, x,\, y)/U(t,\, x)$:

$$\begin{aligned}&\nu w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w \\ &+2\nu_\eta w^2 w_\eta + \nu_{\eta\eta} w^3 = 0,\, (\tau,\, \xi,\, \eta) \in \Omega,\end{aligned} \tag{9}$$

$$\begin{cases} w|_{\tau=0} = \frac{u_{0y}}{U} = w_0(\xi,\, \eta),\, w|_{\eta=1} = 0, \\ (\nu w w_\eta + \nu_\eta w^2 - v_0 w + C)|_{\eta=0} = 0, \end{cases} \tag{10}$$

in the domain $\Omega = \{(\tau,\, \xi,\, \eta) \mid 0 < \tau < T,\, 0 < \xi < X,\, 0 < \eta < 1\}$, where

$$A = (\eta^2 - 1)U_\xi + (\eta - 1)U_\tau/U,$$

$$B = -\eta U_\xi - U_\tau/U,$$

$$C = U_\xi + U_\tau/U.$$

Solutions of problem (9)-(10) are understood in the weak sense.

Definition 3 *A solution of problem (9) is a function $w(\tau,\, \xi,\, \eta)$ with the following properties: $w(\tau,\, \xi,\, \eta)$ is continuous in $\overline{\Omega}$; the weak derivatives w_τ, w_ξ, w_η are bounded functions in Ω, w_η is continuous with respect to η at $\eta = 0$ and its weak derivative $w_{\eta\eta}$ is such that $w w_{\eta\eta}$ is bounded in $\overline{\Omega}$; equation (9) hold almost everywhere in Ω for w, and conditions of (10) are satisfied.*

For any function $f(\tau,\, \xi,\, \eta)$, we use the following notation:

$$f^{m,k}(\eta) = f(\eta,\, mh,\, kh),\, h = const > 0.$$

Instead of equation (9), let us consider the following system of ordinary differential equations:

$$\begin{aligned}&\nu(w^{m-1,k} + h)^2 w^{m,k}_{\eta\eta} - \tfrac{w^{m,k} - w^{m-1,k}}{h} \\ &-\eta U^{m,k} \tfrac{w^{m,k} - w^{m,k-1}}{h} + A^{m,k} w^{m,k}_\eta \\ &+B^{m,k} w^{m,k} + 2\nu_\eta (w^{m-1,k})^2 w^{m,k}_\eta \\ &+\nu_{\eta\eta}(w^{m-1,k})^2 w^{m,k} = 0,\end{aligned} \tag{11}$$

$$w^{0,k} = w^h_0(kh,\, \eta),\, w(1)^{m,k} = 0, \tag{12}$$

$$\begin{aligned}&[\nu w^{m-1,k} w^{m,k}_\eta + \nu_\eta (w^{m-1,k})^2 \\ &-v^{m,k}_0 w^{m-1,k} + C^{m,k}]|_{\eta=0} = 0,\end{aligned} \tag{13}$$

where $m = 1,\, 2,\, \cdots,\, [T/h]$; $k = 0,\, 1,\, 2,\, \cdots,\, [X/h]$.

We take $w^h_0 \equiv w_0(\xi,\, \eta)$ if w_0 has bounded derivatives $w_{0\xi}$, $w_{0\eta}$ and $w_{0\eta\eta}$. If $w_0(\xi,\, \eta)$ is not so smooth, we take for w^h_0 a certain smooth function (to be constructed below) which uniformly converges to w_0 in the domain $\{0 < \xi < X,\, 0 < \eta < 1\}$ as $h \to 0$.

In what follows K_i, M_i, C_i stand for positive constants independent of h.

Lemma 4 *Assume that A, B, C, v_0 are bounded functions in Ω. Let w^h_0 be continuous in $\eta \in [0,\, 1]$ and such that $K_1(1 - \eta) \le w^h_0 \le K_2(1 - \eta)$, $\nu(y)$ is bounded function having bounded first and second order derivatives in Ω. Then problem (12)-(13) for ordinary differential equations admits a unique solution for $mh \le T_0$ and small enough h, where $T_0 > 0$ is a constant which depends on the data of problem (7). The solution $w^{m,k}$ of problem (12)-(13) satisfies the following estimate:*

$$V(mh,\, \eta) \le w^{m,k}(\eta) \le V_1(mh,\, \eta), \tag{14}$$

where V and V_1 are continuous functions in $\overline{\Omega}$, positive for $\eta < 1$ and such that $V \equiv K_3(1 - \eta)$ and $V_1 \equiv K_4(1 - \eta)$ in a neighborhood of $\eta = 1$.

Proof: The existence of this solution follows from its uniqueness which, in its turn, can be established on the basis of the maximum principle and the fact that this problem can be reduced, with the help of the $Green$ function, to a $Fredholm$ integral equation of the second kind.

Indeed, let $Q^{m,k}$ be the difference of two solutions $w^{m,k}$ of problem (12)-(13). Then $Q^{m,k}$ can attain neither a positive maximum nor a negative minimum at $\eta = 0$, since otherwise $Q^{m,k}_\eta(0) \neq 0$ (see [6]), whereas the boundary condition

$$\left[\nu w^{m-1,k} w^{m,k}_\eta + \nu_\eta (w^{m-1,k})^2 - v^{m,k}_0 w^{m-1,k} + C^{m,k}\right]\Big|_{\eta=0} = 0,$$

implies that $Q^{m,k}_\eta(0) = 0$. We also have $Q^{m,k}(1) = 0$, and at the interior points of $[0,\, 1]$, $Q^{m,k}$ can neither attain a positive maximum nor a negative minimum. Consequently, under our assumptions, problem (12)-(13) can not have more than one solution. Therefore, we shall have a fortiori establish the solvability of problem (12)-(13) for m and k such that the solutions $w^{m-1,k}$ of problem (12)-(13) admit the following a priori estimate:

$$w^{m-1,k}(\eta) \geq V((m-1)h, \eta) \tag{15}$$

In order to prove the priori estimate (14) for $\tau = mh$, it suffices to show that there exist functions V and V_1 with the properties in Lemma 4 and such that ordinary differential equations:

$$\begin{array}{l} L_m(V) \equiv \nu(w^{m-1,k}+h)^2 V^{m,k}_{\eta\eta} \\ -\frac{V^{m,k}-V^{m-1,k}}{h} - \eta U^{m,k}\frac{V^{m,k}-V^{m,k-1}}{h} \\ +A^{m,k}V^{m,k}_\eta + B^{m,k}V^{m,k} + 2\nu_\eta(w^{m-1,k})^2 V^{m,k}_\eta \\ +\nu_{\eta\eta}(w^{m-1,k})^2 V^{m,k} \geq 0, \end{array} \tag{16}$$

$$\begin{array}{l} \lambda_m(V) = \nu(0)w^{m-1,k}(0)V^{m,k}_\eta(0) \\ +\nu(0)_\eta(w^{m-1,k}(0))^2 - v^{m,k}_0 w^{m-1,k}(0) + C^{m,k} \\ > 0, \end{array} \tag{17}$$

$$\begin{array}{c} L_m(V_1) \leq 0, \lambda_m(V_1) < 0, \\ k = 0,\, 1,\, 2,\, \cdots,\, [X/h], \end{array} \tag{18}$$

under the assumption that

$$V((m-1)h,\, \eta) \leq w^{m-1,k}(\eta) \leq V_1((m-1)h,\, \eta). \tag{19}$$

Then the inequalities (14) can be proved by induction with respect to m.

Indeed, consider the function $q^{m,k} = V(mh,\, \eta) - w^{m,k}$, where $w^{m,k}$ is the solution of problem (12)-(13). We have

$$L_m(q) \geq 0, \tag{20}$$

$$\lambda_m(V) - \lambda_m(w) = \nu(0)w^{m-1,k}(0)q^{m,k}_\eta(0) > 0. \tag{21}$$

Moreover, by assumption we have $q^{m',k} \leq 0$ for $m' \leq m-1$, and $q^{m,k}(1) = 0$. Let us show that $q^{m,k} \leq 0$. To this end , we introduce new functions by $q^{m,k} = e^{\alpha mh}S^{m,k}$, $\alpha = const. > 0$, Then

$$\begin{array}{l} L_m(q) \equiv e^{\alpha mh}\Big[\nu(w^{m-1,k}+h)^2 S^{m,k}_{\eta\eta} \\ -\eta U^{m,k}\frac{S^{m,k}-S^{m,k-1}}{h} + A^{m,k}S^{m,k}_\eta \\ +2\nu_\eta(w^{m-1,k})^2 S^{m,k}_\eta \\ +\nu_{\eta\eta}(w^{m-1,k})^2 S^{m,k} + B^{m,k}S^{m,k} \\ -\frac{1}{h}(1-e^{-\alpha h})S^{m,k} - e^{-\alpha h}\frac{S^{m,k}-S^{m-1,k}}{h}\Big] \\ \geq 0, \end{array} \tag{22}$$

$$\lambda_m(q) = e^{\alpha mh}\nu(0)w^{m-1,k}(0)S^{m,k}_\eta(0) > 0. \tag{23}$$

It follows that $S^{m,k}_\eta(0) > 0$, then $S^{m,k}$ cannot take its maximum positive value at $\eta = 0$, moreover, $S^{m,k}(1) = 0$. If $S^{m,k}$ attains the maximum positive value at an interior point of the interval $0 \leq \eta \leq 1$, then at this point we must have

$$S^{m,k}_{\eta\eta} \leq 0, \tag{24}$$

$$\frac{S^{m,k}-S^{m-1,k}}{h} \geq 0, \tag{25}$$

$$\eta U^{m,k}\frac{S^{m,k}-S^{m,k-1}}{h} \geq 0, \tag{26}$$

$$S^{m,k}_\eta = 0, \tag{27}$$

$$\begin{array}{l} \Big[B^{m,k} + \nu_{\eta\eta}(w^{m-1,k})^2 \\ -\frac{1}{h}(1-e^{-\alpha h})\Big]S^{m,k} < 0, \end{array} \tag{28}$$

provided that the constant α is large enough and h is sufficiently small, such that $1 - e^{-\alpha h} > 1/2$. However, these relations are incompatible with (22). Therefore,

$$q^{m,k} = e^{\alpha mh}S^{m,k} \leq 0, \tag{29}$$

That is to say,

$$V(mh, \eta) \leq w^{m,k}. \tag{30}$$

In a similar way we can show that (18) and (19) imply the inequality $w^{m,k} \leq V_1(mh,\, \eta)$.

Now let us show that there is a positive T_0 such that for $mh \leq T_0$ there exist functions V and V_1 satisfying the inequalities (16), (17) and (18), under the condition (19).

Set

$$V = \mu\kappa(\alpha_1\eta)\kappa_1(\eta)e^{-\alpha_2\tau}$$

where κ is a smooth function such that

$$\kappa(s) = \begin{cases} e^s, & \text{for } 0 \leq s \leq 1; \\ 1 \leq \kappa \leq 3, & \text{for } 1 \leq s \leq 3/2; \\ 1, & \text{for } s \geq 3/2. \end{cases}$$

κ_1 is a smooth function too, such that

$$\kappa_1(s) = \begin{cases} 1, & \text{for } s \leq 1/4; \\ 1/2 \leq \kappa_1 \leq 1, & \text{for } 1/4 < s < 1/2; \\ 1-s, & \text{for } s \geq 1/2. \end{cases}$$

The constant $\mu > 0$ is chosen such that $V(0,\, \xi,\, \eta) \leq w^h_0(\xi,\, \eta)$; the positive α_1, α_2 will be specified later.

Let us verify the inequalities (16) and (17) for V, assuming that (19) holds. Provided that $\alpha_1 > 0$ is large enough and mh is such that $e^{-\alpha_2 mh} > 1/2$, We have

$$\lambda_m(V) = \nu(0)w^{m-1,k}(0)\mu\alpha_1 e^{-\alpha_2 mh} +\nu_\eta(0)(w^{m-1,k}(0))^2 - v_0^{m,k}w^{m-1,k}(0) + C^{m,k} \geq \mu e^{-\alpha_2(m-1)h}\Big[\mu(\nu(0)\alpha_1 e^{-\alpha_2 mh} +\nu_\eta(0)e^{-\alpha_2(m-1)h}) - v_0^{m,k}\Big] + C^{m,k} > 0, \quad (31)$$

$$L_m(V) \equiv \mu e^{-\alpha_2 mh}\Big[\nu(w^{m-1,k}+h)^2(\kappa\kappa_1)_{\eta\eta} +A^{m,k}(\kappa\kappa_1)_\eta + B^{m,k}\kappa\kappa_1 + 2\nu_\eta(w^{m-1,k})^2(\kappa\kappa_1)_\eta +\nu_{\eta\eta}(w^{m-1,k})^2\kappa\kappa_1 + \kappa\kappa_1\alpha_2 e^{\alpha_2\tilde{h}}\Big], \quad (32)$$

where $0 < \tilde{h} < h$.

It may be assumed that $\kappa\kappa_1 = 1-\eta$ for $1-\eta < \sigma$, and small enough $\sigma > 0$. For such η, we have $L_m(V) \geq 0$ if α_2 is sufficiently large. For $\eta < 1-\sigma$, we have $V > \rho = const. > 0$, and therefore, $L_m(V) > 0$ for large enough α_2, provided that $e^{-\alpha_2 mh} > 1/2$.

Set

$$V_1 = M\kappa_1(\eta)\kappa_2(\beta_1\eta)e^{\beta_2 mh}$$

κ_2 is a smooth function such that

$$\kappa_2(s) = \begin{cases} 4-e^s, & \text{for } 0 \leq s \leq 1; \\ 1 \leq \kappa_2 \leq 3, & \text{for } 1 \leq s \leq 2; \\ 1, & \text{for } s \geq 2. \end{cases}$$

The constant $M > 0$ is chosen such that $V_1(0,\xi,\eta) \geq w_0^h(\xi,\eta)$; the positive β_1, β_2 will be specified shortly. In a similar way we can prove that $L_m(V_1) \leq 0$, $\lambda_m(V_1) < 0$, provided that $\beta_1 > 0$ is large enough and mh is such that $e^{-\beta_2 mh} > 1/2$.

Thus, if $mh \leq T_0$ and T_0 is such that $e^{\beta_2 T_0} \leq 2$ and $e^{-\alpha_2 T_0} \geq 1/2$, then problem (12)-(13) with small enough h admits a unique solution which satisfies the inequality (14).

In what follows, we takes as $w_0^h(\xi,\eta)$ the function $w_0(\xi,\eta)$ if $w_{0\eta\eta}$ is bounded in Ω; otherwise we let $w_0^h(\xi,\eta)$ be a function coinciding with w_0 for $\eta \leq 1/2$, equal to $w_0(\xi,\eta-h) - w_0(\xi,1-h)$ for $1/2+h \leq h < 1$, and defined on the interval $1/2 \leq \eta \leq 1/2+h$ in such a way that for $1/4 \leq \eta \leq 3/4$ it has uniformly (in h) bounded derivatives which are known to be bounded for w_0.

Lemma 5 *Assume that the conditions of Lemma 4 are fulfilled and functions A, B, C, v_0, w_0 have bounded first order derivatives, $|w_{0\xi}| \leq K_5(1-\eta)$, $w_0(\xi,1) = 0$, $w_0 w_{0\eta\eta}$ is bounded in Ω, $\nu(y)$ has bounded third order derivative, and the following compatibility condition is satisfied:*

$$\nu(0)w_0 w_{0\eta} + \nu(0)_\eta w_0^2 - v_0 w_0 + C = 0, \quad (33)$$

Then

$$w_\eta^{m,k},\ \frac{w^{m,k}-w^{m,k-1}}{h},\ \frac{w^{m,k}-w^{m-1,k}}{h},$$

and

$$(1-\eta+h)w_{\eta\eta}^{m,k},$$

are bounded in Ω for $mh \leq T_1 \leq T_0$ and $h \leq h_0$, uniformly with respect to h. The positive constants T_1 and h_0 are determined by the data of problem (7)-(8).

Proof: Let us introduce a new unknown function $W^{m,k} = w^{m,k}e^{\alpha\eta}$ in problem (12)-(13), where α is a positive constant which does not depend on h and will be chosen later. We have

$$\nu(w^{m-1,k}+h)^2 W_{\eta\eta}^{m,k} - \frac{W^{m,k}-W^{m-1,k}}{h} -\eta U^{m,k}\frac{W^{m,k}-W^{m,k-1}}{h} + \widetilde{A}^{m,k}W_\eta^{m,k} +\widetilde{B}^{m,k}W^{m,k} + 2\nu_\eta(w^{m-1,k})^2 W_\eta^{m,k} +\nu_{\eta\eta}(w^{m-1,k})^2 W^{m,k} = 0, \quad (34)$$

and

$$\nu(0)W^{m-1,k}(0)W_\eta^{m,k}(0) - \alpha\nu(0)W^{m-1,k}(0)W^{m,k}(0) +\nu_\eta(0)(W^{m-1,k}(0))^2 - -v_0^{m,k}W^{m-1,k}(0) + C^{m,k} = 0, \quad (35)$$

where

$$\widetilde{A}^{m,k} = A^{m,k} - 2\alpha\nu(w^{m-1,k}+h)^2,$$

$$\widetilde{B}^{m,k} = B^{m,k} + \alpha^2\nu(w^{m-1,k}+h)^2 -\alpha A^{m,k} - 2\alpha\nu_\eta(w^{m-1,k})^2.$$

We introduce the functions

$$\rho^{m,k} = \frac{W^{m,k}-W^{m-1,k}}{h},\ \gamma^{m,k} = \frac{W^{m,k}-W^{m,k-1}}{h}.$$

The function $\Phi^{m,k}(\eta)$ defined by

$$\Phi^{m,k}(\eta) = (W_\eta^{m,k})^2 + (\rho^{m,k})^2 +(\gamma^{m,k})^2 + K_6\eta + 1, \quad (36)$$

for $k \geq 1$, $m \geq 1$;

$$\Phi^{m,k}(\eta) = (W_\eta^{m,k})^2 + (\rho^{m,k})^2 + K_6\eta + 1, \quad (37)$$

for $k = 0$, $m \geq 1$. The constant $K_6 > 0$ will be chosen below.

Consider $\Phi_\eta^{m,k}(0)$ for $k \geq 1$, $m \geq 1$. We have

$$\Phi_\eta^{m,k}(0) = 2W_\eta^{m,k}(0)W_{\eta\eta}^{m,k}(0) +2\rho^{m,k}(0)\rho_\eta^{m,k}(0) + 2\gamma^{m,k}(0)\gamma_\eta^{m,k}(0) + K_6. \quad (38)$$

From the boundary condition (36) and the estimates (14), it follows that

$$\left|W_\eta^{m,k}(0)\right| \le K_7, \tag{39}$$

From (35) and the estimates for $w^{m-1,k}$ established in Lemma 2.1, we find that

$$W_{\eta\eta}^{m,k}(0) = R_1^{m,k}\rho^{m,k}(0) + R_2^{m,k}\gamma^{m,k}(0) + R_3^{m,k}, \tag{40}$$

for $0 \le mh \le T_0$, and

$$R_1^{m,k}(0) = \frac{1}{\nu(w^{m-1,k}+h)^2},$$

$$R_2^{m,k}(0) = \frac{\eta U^{m,k}}{\nu(w^{m-1,k}+h)^2} \equiv 0,$$

$$R_3^{m,k}(0) = \frac{-1}{\nu(w^{m-1,k}+h)^2}\Big[\widetilde{A}^{m,k}W_\eta^{m,k} + \widetilde{B}^{m,k}W^{m,k} + 2\nu_\eta(w^{m-1,k})^2W_\eta^{m,k} + \nu_{\eta\eta}(w^{m-1,k})^2W^{m,k}\Big],$$

where $R_i^{m,k}$ are bounded uniformly in h.

Using the boundary condition (36), we find $\rho_\eta^{m,k}(0)$, $\gamma_\eta^{m,k}(0)$ for $m \ge 1$; in order to calculate $\rho_\eta^{m,k}(0)$ we have to utilize the compatibility condition (2.11) which holds for w_0, and therefore, $w_0^h = W_0^k e^{-\alpha\eta} = w^{0,k}$ for $\eta = 0$, since $w_0^h \equiv w_0$ for $\eta \le 1/2$ by construction. We have

$$\begin{array}{l}\rho_\eta^{m,k}(0) = \alpha\rho^{m,k}(0) + \frac{v_0^{m,k}-v_0^{m-1,k}}{\nu(0)h} \\ +\frac{C^{m,k}\rho^{m-1,k}(0)}{\nu(0)W^{m-1,k}(0)W^{m-2,k}(0)} \\ -\frac{\nu(0)_\eta\rho^{m-1,k}(0)}{\nu(0)} - \frac{C^{m,k}-C^{m-1,k}}{\nu(0)hW^{m-2,k}(0)},\end{array} \tag{41}$$

for $m > 1$, and

$$\rho_\eta^{1,k}(0) = \alpha\rho^{1,k}(0) + \frac{v_0^{1,k}-v_0^{0,k}}{\nu(0)h} - \frac{C^{1,k}-C^{0,k}}{\nu(0)hW^{0,k}(0)},$$

Similarly, for $k \ge 1$

$$\begin{array}{l}\gamma_\eta^{m,k}(0) = \alpha\gamma^{m,k}(0) + \frac{v_0^{m,k}-v_0^{m,k-1}}{\nu(0)h} \\ +\frac{C^{m,k}\gamma^{m-1,k}(0)}{\nu(0)W^{m-1,k}(0)W^{m-1,k-1}(0)} \\ -\frac{\nu(0)_\eta\gamma^{m-1,k}(0)}{\nu(0)} - \frac{C^{m,k}-C^{m,k-1}}{\nu(0)hW^{m-1,k-1}(0)},\end{array} \tag{42}$$

Substituting the expressions found for $w_{\eta\eta}^{m,k}(0)$, $\rho_\eta^{m,k}(0)$, $\gamma_\eta^{m,k}(0)$ into the right-hand side of (39), we obtain the following relation for $k \ge 1$

$$\begin{array}{l}\Phi_\eta^{m,k}(0) = K_6 + 2\alpha[\rho^{m,k}(0)]^2 + 2\alpha[\gamma^{m,k}(0)]^2 \\ +R_4^{m,k}\rho^{m,k}(0)\rho^{m-1,k}(0) + R_5^{m,k}\rho^{m,k}(0) \\ +R_6^{m,k}\gamma^{m,k}(0)\gamma^{m-1,k}(0) + R_7^{m,k}\gamma^{m,k}(0) + R_8^{m,k},\end{array} \tag{43}$$

where

$$R_4^{m,k}(0) = 2\frac{C^{m,k} - \nu_\eta W^{m-1,k}W^{m-2,k}}{\nu W^{m-1,k}W^{m-2,k}},$$

$$R_5^{m,k}(0) = 2\frac{v_0^{m,k} - v_0^{m-1,k}}{\nu h} - 2\frac{C^{m,k} - C^{m-1,k}}{\nu h W^{m-2,k}} + 2W_\eta^{m,k}R_1^{m,k},$$

$$R_6^{m,k}(0) = 2\frac{C^{m,k} - \nu_\eta W^{m-1,k}W^{m-1,k-1}}{\nu W^{m-1,k}W^{m-1,k-1}},$$

$$R_7^{m,k}(0) = 2\frac{v_0^{m,k} - v_0^{m-1,k}}{\nu h} - 2\frac{C^{m,k} - C^{m-1,k}}{\nu h W^{m-1,k-1}},$$

$$R_8^{m,k}(0) = 2W_\eta^{m,k}R_3^{m,k},$$

$R_i^{m,k}$ are bounded uniformly in h, while $R_4^{1,k} = 0$. These relations imply that for $m > 1$, $k \ge 1$, we have

$$\begin{array}{l}\Phi_\eta^{m,k}(0) \ge K_6 + \alpha\Phi^{m,k}(0) \\ -K_8\Phi^{m,k}(0) - K_9\Phi^{m-1,k}(0) - K_{10},\end{array} \tag{44}$$

where the constant $K_{10} = K_{10}(\alpha)$. Let us choose α such that $\alpha/2 > K_8$, $\alpha/4 > K_9$, $K_6 > K_{10}$. Then

$$\Phi_\eta^{m,k}(0) \ge \frac{\alpha}{2}\Phi^{m,k}(0) - \frac{\alpha}{4}\Phi^{m-1,k}(0), \tag{45}$$

for $m = 2, 3, \cdots; k = 1, 2, \cdots, [X/h]$, $mh \le T_0$. Note that $\gamma^{m-1,k}(0)$, with $m = 1$, is uniformly bounded in h, and since $R_4^{1,k} = 0$, it follows from (44) that

$$\Phi_\eta^{1,k}(0) \ge \frac{\alpha}{2}\Phi^{1,k}(0), \tag{46}$$

if K_6 is sufficiently large. Likewise, the inequalities (45) and (46) can be proved for $k = 0$, $m \ge 1$.

Let us define the functions $\Phi^{0,k}$. For this purpose, we introduce the functions $W^{-1,k}$ by

$$\begin{array}{l}\frac{W^{0,k}-W^{-1,k}}{h} = \nu(w^{0,k}+h)^2W_{\eta\eta}^{0,k} \\ -\eta U^{0,k}\frac{W^{0,k}-W^{0,k-1}}{h} + \widetilde{A}^{0,k}W_\eta^{0,k} + \widetilde{B}^{0,k}W^{0,k} \\ +2\nu_\eta(w^{0,k})^2W_\eta^{0,k} + \nu_{\eta\eta}(w^{0,k})^2W^{0,k},\end{array} \tag{47}$$

where

$$\widetilde{A}^{0,k} = A^{0,k} - 2\alpha\nu(W^{0,k}e^{-\alpha\eta}+h)^2,$$

$$\widetilde{B}^{0,k} = B^{0,k} + \alpha^2\nu(W^{0,k}e^{-\alpha\eta}+h)^2 - \alpha A^{0,k} - 2\alpha\nu_\eta(W^{0,k}e^{-\alpha\eta})^2.$$

Then, we can define the function $\Phi^{0,k}$ for $k \ge 1$, and $k = 0$, respectively, by (37) and (38). It is easy to see that $\Phi^{0,k}$ is bounded uniformly in h. Indeed, the functions

$$W_\eta^{0,k} = (w^{0,k}e^{\alpha\eta})_\eta = (w_0^h(kh,\,\eta)e^{-\alpha\eta})_\eta,$$
$$\gamma^{0,k} = e^{\alpha\eta}\frac{w^{0,k}-w^{0,k-1}}{h}$$
$$= e^{\alpha\eta}\frac{w_0^h(kh,\eta)-w_0^h\big((k-1)h,\eta\big)}{h},$$

are bounded uniformly in h, since the first derivatives of w_0^h are uniformly bounded in h. The function $w_0 w_{0\eta\eta}$ is bounded in Ω and

$$K_1(1-\eta) \le w_0 \le K_2(1-\eta),$$

Therefore, $(1-\eta+h)w_{0\eta\eta}^h(\xi,\,\eta)$ which, for $\eta > 1/2+h$, coincides with the function $(1-\eta+h)w_{0\eta\eta}(\xi,\,\eta-h)$ is also uniformly bounded in h. It follows that

$$|(w^{0,k}+h)W_{\eta\eta}^{0,k}| \le K_{11}(1-\eta+h)|(w^{0,k}e^{\alpha\eta})_{\eta\eta}|$$
$$\le K_{12} + K_{13}(1-\eta+h)|w_{0\eta\eta}^{0,k}|$$
$$= K_{12} + K_{13}(1-\eta+h)|w_{0\eta\eta}^{h}|$$
$$\le K_{14}.$$

Consequently, the ratio $(W^{0,k}-W^{-1,k})/h$ defined by (48) is uniformly bounded in h, and

$$|\Phi^{0,k}| \le K_{15}. \qquad (48)$$

Now, we will deduce the equation for $\Phi^{m,k}(\eta)$ on the interval $0 \le \eta < 1$. To this end, we differentiate equation (35) in η and multiply the result by $2W_\eta^{m,k}$; then, we subtract from equation (35) for $W^{m,k}$ equation (35) for $W^{m-1,k}$ and multiply the result by $2\rho^{m,k}/h$; from equation (35) for $W^{m,k}$ we subtract (35) for $W^{m,k-1}$ and multiply the result by $2r^{m,k}/h$. Taking the sum of the three equations just obtained we get the equation for $\Phi^{m,k}(\eta)$, $m = 1,\,2,\,\cdots,\,[T/h]$; $k = 0,\,1,\,2,\,\cdots,\,[X/h]$.

We find the equations for $\Phi^{m,k}(\eta)$ with $k = 0, m \ge 1$ by taking only the first and the second of these equations. In order to derive the equation for $\Phi^{m,k}(\eta)$ with $m = 1$, we utilize the relation (48) which determines the values of $W^{-1,k}$.

For $m \ge 1$, $k \ge 1$, the equation for $\Phi^{m,k}(\eta)$ has the form

$$\nu(w^{m-1,k}+h)^2\Phi_{\eta\eta}^{m,k} - \frac{\Phi^{m,k}-\Phi^{m-1,k}}{h}$$
$$-\eta U^{m,k}\frac{\Phi^{m,k}-\Phi^{m,k-1}}{h} + \widetilde{A}^{m,k}\Phi_\eta^{m,k}$$
$$+2\widetilde{B}^{m,k}\Phi^{m,k} + 2\nu_\eta(w^{m-1,k})^2\Phi_\eta^{m,k}$$
$$+2\nu_{\eta\eta}(w^{m-1,k})^2\Phi^{m,k} + N_1^{m,k} - N_2^{m,k} = 0, \qquad (49)$$

where $N_2^{m,k}$ is a sum of non-negative terms:

$$N_2^{m,k} = 2\nu a^{m-1,k}(W_{\eta\eta}^{m,k}) + 2\nu a^{m-1,k}(\rho_\eta^{m,k})^2$$
$$+2\nu a^{m-1,k}(\gamma_\eta^{m,k})^2 + \frac{\eta U^{m,k}}{h}(\gamma_\eta^{m,k})^2$$
$$+\frac{1}{h}(\rho_\eta^{m,k}-\rho_\eta^{m-1,k})^2 + \frac{\eta U^{m,k}}{h}(\gamma_\eta^{m,k}$$
$$-\gamma_\eta^{m-1,k})^2 + \frac{1}{h}(\rho_\eta^{m,k})^2 + \frac{1}{h}(\rho_\eta^{m,k}$$
$$-\rho_\eta^{m,k-1})^2 + \frac{\eta U^{m,k}}{h}(\gamma_\eta^{m,k}-\gamma_\eta^{m,k-1})^2, \qquad (50)$$

and $N_1^{m,k}$ is a linear function whose coefficients are uniformly bounded in h and can be expressed in terms of the following quantities:

$$\nu_\eta a^{m-1,k}W_\eta^{m,k}W_{\eta\eta}^{m,k},\ \nu a_\eta^{m-1,k}W_\eta^{m,k}W_{\eta\eta}^{m,k},$$

$$W_\eta^{m,k}\gamma^{m,k},\ W_\eta^{m-1,k}\rho^{m,k}\rho^{m-1,k}, W_\eta^{m,k-1}\gamma^{m,k}\gamma^{m-1,k},$$

$$W_\eta^{m,k-1}\gamma^{m,k},\ W_\eta^{m,k}W_\eta^{m-1,k},$$

$$W_\eta^{m,k},\ (W_\eta^{m,k})^2W_\eta^{m-1,k}, W_\eta^{m-1,k}\rho^{m,k},\ (W_\eta^{m,k})^2,$$

$$\rho_\eta^{m,k}\gamma^{m-1,k},\ \rho^{m,k}\rho^{m-1,k},\ \gamma^{m,k}\gamma^{m,k-1},\ \gamma^{m,k},\ \rho^{m,k},$$

$$\frac{1}{h}\nu_\eta W_\eta^{m-1,k}\rho^{m,k}[(w^{m-1,k})^2-(w^{m-2,k})^2],$$

$$\frac{1}{h}W_{\eta\eta}^{m,k-1}\gamma^{m,k}(a^{m-1,k}-a^{m-1,k-1}),$$

$$\nu_{\eta\eta}w_\eta^{m-1,k}w^{m-1,k}W^{m,k}W_\eta^{m,k},$$

$$\frac{1}{h}\nu_\eta W_\eta^{m,k-1}\gamma^{m,k}[(w^{m-1,k})^2-(w^{m-1,k-1})^2],$$

$$\frac{1}{h}W_{\eta\eta}^{m-1,k}\rho^{m,k}(a^{m-1,k}-a^{m-2,k}),\ \gamma^{m,k}\gamma^{m-1,k},$$

$$\nu_{\eta\eta}(w^{m-1,k})^2(W_\eta^{m,k})^2, \nu_\eta w_\eta^{m-1,k}w^{m-1,k}(W_\eta^{m,k})^2,$$

$$\frac{1}{h}\nu_{\eta\eta}W^{m-1,k}\rho^{m,k}[(w^{m-1,k})^2-(w^{m-2,k})^2],$$

$$\frac{1}{h}\nu_{\eta\eta}W^{m,k-1}\gamma^{m,k}[(w^{m-1,k})^2-(w^{m-1,k-1})^2],$$

$$\nu_{\eta\eta\eta}(w^{m-1,k})^2W^{m,k}W_\eta^{m,k},$$

where $a^{m,k} = (w^{m,k}+h)^2$. Using the inequality

$$2ab \le \beta a^2 + \frac{b^2}{\beta}, (\beta > 0), \qquad (51)$$

to estimate the terms that make up $N_1^{m,k}$, we obtain from (50)

$$\nu(w^{m-1,k}+h)^2\Phi_{\eta\eta}^{m,k} - \frac{\Phi^{m,k}-\Phi^{m-1,k}}{h}$$
$$-\eta U^{m,k}\frac{\Phi^{m,k}-\Phi^{m,k-1}}{h} + \widetilde{A}^{m,k}\Phi_\eta^{m,k} + 2\widetilde{B}^{m,k}\Phi^{m,k}+$$
$$+2\nu_\eta(w^{m-1,k})^2\Phi_\eta^{m,k} + 2\nu_{\eta\eta}(w^{m-1,k})^2\Phi^{m,k}$$
$$+\widetilde{C}^{m,k}\Phi^{m,k} \ge 0, \qquad (52)$$

where $\widetilde{C}^{m,k}$ depend on

$$W_\eta^{m-1,k},\ \rho^{m-1,k},\ \gamma^{m-1,k},\ \gamma^{m,k-1},$$
$$(w^{m-1,k}+w^{m-2,k}+2h)W_{\eta\eta}^{m-1,k},$$
$$(w^{m-1,k}+w^{m-1,k-1}+2h)W_{\eta\eta}^{m,k-1},$$
$$\nu_{\eta\eta}(w^{m-1,k})^2,\ \nu_\eta w_\eta^{m-1,k}w^{m-1,k},$$
$$\nu_\eta W_\eta^{m,k-1}(w^{m-1,k}+w^{m-1,k-1}),$$
$$\nu_{\eta\eta\eta}(w^{m-1,k})^2W^{m,k},\ W_\eta^{m,k-1},$$
$$\nu_{\eta\eta}W^{m,k-1}(w^{m-1,k}+w^{m-1,k-1}),$$
$$\nu_\eta W_\eta^{m-1,k}(w^{m-1,k}+w^{m-2,k}),$$
$$\nu_{\eta\eta}W^{m-1,k}(w^{m-1,k}+w^{m-2,k}),$$
$$\nu_{\eta\eta}w_\eta^{m-1,k}w^{m-1,k}W^{m,k}$$

It is easy to see that for $k=1$ the coefficient $\widetilde{C}^{m,k}$ does not depend on $\gamma^{m,k-1}$, since $U^{m,0}=0$. The inequality (53) for $\Phi^{m,0}$ is obtained in exactly the same manner as for $k\geq 1$. Obviously, in this case the coefficient $\widetilde{C}^{m,k}$ depends only on

$$W_\eta^{m-1,k},\ \ \rho^{m-1,k},\ \ \gamma^{m-1,k},\ \ \gamma^{m,k-1},$$
$$(w^{m-1,k}+w^{m-2,k}+2h)W_{\eta\eta}^{m-1,k},$$
$$(w^{m-1,k}+w^{m-1,k-1}+2h)W_{\eta\eta}^{m,k-1},\ \ \nu_{\eta\eta}(w^{m-1,k})^2,$$
$$\nu_\eta w_\eta^{m-1,k}w^{m-1,k},\nu_{\eta\eta\eta}(w^{m-1,k})^2W^{m,k},\ \ W_\eta^{m,k-1},$$
$$\nu_\eta W_\eta^{m-1,k}(w^{m-1,k}+w^{m-2,k}),$$
$$\nu_{\eta\eta}W^{m-1,k}(w^{m-1,k}+w^{m-2,k}),$$
$$\nu_{\eta\eta}w_\eta^{m-1,k}w^{m-1,k}W^{m,k}$$

Now, consider the functions

$$Y^{m,k}(\eta)=(\rho^{m,k})^2+(\gamma^{m,k})^2+f(\eta), \qquad (53)$$

when $k\geq 1,\ m>0$,

$$Y^{m,k}(\eta)=(\rho^{m,k})^2+f(\eta), \qquad (54)$$

when $k=0,\ m>0$. Where $f(\eta)=\kappa(\beta\eta)\kappa_1^2(\eta)$, and the functions $\kappa,\ \kappa_1$ are those constructed in the proof of Lemma 4; β is a positive constant. Just as we have proved inequalities (45) and (46), we find that

$$Y_\eta^{m,k}(0)\geq\frac{\alpha}{2}Y^{m,k}(0)-\frac{\alpha}{4}Y^{m-1,k}(0), \qquad (55)$$

for $m>1,\ k\geq 1$,

$$Y_\eta^{m,k}(0)\geq\frac{\alpha}{2}Y^{m,k}(0), \qquad (56)$$

for $m=1,\ k\geq 1$, provided that $\alpha,\ \beta$ are sufficiently large. We clearly have $Y^{m,k}(1)=0$. Consider the functions $Y^{0,k}$. Obviously,

$$\left|\frac{W^{0,k}-W^{0,k-1}}{h}\right|\leq K_{16}(1-\eta+h),$$

since $|w_{0\xi}|\leq K_5(1-\eta)$ by the assumption of Lemma 4, and the functions $W^{0,k}=w_0^h(kh,\,\eta)e^{\alpha\eta}$, $w_{0\xi}^h$ are uniformly bounded in h. It follows from (48) that

$$\left|\frac{W^{0,k}-W^{-1,k}}{h}\right|\leq K_{17}(1-\eta+h),$$

since by virtue of (14) and the properties of $w_0^h=w^{0,k}=W^{0,k}e^{-\alpha\eta}$ we have

$$\left|\nu(w^{0,k}+h)^2W_{\eta\eta}^{0,k}(\eta)\right|\leq K_{18}(1-\eta+h), \qquad (57)$$

$$\left|\widetilde{A}^{0,k}W_\eta^{0,k}(\eta)\right|\leq K_{19}(1-\eta+h), \qquad (58)$$

$$\left|\widetilde{B}^{0,k}W^{0,k}(\eta)\right|\leq K_{20}(1-\eta+h), \qquad (59)$$

$$\left|\nu_\eta(w^{0,k})^2W_\eta^{0,k}(\eta)\right|\leq K_{21}(1-\eta+h), \qquad (60)$$

$$\left|\nu_{\eta\eta}(w^{0,k})^2W^{0,k}(\eta)\right|\leq K_{22}(1-\eta+h), \qquad (61)$$

Consequently,

$$\left|Y^{0,k}(\eta)\right|\leq K_{23}(1-\eta+h)^2. \qquad (62)$$

Let us write out the equation for $Y^{m,k}$. For $m\geq 1,\ k\geq 1$, this equation has the form

$$\begin{aligned}&\nu(w^{m-1,k}+h)^2Y_{\eta\eta}^{m,k}-\tfrac{Y^{m,k}-Y^{m-1,k}}{h}\\&-\eta U^{m,k}\tfrac{Y^{m,k}-Y^{m,k-1}}{h}+\widetilde{A}^{m,k}Y_\eta^{m,k}\\&+2\widetilde{B}^{m,k}Y^{m,k}+2\nu_\eta(w^{m-1,k})^2Y_\eta^{m,k}\\&+2\nu_{\eta\eta}(w^{m-1,k})^2Y^{m,k}-\nu(w^{m-1,k}+h)^2f_{\eta\eta}\\&-\widetilde{A}^{m,k}f_\eta-2\widetilde{B}^{m,k}f-2\nu_\eta(w^{m-1,k})^2f_\eta\\&-2\nu_{\eta\eta}(w^{m-1,k})^2f+N_3^{m,k}-N_4^{m,k}=0,\end{aligned} \qquad (63)$$

where $N_4^{m,k}$ is a sum of non-negative terms, namely

$$\begin{aligned}N_4^{m,k}&=2\nu a^{m-1,k}(W_{\eta\eta}^{m,k})+\tfrac{1}{h}(\rho_\eta^{m,k}\\&-\rho_\eta^{m-1,k})^2+\tfrac{\eta U^{m,k}}{h}(\gamma_\eta^{m,k}\\&-\gamma_\eta^{m-1,k})^2+2\nu a^{m-1,k}(\gamma_\eta^{m,k})^2\\&+\tfrac{1}{h}(\rho_\eta^{m,k}-\rho_\eta^{m,k-1})^2+\tfrac{\eta U^{m,k}}{h}(\gamma_\eta^{m,k}-\gamma_\eta^{m,k-1})^2,\end{aligned} \qquad (64)$$

and $N_3^{m,k}$ is a linear functions whose coefficients are uniformly bounded in h and can be expressed in terms of the following quantities:

$$\begin{array}{l}\rho^{m,k}\gamma^{m-1,k},\ W_\eta^{m-1,k}\rho^{m,k}\rho^{m-1,k},\\ \rho^{m,k}W_\eta^{m-1,k}(1-\eta),\ \rho^{m,k}w^{m-1,k},\\ \gamma^{m,k}\gamma^{m-1,k}W_\eta^{m,k-1},\ \rho^{m,k}\rho^{m-1,k},\\ \frac{1}{h}\nu_\eta\gamma^{m,k}[(w^{m-1,k-1})^2-(w^{m-1,k-1})^2]W_\eta^{m,k-1},\\ \frac{1}{h}\nu_{\eta\eta}\gamma^{m,k}[(w^{m-1,k-1})^2-(w^{m-1,k-1})^2]W^{m,k-1},\\ \gamma^{m,k}\gamma^{m-1,k},\ \gamma^{m,k}\gamma^{m,k-1},\ \rho^{m,k}w^{m-1,k},\\ \rho^{m,k}\rho^{m-1,k}(w^{m-1,k}+w^{m-2,k}+2h)W_{\eta\eta}^{m-1,k},\\ \frac{1}{h}\nu_\eta\rho^{m,k}[(w^{m-1,k})^2-(w^{m-2,k})^2]W_\eta^{m-1,k},\\ \frac{1}{h}\nu_{\eta\eta}\rho^{m,k}[(w^{m-1,k})^2-(w^{m-2,k})^2]W^{m-1,k},\\ \gamma^{m,k}\gamma^{m-1,k}(w^{m-1,k}+w^{m-1,k-1}+2h)W_{\eta\eta}^{m,k-1},\\ \gamma^{m,k}W_\eta^{m,k-1}(1-\eta).\end{array} \tag{65}$$

Using (52) to estimate the terms that make up $N_3^{m,k}$, we obtain the following inequality

$$\begin{array}{l}\nu(w^{m-1,k}+h)^2Y_{\eta\eta}^{m,k}-\frac{Y^{m,k}-Y^{m-1,k}}{h}\\ -\eta U^{m,k}\frac{Y^{m,k}-Y^{m,k-1}}{h}+\widetilde{A}^{m,k}Y_\eta^{m,k}\\ +2\widetilde{B}^{m,k}Y^{m,k}+2\nu_\eta(w^{m-1,k})^2Y_\eta^{m,k}\\ +2\nu_{\eta\eta}(w^{m-1,k})^2Y^{m,k}+Q_1^{m,k}Y^{m,k}+Q_2^{m,k}+Q_3\\ \geq 0,\end{array} \tag{66}$$

where $Q_1^{m,k}\geq 0$, and depends on

$$\begin{array}{l}W_\eta^{m-1,k},\ (w^{m-1,k}+w^{m-2,k}+2h)W_{\eta\eta}^{m-1,k},\\ \nu_\eta(w^{m-1,k}+w^{m-2,k})W_\eta^{m-1,k},\ W_\eta^{m,k-1},\\ (w^{m-1,k}+w^{m-1,k-1}+2h)W_{\eta\eta}^{m,k-1},\\ \nu_{\eta\eta}(w^{m-1,k}+w^{m-2,k})W^{m-1,k},\\ \nu_\eta(w^{m-1,k-1}+w^{m-1,k-1})W_\eta^{m,k-1},\\ \nu_{\eta\eta}(w^{m-1,k-1}+w^{m-1,k-1})W^{m,k-1},\end{array} \tag{67}$$

and $Q_2^{m,k}$ is a linear combination of the following functions:

$$\begin{array}{l}(\gamma^{m-1,k})^2,\ (\gamma^{m,k-1})^2,\ (\rho^{m-1,k})^2,\ (w^{m-1,k})^2,\\ (w^{m,k-1})^2,\ [(W_\eta^{m-1,k}(1-\eta)]^2,\\ [(W_\eta^{m,k-1}(1-\eta)]^2.\end{array} \tag{68}$$

We also have

$$\begin{array}{l}Q_3\equiv K_{24}(1-\eta+h)^2\geq\Big|\nu(w^{m-1,k}+h)^2f_{\eta\eta}\\ +\widetilde{A}^{m,k}f_\eta+2\widetilde{B}^{m,k}f\\ +2\nu_\eta(w^{m-1,k})^2f_\eta+2\nu_{\eta\eta}(w^{m-1,k})^2f\Big|.\end{array} \tag{69}$$

It is easy to see that for $k=1$ the function $Q_2^{m,k}$ does not depend on $(\rho^{m-1,k})^2$. For $k=0$, an inequality of the form (67) holds for $Y^{m,k}$, but in this case, $Q_1^{m,k}$ depends only on $W_\eta^{m-1,k}$,

$$(w^{m-1,k}+w^{m-2,k}+2h)W_{\eta\eta}^{m-1,k},$$

$$\nu_\eta(w^{m-1,k}+w^{m-2,k})W_\eta^{m-1,k},$$

$$\nu_{\eta\eta}(w^{m-1,k}+w^{m-2,k})W^{m-1,k}$$

, and $Q_2^{m,k}$ is a linear combination of $(\rho^{m-1,k})^2$, $[W_\eta^{m-1,k}(1-\eta)]^2$, $(w^{m,k-1})^2$.

Let us show by induction that

$$\Phi^{m,k}\leq M_1,\ Y^{m,k}\leq M_2(1-\eta+h), \tag{70}$$

for $mh\leq T_1$ and some $T_1\leq T_0$, the constants T_1, M_i being independent of h. To show this, assume that for $m<m'$ and $m=m'$, $k<k'$ the inequalities (71) hold with constants M_1, M_2 specified below. Let us show that if $mh\leq T_1$, the same inequalities are valid for $m=m'$, $k=k'$. Note that under the induction assumptions we can claim that for $m<m'$ or $m=m'$, $k<k'$ the following inequalities hold:

$$\begin{array}{l}\Big|(w^{m-1,k}+w^{m-2,k}+2h)W_{\eta\eta}^{m-1,k}\Big|\\ \leq K_{25}(1-\eta+h)\Big|W_{\eta\eta}^{m-1,k}\Big|\\ \leq K_{26}\nu(w^{m-2,k}+h)\Big|W_{\eta\eta}^{m-1,k}\Big|\\ =K_{26}\Big|\rho^{m-1,k}+\eta U^{m-1,k}\gamma^{m-1,k}\\ -\widetilde{A}^{m-1,k}W_\eta^{m-1,k}-\widetilde{B}^{m-1,k}W^{m-1,k}-\\ -2\nu_\eta(w^{m-2,k})^2W_\eta^{m-1,k}\\ -\nu_{\eta\eta}(w^{m-2,k})^2W^{m-1,k}\Big|(w^{m-2,k}+h)^{-1}\\ \leq K_{27}\end{array} \tag{71}$$

In exactly the same manner, we get that

$$\Big|(w^{m-1,k}+w^{m-1,k-1}+2h)W_{\eta\eta}^{m,k-1}\Big|\leq K_{28}$$

where the constants K_{26}, K_{27} depend on M_1 and M_2. Therefore, if the inequalities (71) hold for $m<m'$ and $m=m'$, $k<k'$, then it can be seen that in (53) and (67) we have

$$\begin{array}{l}|C^{m,k}|\leq K_{29}(M_1,\ M_2),\\ |Q_1^{m,k}|\leq K_{30}(M_1,\ M_2),\\ |Q_2^{m,k}|\leq K_{31}(1-\eta+h)^2.\end{array} \tag{72}$$

Let us pass to new functions in (2.25) and (2.32) by

$$\Phi^{m,k}=\widetilde{\Phi}^{m,k}e^{\gamma mh}, \tag{73}$$

$$Y^{m,k}=\widetilde{Y}^{m,k}e^{\gamma mh}, \tag{74}$$

The constant $\gamma(M_1,\ M_2)$ will be chosen later. For $1\leq m\leq m'$ and $m=m'$, we have

$$\begin{array}{l}\nu(w^{m-1,k}+h)^2\widetilde{\Phi}_{\eta\eta}^{m,k}-\eta U^{m,k}\frac{\widetilde{\Phi}^{m,k}-\widetilde{\Phi}^{m,k-1}}{h}\\ -e^{-\gamma h}\frac{\widetilde{\Phi}^{m,k}-\widetilde{\Phi}^{m-1,k}}{h}\\ +\widetilde{A}^{m,k}\widetilde{\Phi}_\eta^{m,k}+2\nu_\eta(w^{m-1,k})^2\widetilde{\Phi}_\eta^{m,k}\\ +\Big[2\nu_{\eta\eta}(w^{m-1,k})^2+2\widetilde{B}^{m,k}+\widetilde{C}^{m,k}-\gamma e^{-\gamma\hbar}\Big]\widetilde{\Phi}^{m,k}\\ \geq 0,\end{array} \tag{75}$$

for $0 < \hbar < h$, and also

$$\begin{aligned}&\nu(w^{m-1,k}+h)^2\widetilde{Y}^{m,k}_{\eta\eta}\\&-\eta U^{m,k}\tfrac{\widetilde{Y}^{m,k}-\widetilde{Y}^{m,k-1}}{h}\\&-e^{-\gamma h}\tfrac{\widetilde{Y}^{m,k}-\widetilde{Y}^{m-1,k}}{h}+\widetilde{A}^{m,k}\widetilde{Y}^{m,k}_{\eta}\\&+2\nu_\eta(w^{m-1,k})^2\widetilde{Y}^{m,k}_{\eta}+\Big[2\nu_{\eta\eta}(w^{m-1,k})^2\\&+2\widetilde{B}^{m,k}+\widetilde{Q}^{m,k}_1-\gamma e^{-\gamma\hbar}\Big]\widetilde{Y}^{m,k}+\\&+K_{32}(M_1,\,M_2)(1-\eta+h)^2\\&\geq 0.\end{aligned} \tag{76}$$

Let us choose $\gamma(M_1,\,M_2)$ such that for small enough h the following inequalities are valid:

$$\begin{aligned}&2\nu_{\eta\eta}(w^{m-1,k})^2+2\widetilde{B}^{m,k}\\&+\widetilde{C}^{m,k}-\gamma e^{-\gamma\hbar}<0,\end{aligned} \tag{77}$$

$$\begin{aligned}&2\nu_{\eta\eta}(w^{m-1,k})^2+2\widetilde{B}^{m,k}+\widetilde{Q}^{m,k}_1\\&-\gamma e^{-\gamma\hbar}<-K_{33}(M_1,\,M_2),\end{aligned} \tag{78}$$

where

$$\begin{aligned}&K_{33}=2\tfrac{K_{32}}{M_2}+K_{34},\\&K_{34}(1-\eta+h)^2\geq\Big|2\nu(w^{m-1,k}+h)^2\\&-2\widetilde{A}^{m,k}(1-\eta+h)\Big|.\end{aligned} \tag{79}$$

Consider the point at which $\widetilde{\Phi}^{m,k}(\eta)$, for $0\leq\eta\leq 1$, $m<m'$, or $m=m'$, $k\leq k'$, attains its largest value. In view of (76) and (78), this point cannot belong to the interval $0<\eta<1$ for $m\geq 1$. Moreover, if $\widetilde{\Phi}^{m,k}(\eta)$ attains its maximum at $\eta=0$, $m\geq 1$, we should have $\widetilde{\Phi}^{m,k}_\eta(\eta)\leq 0$, whereas relations (45) and (46) imply that

$$\begin{aligned}&0\geq\widetilde{\Phi}^{m,k}_\eta(0)\\&\geq\tfrac{\alpha}{2}\widetilde{\Phi}^{m,k}(0)-\tfrac{\alpha}{4}e^{-\gamma h}\widetilde{\Phi}^{m-1,k}(0)\\&\geq\tfrac{\alpha}{2}\widetilde{\Phi}^{m,k}(0)-\tfrac{\alpha}{4}\widetilde{\Phi}^{m-1,k}(0),\end{aligned} \tag{80}$$

and therefore,

$$\widetilde{\Phi}^{m,k}(0)\leq\frac{1}{2}\widetilde{\Phi}^{m-1,k}(0),$$

which is impossible.

For $\eta=1$, we have

$$\widetilde{\Phi}^{m,k}=\Phi^{m,k}e^{-\gamma mh}=e^{-\gamma mh}\Big[(W^{m,k}_\eta)^2+K_6+1\Big], \tag{81}$$

Estimates (14) for $w^{m,k}$ imply that

$$|W^{m,k}_\eta(1)|\leq K_{35}. \tag{82}$$

Therefore, if $\widetilde{\Phi}^{m,k}(\eta)$ attains its maximum value at $\eta=1$, we have $\widetilde{\Phi}^{m,k}(\eta)\leq K_{36}$ for $m<m'$ and $m=m'$, $k\leq k'$, where the constant K_{36} depends neither on M_1 nor on M_2. If $\widetilde{\Phi}^{m,k}(\eta)$ attains its maximum value at $m=0$, we have already shown that $\widetilde{\Phi}^{m,k}(\eta)\leq\max\widetilde{\Phi}^{0,k}\leq\max\Phi^{0,k}\leq K_{15}$. It follows that

$$\widetilde{\Phi}^{m,k}(\eta)\leq\max\{K_{15},\,K_{36}\} \tag{83}$$

Choose $M_1>\max\{2K_{15},\,2K_{36}\}$. Then

$$\widetilde{\Phi}^{m,k}(\eta)\leq\frac{M_1}{2}, \tag{84}$$

$$\Phi^{m,k}(\eta)\leq\frac{M_1}{2}e^{\gamma mh}, \tag{85}$$

If

$$e^{\gamma T_1}\leq 2,\ mh\leq T_1, \tag{86}$$

then

$$\Phi^{m,k}(\eta)\leq M_1. \tag{87}$$

for $m<m'$ and $m=m'$, $k\leq k'$, as required.

Now, consider the functions

$$X^{m,k}=\widetilde{Y}^{m,k}-\frac{M_2}{2}(1-\eta+h)^2, \tag{88}$$

it follows from (77) and (78) that

$$\begin{aligned}&\nu(w^{m-1,k}+h)^2X^{m,k}_{\eta\eta}-\eta U^{m,k}\tfrac{X^{m,k}-X^{m,k-1}}{h}\\&-e^{-\gamma h}\tfrac{X^{m,k}-X^{m-1,k}}{h}+\widetilde{A}^{m,k}X^{m,k}_\eta\\&+2\nu_\eta(w^{m-1,k})^2X^{m,k}_\eta+\Big[2\nu_{\eta\eta}(w^{m-1,k})^2+2\widetilde{B}^{m,k}\\&+\widetilde{Q}^{m,k}_1-\gamma e^{-\gamma\hbar}\Big]X^{m,k}\\&\geq-K_{32}(M_1,\,M_2)(1-\eta+h)^2\\&-\tfrac{M_2}{2}\Big[2\nu(w^{m-1,k}+h)^2-2\widetilde{A}^{m,k}(1-\eta+h)]\\&+\Big(2\nu_{\eta\eta}(w^{m-1,k})^2+2\widetilde{B}^{m,k}+\widetilde{Q}^{m,k}_1\\&-\gamma e^{-\gamma\hbar}\Big)(1-\eta+h)^2\Big]\\&\geq-K_{32}(M_1,\,M_2)(1-\eta+h)^2\\&-\tfrac{M_2}{2}\Big[K_{34}-K_{33}\Big](1-\eta+h)^2\\&\geq 0\end{aligned} \tag{89}$$

if $m<m'$ or $m=m',k\leq k'$. Let us show that $X^{m,k}(\eta)\leq 0$ for such m and k. If $X^{m,k}(\eta)$ takes positive values, then there is a point η at which, for $m<m'$ or $m=m',k\leq k'$, the function $X^{m,k}(\eta)$ attains its largest positive value. This point cannot belong to the interval $0<\eta<1$ for $m\geq 1$ because of (90). For $m=0$, if $M_2/2>K_{23}$, taking into account of the estimate (90) for $Y^{0,k}$, we find that

$$X^{m,k}(\eta)\leq\Big[K_{23}-\frac{M_2}{2}\Big](1-\eta+h)^2\leq 0,$$

Since $\widetilde{Y}^{m,k}(1)=0$, we have $X^{m,k}(1)\leq 0$. For $\eta=0$, $m\geq 1$, the function $X^{m,k}$ cannot attain its largest positive value, since inequalities (56) and (57) show that

$$\begin{aligned}&X_\eta^{m,k}(0)=\widetilde{Y}_\eta^{m,k}(0)+M_2(1+h)\\&\geq\tfrac{\alpha}{2}\widetilde{Y}^{m,k}(0)-\tfrac{\alpha}{4}\widetilde{Y}^{m-1,k}(0)+M_2(1+h)\\&\geq\tfrac{\alpha}{2}X^{m,k}(0)-\tfrac{\alpha}{4}X^{m-1,k}(0)\\&+\tfrac{\alpha}{8}M_2(1+h)^2+M_2(1+h)\\&\geq\tfrac{\alpha}{2}X^{m,k}(0)-\tfrac{\alpha}{4}X^{m-1,k}(0).\end{aligned}$$

So, $X^{m,k}\leq 0$ and

$$\widetilde{Y}^{m,k}\leq\frac{1}{2}M_2(1-\eta+h)^2, \tag{90}$$

when $m<m'$ and $m=m'$, $k\leq k'$.

Therefore, if $M_2/2>K_{23}$, $e^{\gamma T_1}<2$, $mh\leq T_1$, we have

$$Y^{m,k}(\eta)=\widetilde{Y}^{m,k}e^{\gamma mh}\leq M_2(1-\eta+h)^2. \tag{91}$$

It follows from (71), (74), (75) and (35) that $(1-\eta+h)|W_{\eta\eta}^{m,k}|$ are uniformly bounded in h.

Lemma 6 *Under the assumptions of Lemmas 4 and 5, problem (9) in Ω, with $T=T_1$, admits a unique solution w with the following properties: w is continuous in Ω;*

$$C_1(1-\eta)\leq w\leq C_2(1-\eta), \tag{92}$$

w has bounded weak derivatives w_η, w_ξ,w_τ;

$$|w_\xi|\leq C_3(1-\eta),\ \ |w_\tau|\leq C_4(1-\eta), \tag{93}$$

the derivative w_η is continuous in $\eta<1$; conditions 0f (8) hold for w; the weak derivative $w_{\eta\eta}$ exists and $ww_{\eta\eta}$ is bounded in Ω; equation (9) holds almost everywhere in the same domain.

Proof: First, let us prove the uniqueness of the solution. Assume the contrary, namely, that w_1 and w_2 are two solutions of problem (9) with the properties specified in Lemma 6. Then, almost everywhere in Ω, the function $z=w_1-w_2$ satisfies the following equation and the boundary conditions:

$$\begin{cases}\nu w_1^2z_{\eta\eta}-z_\tau-\eta Uz_\xi+(A+2\nu_\eta w_1^2)z_\eta\\+[B+3\nu_{\eta\eta}w_1w_2+(\nu w_{2\eta\eta}+2\nu_\eta w_{2\eta})(w_1+w_2)]z\\+\nu_{\eta\eta}z^3=0,\\z|_{\tau=0}=0,\ \ z|_{\eta=1}=0,\\ [\nu w_1z_\eta+\nu w_{2\eta}z+\nu_\eta(w_1+w_2)z-v_0z]|_{\eta=0}=0,\end{cases} \tag{94}$$

Set

$$z=e^{\alpha\tau-\beta\eta}\bar{z},$$

where $\alpha,\beta=const.>0$. Then

$$\begin{aligned}&\nu w_1^2\bar{z}_{\eta\eta}-\bar{z}_\tau-\eta U\bar{z}_\xi+E\bar{z}_\eta+F\bar{z}\\&+\nu_{\eta\eta}(w_1-w_2)^2\bar{z}=0,\end{aligned} \tag{95}$$

$$\bar{z}|_{\tau=0}=0,\ \bar{z}|_{\eta=1}=0, \tag{96}$$

$$\begin{aligned}&[\nu w_1\bar{z}_\eta+\nu w_{2\eta}\bar{z}+\nu_\eta(w_1+w_2)\bar{z}\\&-\beta\nu w_1\bar{z}-v_0\bar{z}]|_{\eta=0}=0,\end{aligned} \tag{97}$$

where

$$\begin{aligned}&E=A+2\nu_\eta w_1^2-2\beta\nu w_1^2,\\&F=B+3\nu_{\eta\eta}w_1w_2+(\nu w_{2\eta\eta}+2\nu_\eta w_{2\eta})(w_1+w_2)\\&+\nu\beta^2w_1^2-\alpha-A\beta-2\nu_\eta\beta w_1^2,\end{aligned}$$

The constant β is chosen suitable large such that

$$[\beta\nu w_1+v_0-\nu w_{2\eta}-\nu_\eta(w_1+w_2)]|_{\eta=0}>1. \tag{98}$$

Let us multiply equation (96) by $\bar{z}$ and integrate the result over Ω. Integrating by parts in some of the terms, we find that

$$\begin{aligned}&\textstyle\int_\Omega E\bar{z}_\eta\bar{z}\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau+\int_\Omega F\bar{z}^2\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau\\&+\textstyle\int_\Omega\frac{1}{2}\eta U_\xi\bar{z}^2\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau+\int_\Omega\nu_{\eta\eta}(w_1-w_2)^2\bar{z}^2\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau\\&-2\textstyle\int_\Omega\nu w_1w_{1\eta}\bar{z}_\eta\bar{z}\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau-\int_\Omega\nu_\eta w_1^2\bar{z}_\eta\bar{z}\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau\\&-\textstyle\int_\Omega\nu w_1^2\bar{z}_\eta^2\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau-\frac{1}{2}\int\limits_{\tau=T}\bar{z}^2\mathrm{d}\eta\mathrm{d}\xi-\\&-\frac{1}{2}\int\limits_{\xi=X}\eta U\bar{z}^2\mathrm{d}\eta\mathrm{d}\tau-\int\limits_{\eta=0}\nu w_1^2\bar{z}_\eta\bar{z}\mathrm{d}\xi\mathrm{d}\tau=0,\end{aligned} \tag{99}$$

Using the boundary condition (98) at $\eta=0$, we can write the last integral in (100) as

$$\int\limits_{\eta=0}w_1\bar{z}^2\Big[\beta\nu w_1+v_0-\nu w_{2\eta}-\nu_\eta(w_1+w_2)\Big]\mathrm{d}\xi\mathrm{d}\tau. \tag{100}$$

By our choice of β, this integral is non-negative. Let us estimate the integral over Ω containing $\bar{z}\bar{z}_\eta$. Taking into account $|E|<1/h$ in Ω, we get

$$\begin{aligned}&\Big|\textstyle\int_\Omega(E-2\nu w_1w_{1\eta}-\nu_\eta w_1^2)\bar{z}_\eta\bar{z}\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau\Big|\\&\leq\textstyle\int_\Omega\nu w_1^2\bar{z}_\eta^2\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau\\&+C_5\textstyle\int_\Omega\bar{z}^2\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau,\end{aligned} \tag{101}$$

where $C_5=const.$ Therefore, it follows from (100) that

$$\begin{aligned}&\textstyle\int_\Omega\Big[F+\frac{1}{2}\eta U_\xi+\nu_{\eta\eta}(w_1-w_2)^2\\&+C_5\Big]\bar{z}^2\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau\geq 0,\end{aligned} \tag{102}$$

Because

$$C_1(1-\eta)\leq w_i\leq C_2(1-\eta),$$

$$|w_iw_{i\eta\eta}|\leq C_6,\ \ |w_{i\eta}|\leq C_7,\ \ (i=1,2)$$

and assumptions, we can choose α, β to sure the inequality

$$\begin{aligned}&B+3\nu_{\eta\eta}w_1w_2+(\nu w_{2\eta\eta}+2\nu_\eta w_{2\eta})(w_1+w_2)\\&+\beta^2\nu w_1^2+\tfrac{1}{2}\eta U_\xi+(w_1-w_2)^2+\\&+C_5-\alpha-A\beta-2\beta\nu_\eta w_1^2\leq -1.\end{aligned} \tag{103}$$

Then, it follows from (103) that

$$-\int_\Omega \bar{z}^2 \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}\tau \geq 0, \tag{104}$$

Therefore, $z \equiv 0$ in Ω and $w_1 = w_2$ in Ω, as required.

Now, we will prove the existence of the solution of (9)-(10). The solutions $w^{m,k}$ of problem (12)-(13) should be linearly extended to the domain Ω.

First, when $(k-1)h < \xi \leq kh$, $k = 1, 2, \cdots, k(h)$; $k(h) = [X/h]$, let

$$w_h^m(\xi, \eta) = w_h^m((k-1)h\lambda + kh(1-\lambda),$$
$$\eta) = (1-\lambda)w^{m,k}(\eta) + \lambda w^{m,k-1}(\eta),$$

Secondly, when $(m-1)h < \tau < mh$, $m = 1, 2, \cdots, m(h)$; $m(h) = [T/h]$, let

$$w_h(\tau, \xi, \eta) = w_h((m-1)h\sigma + mh(1-\sigma), \xi,$$
$$\eta) = (1-\sigma)w_k^m(\xi, \eta) + \sigma w_k^{m-1}(\xi, \eta),$$

According to Lemma 4, Lemma 5, the functions $w_h(\tau, \eta, \xi)$ obtained in this manner form this family satisfy the $Lipschitz$ condition with respect to ξ, τ, and have uniformly (in h) bounded first derivative in η in Ω. By the $Arzel\grave{a}$ Theorem, there is a sequence $h_i \to 0$ such that w_{h_i} uniformly converge to some $w(\eta, \xi, \tau)$. It follows from Lemma 4, Lemma 5 that $w(\eta, \xi, \tau)$ has bounded weak derivatives w_η, w_ξ, w_τ, and its weak derivative $w_{\eta\eta}$ is such that $(1-\eta)w_{\eta\eta}$ is bounded, since the weak limit of a bounded sequence is bounded by the same constant. Consequently, w_η is continuous in $\eta < 1$. The sequence w_{h_i} may be assumed such that the derivatives w_η, w_ξ, w_τ, $w_{\eta\eta}$ in the domain Ω coincide with weak limits in $L^2(\Omega)$ of the respective functions

$$\frac{w(\tau+h_i, \xi, \eta) - w(\tau, \xi, \eta)}{h_i},$$

$$\frac{w(\tau, \xi+h_i, \eta) - w(\tau, \xi, \eta)}{h_i},$$

$$w_{h_i\eta}, \, w_{h_i\eta\eta},$$

Denoting

$$w_h^{m,k} = w_h(\tau, \xi, \eta) = w(mh, kh, \eta),$$

By the first term of (11)

$$\begin{aligned}&\nu(w_h^{m-1,k}+h)^2 w_{h\eta\eta}^{m,k} - \tfrac{w_h^{m,k}-w_h^{m-1,k}}{h} \\ &-\eta U^{m,k}\tfrac{w_h^{m,k}-w_h^{m,k-1}}{h} + A^{m,k}w_{h\eta}^{m,k} + B^{m,k}w_h^{m,k} \\ &+2\nu_\eta(w_h^{m-1,k})^2 w_{h\eta}^{m,k} \\ &+\nu_{\eta\eta}(w_h^{m-1,k})^2 w_h^{m,k} = 0,\end{aligned} \tag{105}$$

Now, suppose that $\varphi(\tau, \xi, \eta)$ be a smooth function,which support set is compact in Ω. Let

$$\varphi^{m,k}(\eta) = \varphi(mh, kh, \eta),$$

Let us multiply with $h\varphi^{m,k}$ at the two sides of (106), integrating the resulting equation in η from 0 to 1, and taking the sum over k, m from 1 to $k(h)$, $m(h)$ respectively, we obtain

$$\begin{aligned}&\sum_{m,k} h \int_0^1 \varphi^{m,k}\Big[\nu(w_h^{m-1,k}+h)^2 w_{h\eta\eta}^{m,k} \\ &-\tfrac{w_h^{m,k}-w_h^{m-1,k}}{h} - \eta U^{m,k}\tfrac{w_h^{m,k}-w_h^{m,k-1}}{h} \\ &+A^{m,k}w_{h\eta}^{m,k} + B^{m,k}w_h^{m,k} + 2\nu_\eta(w_h^{m-1,k})^2 w_{h\eta}^{m,k} \\ &+\nu_{\eta\eta}(w^{m-1,k})^2 w^{m,k}\Big]\mathrm{d}\eta = 0,\end{aligned} \tag{106}$$

Denote the function $\bar{f}(\tau, \xi, \eta)$ on Ω as

$$\bar{f}(\tau, \xi, \eta) = f(mh, kh, \eta),$$

when $(k-1)h < \xi \leq kh, (m-1)h < \tau < mh.$ and denote

$$\Big(\frac{\triangle w_h}{h}\Big)_1^m = \frac{w_h^{m,k} - w_h^{m-1,k}}{h},$$

$$\Big(\frac{\triangle w_h}{h}\Big)_1^k = \frac{w_h^{m,k} - w_h^{m,k-1}}{h},$$

Then we can rewrite (107) as

$$\begin{aligned}&\int_\Omega \Big[\bar{\nu}(\bar{w}_h^{m-1,k}+h)^2\bar{w}_{h\eta\eta}\bar{\varphi} - \overline{\Big(\tfrac{\triangle w_h}{h}\Big)}_1^m\bar{\varphi} \\ &-\bar{\eta}\bar{U}\overline{\Big(\tfrac{\triangle w_h}{h}\Big)}_1^k\bar{\varphi} + \bar{A}\bar{w}_{h\eta}\bar{\varphi} + \bar{B}\bar{w}\bar{\varphi} \\ &+2\bar{\nu}_\eta(\bar{w}_h^{m-1,k})^2\bar{w}_{h\eta}\bar{\varphi} + \bar{\nu}_{\eta\eta}(\bar{w}^{m-1,k})^2\bar{w}^{m,k}\bar{\varphi}\Big]\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}\tau \\ &= 0,\end{aligned} \tag{107}$$

Because

$$|\bar{w} - w| \leq |\bar{w} - w_h| + |w_h - w| \leq Mh + |w_h - w|,$$

when $h \to 0$, $\bar{w}$ uniformly convergent to w, i.e. $\bar{w} \Longrightarrow w$. Just likely

$$\bar{\varphi} \Longrightarrow \varphi, \, \bar{A}\bar{\varphi} \Longrightarrow A\varphi,$$
$$\bar{B}\bar{\varphi} \Longrightarrow B\varphi, \, \bar{\eta}\bar{U}\bar{\varphi} \Longrightarrow \eta U\varphi,$$
$$\bar{\nu}_\eta\bar{\varphi} \Longrightarrow \nu_\eta\varphi,$$

$$\bar{\nu}_{\eta\eta}\bar{\varphi} \Longrightarrow \nu_{\eta\eta}\varphi, \, \bar{\nu}(\bar{w}_h^{m-1,k}+h)^2 \Longrightarrow \nu w^2,$$

At the same time, on account of that

$$\overline{\Big(\frac{\triangle w_h}{h}\Big)}_1^m \rightharpoonup w_\tau, \, \overline{\Big(\frac{\triangle w_h}{h}\Big)}_1^k \rightharpoonup w_\xi,$$

$$\bar{w}_{h\eta\eta} \rightharpoonup w_{\eta\eta}, \text{ in } L^2(\Omega).$$

So, if let $h \to 0$ in (108), then

$$\begin{aligned}&\int_{\Omega}(\nu w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + Bw \\ &+2\nu_\eta w^2 w_\eta + \nu_{\eta\eta} w^3)\varphi \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}\tau = 0.\end{aligned} \quad (108)$$

By the arbitrary of φ, we get ours result.

Acknowledgements: The research was supported by the NSF of Fujian Province of China (No: 2009J01009) and the NSF of Jimei University of China.

References:

[1] Prandtl L. *Über Flüssigkeisbewegungen bei sehr kleine Reibung*, In: Verh. Int. Math. Kongr, Heidelberg, 1904.

[2] Anderson I.and Toften H. Numerical solutions of the laminar boundary layer equations for power-law fluids, *Non-Newton,Fluid Mech.*, 32, 1989, pp.175-195.

[3] Schliching H. and Gersten K., and Krause E. *Boundary Layer Theory* , Springer, 2004.

[4] Shuming Wen, *Micro-Fluid Boundary Layer Theory and Its Applications*, Metallurgical Industry Press, 2002.

[5] Oleinik O.A and Samkhin V.N., *Mathematical Models in Boundary Layer Theorem*, Chapman and Hall/CRC,1999.

[6] Gilbarg.D and Trudinger.N.S, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1977.

[7] Zhang J.W. and Zhao J.N., On the Global Existence and Uniqueness of Solutions to Non-stationary Boundary Layer System, *Science in China, Ser. A*, 36, 2006,pp. 870-900.

[8] E.W.N., Blow up of solutions of the unsteadly Pradtl's equations, *Comm. pure Appl. Math.*, 1997, 1, pp.1287-1293.

[9] Xin Z. and Zhang L., On the global existence of solutions to the Pradtl's system, *Adv. in Math.*, 2004, 191, pp. 88-133.