Optimal control applied to a Ramsey model with taxes and exponential utility

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Abstract: In this paper we analyze an economical growth model with taxes and exponential utility in continuous and infinite time. This economical growth model leads to an optimal control problem. The necessary and sufficient conditions for optimality are given. Using the optimality conditions we prove the existence, uniqueness and stability of the study state for a differential equations system. Also we have investigated the dependence of the steady state (k^*, c^*) on the growth rate of the labor force n and the effects of fiscal policy changes on welfare.

Key–Words: mathematical models applied in economies, endogenous growth, optimal policy

1 Introduction

In this paper, we consider a version of the Ramsey growth model with taxes in infinite and continuous time. This economy consists of a firm, a household and a government. The firm produces goods using two factors of production, capital and labor, it rents capital at the rate of interest r and it hires labor at the wage rate w. Also, the firm seeks to maximize the present value of its profits. The government imposes taxes on capital and labor income, and it provides lump sum transfer and public-consumption expenditures. In this economy, the consumer chooses at any moment in time the level of consumption so as to maximize the global utility on the infinite time taking into account the budget constraint for household and the government. The utility function is given by an exponential function. This economical growth model leads to an optimal control problem. We prove that a necessary condition for the control function to solve our optimal control problem is that it is a solution of Euler equation. Also, we give the sufficient conditions for the optimal solution of the optimal control problem. An analogous result was formulated by Ş.Cruceanu and C. Vârsan [5]. Finally we investigate the effects of the fiscal policy changes on the steady-state values of capital and consumption and on welfare.

The outline of this paper is as follows. In Section 2, we present the model which we use in this article. In Section 3, we give the necessary and sufficient conditions for the optimal solution of the economical growth problem with public intervention. In Section 4, we determine the steady state of the optimal control problem and we show that it is a saddle point. Also, we examine the qualitative dynamic behavior of the optimal solution. In Section 5,we analyze the dependence of the steady state (k^*, c^*) on the growth rate of the labor force n. In Section 6, we investigate the effects of the fiscal policy changes on welfare. Some conclusions are given in section 7.

2 The economical growth model

In this paper, based on [1], [3], [2] and [8], we consider an economical growth model with public intervention. The economy consists of a household, a firm and a government. The competitive equilibrium achieved in a decentralized manner through perfect competition between the firm and the household. Also, we assume the economy closed (i.e. all of the stock capital must be owned by someone in economy and the net foreign debt is zero.)

The firm produces goods, pays wages for labor, and makes rental payments for capital on the competitive market at equilibrium prices given by the net marginal products of labor and capital.

Output is determined according to a constant returns to scale technology using the following production function

$$Y = F(K, L) \tag{1}$$

where K is the capital, L is the labor force, $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function of class C^2 having the following properties

$$\frac{\partial F(K,L)}{\partial K} > 0,\ \frac{\partial F(K,L)}{\partial L} > 0, \forall K > 0, L > 0;$$

$$\frac{\partial^2 F(K,L)}{\partial K^2} < 0,\ \frac{\partial^2 F(K,L)}{\partial L^2} < 0, \forall K > 0, L > 0;$$
$$F(\lambda K, \lambda L) = \lambda F(K,L) = \lambda Y, \forall \lambda > 0;$$

$$\lim_{K\to 0}\frac{\partial F(K,L)}{\partial K} = \lim_{L\to 0}\frac{\partial F(K,L)}{\partial L} = \infty;$$
$$\lim_{K\to\infty}\frac{\partial F(K,L)}{\partial K} = \lim_{L\to\infty}\frac{\partial F(K,L)}{\partial L} = 0$$
$$F(K,0) = F(0,L) = 0$$

The firm seeks to maximize the present value of profits, given by

$$Profit = F(K(t), L(t)) - (r(t) + \delta)K(t) - w(t)L(t), \quad (2)$$

taking $r(t)$ and $w(t)$ as given, r is the rental rate of capital, w is wage per worker, and δ is depreciation of capital.
We define

$$k(t) = \frac{K(t)}{L(t)} - \text{ the capital per worker;}$$
$$y(t) = \frac{Y(t)}{L(t)} - \text{ the output per worker.}$$

Using the homogeneity condition we have

$$Y = F(K,L) = L\cdot F(\frac{K}{L}, 1) = L\cdot f(k),$$

where
$$f(k) = F(k,1).$$

Then, the output per worker is

$$y = f(k)$$

Using the properties of function F we obtain the properties of function of class C^2, f :
$f(0) = 0; f'(k) > 0, f''(k) < 0, \forall k > 0;$
$\lim\limits_{k\to 0} f'(k) = \infty; \lim\limits_{k\to\infty} f'(k) = 0.$
Profit for this firm can be written as

$$Profit = L(t)(f(k(t)) - (r(t)+\delta)k(t) - w(t)) \quad (3)$$

The economical condition is that on a competitive market the maximum profit obtained by the firm is zero, and is given by

$$\begin{aligned} f'(k) &= r + \delta \\ f(k) - kf'(k) &= w. \end{aligned} \quad (4)$$

We assume that the government taxes labor and net capital incomes at proportional rates τ_w and τ_r. We also assume that the tax revenue is allocated between lump-sum transfer S and public consumption goods X. Assuming that the government must run a balanced budget at each moment in time and that tax rates, transfers, and public-consumption expenditures per unit of labor ($x = \frac{X}{L}, s = \frac{S}{L}$) are constant over time, then we have the government budget constraint written in terms per capita

$$\tau_r(f'(k) - \delta)k + \tau_w(f(k) - kf'(k)) = x + s \quad (5)$$

where the left-hand side is the total tax revenue per unit of labor, and the right-hand side gives total expenditure. Given constant values of $x + s$ and τ_r, equation can be solved for the value of τ_w that will preserve budget balance for each value of k.

In this economy the consumer chooses at any moment in time the level of consumption $c(t)$ so that to maximize the global utility, given by

$$U = -\frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}(1 - e^{-\theta c(t)})dt, \quad (6)$$

and subject to the budget constraint for the household, given by

$$\dot{K}(t) = (1-\tau_r)r(t)K(t) + (1-\tau_w)w(t)L(t) + S - C(t).$$

Where c is the consumption, $\rho > 0$ is the discount rate and $\theta > 0$.

The size of the household grows at rate n.

The budget constraint for the household can be rewritten in per capita terms as

$$\dot{k}(t) = (1-\tau_r)r(t)k(t) + (1-\tau_w)w(t) + s - c(t) - nk(t).$$

where $c(t) = \frac{C(t)}{L(t)}$ is the consumption per capita.

The initial stock of capital for the household is K_0. Thus, the initial stock of capital per capita is k_0.

In the conditions of a competitive equilibrium and taking into account the government budget constraint , the budget constraint for household in per capita terms can be rewritten as:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - x - c(t). \quad (7)$$

3 Determination of optimality conditions

The economical problem is to choose in every moment t, the size of consumption so as to maximize the global utility taking into account the government budget constraint and the budget constraint for household and the initial stock of capital k_0, in the conditions of competitive equilibrium, leads us to the following mathematical optimization problem (P) :

The problem P. To determine (k^*, c^*) which maximizes the following functional

$$-\frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}(1-e^{-\theta c(t)})dt \qquad (8)$$

in the class of functions $k \in AC([0,\infty),\mathbb{R}_+)$, and $c \in \mathcal{X}$, where
$\mathcal{X} = \{c : [0,\infty) \to [0,A]\,, c-\text{measurable},\ A < \infty\}$ which verifies:

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - (n+\delta)k(t) - x - c(t) \quad (9)\\ k(0) &= k_0 \quad (10)\end{aligned}$$

In our problem **P,** k is the state variable and c is the control variable.

Definition 1 *A trajectory $(k(t), c(t))$ is called an admissible trajectory, with initial capital k_0, for the problem (P) if it verifies the relation (9)-(10).*

Definition 2 *An admissible trajectory, $(k^*(t), c^*(t))$, is called optimal trajectory if:*

$$-\frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}(1-e^{-\theta c(t)})dt \le$$

$$\le -\frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}(1-e^{-\theta c^*(t)})dt.$$

for every admissible trajectory $(k(t), c(t))$ of the problem (P).

In the following theorem we prove that a solution of our optimal control problem must satisfy a certain differential equation called the Euler equation.
We denote:

$$\phi(k(t)) = f(k(t)) - (n+\delta)k(t) - x. \qquad (11)$$

$$U(c(t)) = -\frac{1}{\theta}(1-e^{-\theta c(t)}) \qquad (12)$$

Theorem 1 *If $(c(t), k(t))$ is an optimal trajectory of the problem (P), then it verifies the Euler-Lagrange equation:*

$$-\frac{d}{dt}(e^{-\theta c(t)-(\rho-n)t}) = e^{-\theta c(t)-(\rho-n)t}\phi'(k(t)). \qquad (13)$$

Proof: Let $(c(t), k(t))$ by an optimal trajectory of the problem (P).

From (9) and (11), we have $c(t) = \phi(k(t)) - \dot{k}(t)$.
Choose $[T, T']$ such that $\phi'(k(t))$ is continuous on $[T, T']$. Let h by any C^2-function on $[T, T']$ which satisfies $h(T) = h(T') = 0$.
For each real number $\alpha \in \mathbb{R}$, we define a new function $k_1(t)$ by $k_1(t) = k(t) + \alpha h(t)$, as in fig. 1

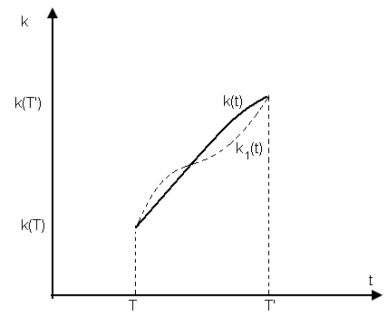

Fig.1

Note that if α is small, the function $k_1(t)$ is "near" the function $k(t)$.
We define

$$J_h(\alpha) = \int_T^{T'} U(\phi(k(t)+\alpha h(t)) - (\dot{k}(t)+\alpha\dot{h}(t)))e^{-(\rho-n)t}dt. \qquad (14)$$

Because $k(t)$ is an optimal trajectory, we have

$$\int_T^{T'} U(\phi(k(t)) - \dot{k}(t))e^{-(\rho-n)t}dt$$

$$\ge \int_T^{T'} U(\phi(k(t)+\alpha h(t)) - (\dot{k}(t)+\alpha\dot{h}(t)))e^{-(\rho-n)t}dt$$

for all $\alpha \in \mathbb{R}$.
Thus, $J_h(\alpha) \le J_h(0)$ for all $\alpha \in \mathbb{R}$. Hence the function $J_h(\alpha)$ has a maximum at $\alpha = 0$, so that

$$J'_h(0) = 0. \qquad (15)$$

Now, looking at (14), we see that to calculate $J'_h(0)$ we must differentiate the integral with respect to a parameter appearing in the integrand.

Conversely, (15) implies

$$\int_T^{T'} U'(\phi(k(t)) - \dot{k}(t))(\phi'(k(t))h(t) - \dot{h}(t))e^{-(\rho-n)t}dt = 0 \qquad (16)$$

We consider

$$\psi(t)=\int_T^t U'(\phi(k(s))-\dot{k}(s))\phi'(k(s))e^{-(\rho-n)s}ds,$$

$t\in[T,T']$ and

$$g(t)=\psi(t)+G,$$

where G is given by

$$\int_T^{T'}[U'(\phi(k(t))-\dot{k}(t))e^{-(\rho-n)t}+\psi(t)]dt+G(T'-T)=0$$

Since

$$h(T)=h(T')=0.$$

we have

$$\int_T^{T'}U'(\phi(k(t))-\dot{k}(t))\phi'(k(t))h(t)e^{-(\rho-n)t}dt$$

$$=\int_T^{T}\dot{g}(t)h(t)dt=-\int_T^{T'}g(t)\dot{h}(t)dt$$

Thus, (16) it becomes:

$$-\int_T^{T'}(g(t)+U'(\phi(k(t))-\dot{k}(t))e^{-(\rho-n)t})\dot{h}(t)dt=0, \quad (17)$$

for all functions h which are $\mathcal{C}^1$ on $[T,T']$ and which satisfy $h(T)=h(T')=0$.
We consider

$$h(t)=-\int_T^t(U'(\phi(k(s))-\cdot k(s))e^{-(\rho-n)s}+\psi(s))ds-G(t-T),$$

$t\le T'$.
From (17) and the definition for g and G, we obtain

$$\int_T^{T'}(g(t)+U'(\phi(k(t))-\dot{k}(t))e^{-(\rho-n)t})^2dt=0.$$

Thus

$$g(t)=-U'(\phi(k(t))-\dot{k}(t))e^{-(\rho-n)t} \text{ for all } t\in[T,T'].$$

Since g is a continuous function on $[T,T']$, we see that $c(t)=\phi(k(t))-\dot{k}(t)$ is a continuous function on $[T,T']$ and $\dot{g}$ is a continuous function on $[T,T']$.
Hence, we conclude

$$\begin{aligned}\dot{g}(t) &= U'(\phi(k(t))-\dot{k}(t))\phi'(k(t))e^{-(\rho-n)t}=\\ &= -\frac{d}{dt}(U'(\phi(k(t))-\dot{k}(t))e^{(-\rho-n)t}).\end{aligned}$$

This is the Euler-Lagrange equation discovered in 1744 by mathematician Euler.

In next theorem, we give the sufficient conditions for the solution of our optimal control problem P.

Theorem 2 *Let $(k^*(t),c^*(t))$ be an admissible trajectory in problem $\mathbf{P}$. If there exists an absolutely continuous function $q(t)$ such that for all t, the following conditions are satisfied*

$$\dot{q}(t) = q(t)(\rho-(f'(k(t))-\delta)(1-\tau_r)) \quad (18)$$

$$q(t) = e^{-\theta c^*(t)} \quad (19)$$

$$\lim_{t\to\infty}e^{-(\rho-n)t}q(t)=0 \quad (20)$$

then $(k^*(t),c^*(t))$ is an optimal trajectory in problem P.

Proof: Let $(k^*(t),c^*(t))$ be an arbitrary admissible trajectory for **P**.
We denote

$$\Delta=\frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}(1-e^{-\theta c^*(t)})dt-\frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}(1-e^{-\theta c(t)})dt \quad (21)$$

and we define the function of Hamilton-Pontryagin $H:[0,\infty)\times[0,A]\times\mathbb{R}\times[0,\infty)\to\mathbb{R}$, given by

$$H(k,c,p,t)=\frac{1}{\theta}e^{-(\rho-n)t}(1-e^{-\theta c})+p(f(k)-(n+\delta)k-c-x) \quad (22)$$

In the following, we simplify our notation and put $k^*(t)=k^*,\ c^*(t)=c^*\ \ k(t)=k, c(t)=c,$ $H(k,c,p,t)=H,\ \ H(k^*,c^*,p,t)=H^*.$
From (22) we have

$$\frac{1}{\theta}e^{-(\rho-n)t}(1-e^{-\theta c})=H(k,c,p,t)-p(f(k)-(n+\delta)k-c-x)$$

and using (9) the relation (21) becomes

$$\Delta=\int_0^\infty[H(k^*,c^*,p,t)-H(k,c,p,t)]dt+\int_0^\infty p(\dot{k}-\dot{k}^*)dt. \quad (23)$$

Since

$$\begin{aligned}H''_{kk}(k,c,p,t) &= pf''(k)\\ H'_c(k,c,p,t) &= e^{-(\rho-n)t}e^{-\theta c}-p(t)\\ H''_{cc}(k,c,p,t) &= -\theta e^{-(\rho-n)t}e^{-\theta c},\end{aligned}$$

we have that $H(k,c,p,t)$ is concave as a function of c and k.
Using a standard result on concave functions, we obtain

$$H(k,c,p,t)-H(k^*,c^*,p,t) \le \frac{\partial H^*}{\partial k}(k-k^*)+\frac{\partial H^*}{\partial c}(c-c^*). \quad (24)$$

Using the transformation $q(t) = e^{(\rho-n)t}p(t)$ in (18) we have the equation of the adjoint variable p , $\frac{\partial H^*}{\partial k} = -\dot{p}$.
Using (24) and $\frac{\partial H^*}{\partial k} = -\dot{p}$ in (23), we obtain that

$$\Delta \ge \int_0^\infty (\dot{p}(k-k^*)+p(\dot{k}-\dot{k}^*))dt+\int_0^\infty \frac{\partial H^*}{\partial c}(c^*-c)dt. \quad (25)$$

We differentiate (22) with respect to c and using (19) we obtain the maximum condition

$$H(k^*(t),c^*(t),p(t),t) \ge H(k^*(t),c,p(t),t), \forall c \in [0,A]$$

i.e.

$$\frac{\partial H^*}{\partial c}(c^*-c) \ge 0, \ \ \forall\, c \in [0,A]\,. \quad (26)$$

From (26) we have that the second integral in (25) is ≥ 0, so

$$\Delta \ge \int_0^\infty (\dot{p}(k-k^*)+p(\dot{k}-\dot{k}^*))dt \quad (27)$$

or equivalently,

$$\Delta \ge \int_0^\infty \frac{d}{dt}(p(k-k^*))dt = p(k-k^*)|_0^\infty\,.$$

Hence, using (20) we obtain the inequality

$$\frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}(1-e^{-\theta c^*(t)})dt \ge \frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}(1-e^{-\theta c(t)})dt,$$

which prove that $(k^*(t),c^*(t))$ is an optimal trajectory in problem P.

Remark 3 *The optimal trajectory of the problem* $(\mathbf{P})$ *in the conditions of a competitive equilibrium is the solution of the following system*

$$\dot{k}(t) = f(k(t))-(n+\delta)k(t)-x-c(t) \quad (28)$$
$$\dot{c}(t) = -\frac{1}{\theta}[\rho-(1-\tau_r)(f'(k(t))-\delta)]. \quad (29)$$

4 Qualitative analysis of the optimal solution

Proposition 4 *The system of differential equations*

$$\dot{k}(t) = f(k(t))-(n+\delta)k(t)-x-c(t) \quad (30)$$
$$\dot{c}(t) = -\frac{1}{\theta}\left[\rho-(1-\tau_r)\left(f'(k(t))-\delta\right)\right] \quad (31)$$

exhibits saddle-path stability.

Proof: In order to determinate the steady state of the above system we choose the stationary solutions $k(t)=k^*$, $c(t)=c^*$. From $\dot{c}(t)=0$ we obtain

$$f'(k) = \frac{\rho}{1-\tau_r}+\delta. \quad (32)$$

Using the properties of the function f, it results that the equation (32) has a unique solution $k=k^*$ with which we may determine

$$c^* = f(k^*)-(n+\delta)k^*-x. \quad (33)$$

The point (k^*,c^*) represents a steady state for the system of differential equations.

In order to investigate the stability of the steady state (k^*,c^*) we linearize the system (30)-(31)in the steady state (k^*,c^*) and we obtain

$$\dot{k}(t) \approx \left(f'(k^*)-n-\delta\right)(k(t)-k^*)-(c(t)-c^*)$$
$$\dot{c}(t) \approx \frac{1}{\theta}(1-\tau_r)f''(k^*)(k(t)-k^*)$$

The matrix of the linearized system is $\begin{pmatrix} f'(k^*)-n-\delta & -1 \\ \frac{1}{\theta}(1-\tau_r)f''(k^*) & 0 \end{pmatrix}$.

The eigenvalues are solutions of equation

$$\lambda^2-\lambda\left(f'(k^*)-n-\delta\right)+\frac{1}{\theta}(1-\tau_r)f''(k^*)=0. \quad (34)$$

Because f is a concave function, it results that the determinant of the equation is positive

$$\Delta = \left(f'(k^*)-n-\delta\right)^2-4\frac{1}{\theta}(1-\tau_r)f''(k^*)>0$$

and

$$p = \frac{1}{\theta}(1-\tau_r)f''(k^*)<0.$$

Hence, the equation (34) has two real roots for the contrary signs

$$\lambda_{1,2} = \frac{(f'(k^*)-n-\delta)\pm\sqrt{\Delta}}{2}. \quad (35)$$

Using (32), it results that the relation (35) which gives the eigenvalues, can be written thus

$$\lambda_{1,2}=\frac{\frac{\rho}{1-\tau_r}\pm\sqrt{\Delta}}{2}$$

The eigenvalues of the linearized system being the real numbers with contrary signs, it results that the steady state (k^*,c^*) is the saddle point.

Because the steady state (k^*,c^*) is saddle point, there are two manifolds passing through the steady state: a stable manifold W_s and an instable manifold W_i. The dynamic equilibrium follows the stable manifold

Theorem 3 *i) The eigenvector, corresponding to the eigenvalue*

$$\lambda_1=\frac{\frac{\rho}{1-\tau_r}-n-\sqrt{(\frac{\rho}{1-\tau_r}-n)^2-\frac{4}{\theta}(1-\tau_r)f''(k^*)}}{2},$$

tangent in the steady state (k^*,c^*) *to the stable manifold* W_s *is given by* $v=\alpha(1,v_2)$, $\alpha\in\mathbb{R}$

$$v_2=\frac{\frac{\rho}{1-\tau_r}-n+\sqrt{(\frac{\rho}{1-\tau_r}-n)^2-\frac{4}{\theta}(1-\tau_r)f''(k^*)}}{2};$$

ii) The eigenvector, corresponding to the eigenvalue

$$\lambda_2=\frac{\frac{\rho}{1-\tau_r}-n+\sqrt{(\frac{\rho}{1-\tau_r}-n)^2-\frac{4}{\theta}(1-\tau_r)f''(k^*)}}{2},$$

tangent in the steady state (k^*,c^*) *to the stable manifold* W_i *is given by* $\omega=\alpha(1,\omega_2)$, $\alpha\in\mathbb{R}$ *where*

$$\omega_2=\frac{\frac{\rho}{1-\tau_r}-n-\sqrt{(\frac{\rho}{1-\tau_r}-n)^2-\frac{4}{\theta}(1-\tau_r)f''(k^*)}}{2}.$$

Proof: The matrix of the linearized system is :

$$A=\begin{pmatrix} f'(k^*)-(n+\delta) & -1\\ \frac{1}{\theta}(1-\tau_r)f''(k^*) & 0\end{pmatrix}$$

Its eigenvalues are the roots of the characteristic equation

$$\lambda^2-trA\lambda+\det A=0$$

where $trA=f'(k^*)$-(n+δ) $=\frac{\rho}{1-\tau_r}-n$ and $\det A=\frac{1}{\theta}(1-\tau_r)f''(k^*)$.

So, the eigenvalues are given by

$$\lambda_1=\frac{\frac{\rho}{1-\tau_r}-n-\sqrt{(\frac{\rho}{1-\tau_r}-n)^2-\frac{4}{\theta}(1-\tau_r)f''(k^*)}}{2}<0.$$

and

$$\lambda_2=\frac{\frac{\rho}{1-\tau_r}-n+\sqrt{(\frac{\rho}{1-\tau_r}-n)^2-\frac{4}{\theta}(1-\tau_r)f''(k^*)}}{2}>0$$

Next, we shall calculate the eigenvector $v=(v_1,v_2)^T$ associated to the eigenvalue λ_1. This vector is tangent in the steady state (k^*,c^*) to the optimal trajectory.

The eigenvector is the solution of the equation

$$Av=\lambda_1 v,\ \ v=(v_1,v_2)^T$$

Therefore,

$$\begin{pmatrix} f'(k^*)-n-\delta-\lambda_1 & -1\\ \frac{1}{\theta}(1-\tau_r)f''(k^*) & -\lambda_1\end{pmatrix}\begin{pmatrix} v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

which means that

$$\begin{aligned}\left(f'(k^*)-n-\delta-\lambda_1\right)v_1-v_2&=0\\ \frac{1}{\theta}(1-\tau_r)f''(k^*)v_1-\lambda_1 v_2&=0.\end{aligned}$$

Normalizing $v_1=1$ and taking into account (32) we obtain $v=(1,v_2)$, where

$$v_2=\frac{\frac{\rho}{1-\tau_r}-n+\sqrt{(\frac{\rho}{1-\tau_r}-n)^2-\frac{4}{\theta}(1-\tau_r)f''(k^*)}}{2}>0.$$

The slope of the stable manifold W_s in the steady state (k^*,c^*) will be given by v_2.

In the same way for the eigenvalue λ_2 we obtain the associated eigenvector $\omega=(1,\omega_2)$, where

$$\omega_2=\frac{\frac{\rho}{1-\tau_r}-n-\sqrt{(\frac{\rho}{1-\tau_r}-n)^2-\frac{4}{\theta}(1-\tau_r)f''(k^*)}}{2}<0.$$

The slope of the instable manifold W_i in the steady state (k^*,c^*) will be given by ω_2.

5 The dependence of the steady state (k^*,c^*) on the growth rate of the labor force n

Proposition 5 *The function* $c^*(n)$ *is strictly decreasing with respect to* n *and* k^* *does not depend on* n.

Proof: Differentiating with respect to n the relation

$$f'(k^*) = \frac{\rho}{1-\tau_r} + \delta$$

we obtain

$$f''(k^*)\frac{dk^*}{dn} = 0$$

consequently,

$$\frac{dk^*}{dn} = 0.$$

Therefore, because $\frac{dk^*}{dn} = 0$, $k^*(n)$ does not depend on n.

Differentiating with respect to n the relation

$$c^* = f(k^*) - (n+\delta)k^* - x$$

we obtain

$$\frac{dc^*}{dn} = f'(k^*)\frac{dk^*}{dn} - (n+\delta)\frac{dk^*}{dn} - k^*$$

or equivalently

$$\frac{dc^*}{dn} = \left(f'(k^*) - n - \delta\right)\frac{dk^*}{dn} - k^*$$

Because $\frac{dk^*}{dn} = 0$, we obtain $\frac{dc^*}{dn} = -k^* < 0$, therefore $c^*(n)$ is strictly decreasing with respect to n.

Due to the fact that the parameter n represents the growth rate of labor force, the result of the above theorem from an economic point of view, would be interpreted as following: the optimum level of capital is not influenced by labor force and if the labor force would grow in time then the optimum level of consumption per capita would shrink in time as well. A reduction of labor force would lead to a rise in the optimum consumption level per capita.

6 Change in Public Policy

We will study the effects of a tax change. We suppose that the economy is initially at the steady state corresponding to the given values of the different tax parameters. Also, we suppose that the government changes the tax rate on interest income from the initial value τ_r to a new one τ_r^1, keeping total expenditure per unit of labor $s+x$ constant, and adjusting the tax rate on wage income τ_w as needed to preserve budget balance at any moment in time.

In the following, we will determine the effects of the fiscal policy change on the steady-state values of capital k and consumption c, given by the solution of the system

$$\dot{c}(t) = 0 \Leftrightarrow f'(k) = \frac{\rho}{1-\tau_r} + \delta \quad (36)$$

$$\dot{k}(t) = 0 \Leftrightarrow c = f(k) - (n+\delta)k - x \quad (37)$$

Let (k^*, c^*) be the steady state value of the capital k and the consumption c.

Proposition 6 *The function $k^*(\tau_r)$ is strictly decreasing with respect to τ_r.*

Proof: Due to the relation (36), k^* verifies

$$f'(k^*) = \frac{\rho}{1-\tau_r} + \delta. \quad (38)$$

Differentiating with respect to τ_r on both members of the relation (38) we obtain

$$f''(k^*)\frac{dk^*}{d\tau_r} = \frac{\rho}{(1-\tau_r)^2}.$$

Hence, we have

$$\frac{dk^*}{d\tau_r} = \frac{1}{f''(k^*)}\frac{\rho}{(1-\tau_r)^2}.$$

This equality and the fact that the function f is strictly concave imply

$$\frac{dk^*}{d\tau_r} < 0.$$

Therefore, $k^*(\tau_r)$ is strictly decreasing with respect to τ_r.

With x fixed, the position of the $\dot{k} = 0$ line does not depend on the value of τ_r.

According to the Proposition 6, we have that a decrease in τ_r increases the steady-state value of the capital from the initial value k^* to a new one $k_1^*$and shifts the $\dot{c}(t) = 0$ isocline to the right, determining the new steady-state value of the consumption, as in fig. 2.

The effect on consumption depends on the slope of the $\dot{k}(t) = 0$ isocline at the steady state (k^*, c^*), given by

$$c'(k^*) = f'(k^*) - (n+\delta) = \frac{\rho}{1-\tau_r} - n \quad (39)$$

Because $c'(k^*) > 0$, we obtain that a decrease in τ_r will grow the steady state value of consumption from the initial value c^* to a new one c_1^*.

Therefore, a decrease in τ_r encourages accumulation by growing the net return on saving. In the long run, therefore, the capital is an decreasing function of τ_r.

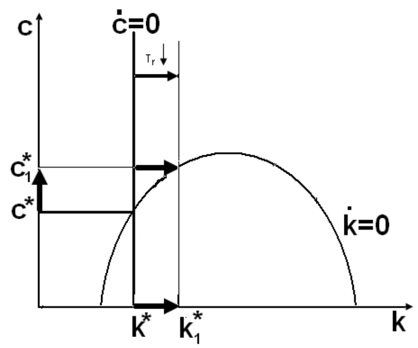

Fig. 2

Because the steady state utility, given by

$$u(c^*) = -\frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}e^{-\theta c^*}dt = -\frac{1}{\theta(\rho-n)}e^{-\theta c^*}$$

is a decreasing function of steady state consumption, decreases in τ_r will grow steady state value of consumption and therefore will grow welfare.

In order to calculate the net changes upon welfare, we shall evaluate the utility function with respect to the tax rate on interest income τ_r. For this we define

$$V(\tau_r) = \frac{1}{\theta}\int_0^\infty e^{-(\rho-n)t}\left(1 - e^{-\theta c_t(\tau_k)}\right)dt \quad (40)$$

which expresses the global utility of the representative household as a function of the tax rate on interest income τ_r.

$c_t(\tau_r)$ represents the consumption along the stable manifold of the system

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - (n+\delta)k(t) - c(t) - x \quad (41)\\ \dot{c}(t) &= -\frac{1}{\theta}[\rho - (1-\tau_r)(f'(k(t)) - \delta)] \quad (42) \end{aligned}$$

corresponding to a given value of the tax rate on interest income τ_r.

The stable manifold of the system (41)-(42) can be looked as the graph of the function

$$c = p(k, \tau_r) \quad (43)$$

which expresses the equilibrium value of consumption c as a function of the current value of the capital k and the tax rate on interest income τ_r. This relation is known in dynamic programming as a *policy function*: it relates the optimal value of a control variable to the state variable.

Because the slope of the stable manifold at the steady state is negative , being given by the slope of the stable eigenvector corresponding to the negative eigenvalue λ_1

$$\begin{aligned} p_k = v_2 &= \frac{1}{2}\left(\frac{\rho}{1-\tau_r} - n\right) + \quad (44)\\ &+ \frac{1}{2}\sqrt{\left(\frac{\rho}{1-\tau_r} - n\right)^2 - \frac{4}{\theta}(1-\tau_r)f''(k^*)} \end{aligned}$$

we have that an increases of the tax rate on interest income τ_r will shift the manifold upward.

Thus,

$$p_{\tau_r} > 0. \quad (45)$$

Substituting the policy function into the evolution equation of the capital, the relation (41), we obtain the evolution equation of the capital along the stable manifold

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - (n+\delta)k(t) - p(k(t), \tau_r) - x = \\ &= \chi(k(t), \tau_r). \quad (46) \end{aligned}$$

This equation has a unique steady state that coincides with the steady state of the system (41)-(42).

Taking into account (44) and (46), we have

$$\begin{aligned} \chi_k(k^*, \tau_r) &= -p_k(k^*, \tau_r) + \quad (47)\\ &+ \left(f'(k^*) - n - \delta\right) = \lambda_1 < 0 \end{aligned}$$

therefore, that the steady state is stable.

The solution of the differential equation (46), denote with $k_t(\tau_r)$, us give the equilibrium value of k as a function of time and the tax rate on interest income τ_r.

Substituting the solution of the differential equation (46) into the policy function we obtain the trajectory of consumption c as a function of time and the tax rate on interest income τ_r

$$c_t(\tau_r) = p(k_t(\tau_r), \tau_r). \quad (48)$$

In order to analyze the effect of the tax rate on interest income τ_r on the consumption we will calculate the marginal change in consumption at the moment t, so that:

$$\frac{dc_t(\tau_r)}{d\tau_r} = p_k(k_t(\tau_r), \tau_r)\frac{dk_t(\tau_r)}{d\tau_r} + p_{\tau_r}(k_t(\tau_r), \tau_r). \quad (49)$$

The solution of the equation (46), $k_t(\tau_r)$, must satisfy the equation (41)

$$\dot{k}(t,\tau_r) = f(k(t,\tau_r)) - (n+\delta)k(t,\tau_r) - p(k(t,\tau_r),\tau_r) - x \quad (50)$$

where we denote $k_t(\tau_r) = k(t, \tau_r)$.

Differentiating the equation (50) with respect to τ_r and taking into account the fact that $\dot{k}_{\tau_r}(t, \tau_r) = \frac{dk_{\tau_r}}{dt}(t, \tau_r)$, we have:

$$\frac{dk_{\tau_r}}{dt}(t, \tau_r) = f'(k(t,\tau_r))\frac{dk}{d\tau_r}(t,\tau_r) - (n+\delta)\frac{dk}{d\tau_r}(t,\tau_r) - -p_k(k(t,\tau_r), \tau_r)\frac{dk}{d\tau_r}(t,\tau_r) - p_{\tau_r}(k(t,\tau_r),\tau_r)$$

$$\frac{dk_{\tau_r}}{dt}(t,\tau_r) = \frac{dk}{d\tau_r}(t, \tau_r)(f'(k(t,\tau_r)) - (n+\delta) - -p_k(k(t,\tau_r),\tau_r)) - p_{\tau_r}(k(t,\tau_r),\tau_r) \quad (51)$$

The equation (51) can be rewritten at the steady state such as:

$$\frac{dk_{\tau_r}}{dt}(t, \tau_r) = \lambda_1 \frac{dk}{d\tau_r}(t, \tau_r) - p_{\tau_r}(k^*, \tau_r)$$

or equivalent with

$$\dot{k}_{\tau_r}(t) = \lambda_1 k_{\tau_r}(t) - p_{\tau_r}(k^*, \tau_r) \quad (52)$$

which is a differential equation with constant coefficients

The general solution of the differential equation (52) is given by:

$$k_{\tau_r}(t) = e^{\lambda_1 t} k_{\tau_r}(0) + \left(1 - e^{\lambda_1 t}\right) \frac{p_{\tau_r}(k^*, \tau_r)}{\lambda_1} \quad (53)$$

Because $\lambda_1 < 0$, the steady state value of this equation is stabile and k_{τ_r} asymptotically converges at the steady state value, given by:

$$k^*_{\tau_r} = \frac{p_{\tau_r}(k^*, \tau_r)}{\lambda_1} < 0 \quad (54)$$

which is the derived of the steady state value of the capital k with respect to τ_r.

Due to the fact that $k_{\tau_r}(0)$ is predetermined, it results that the changes in τ_r does not produce changes in $k_{\tau_r}(0)$.

Hence, the trajectory of the capital k_{τ_r} is given by:

$$k_{\tau_r}(t) = \left(1 - e^{\lambda_1 t}\right) \frac{p_{\tau_r}(k^*, \tau_r)}{\lambda_1} \quad (55)$$

From the equation (55), we note that a reduction in the tax rate on interest income τ_r determines a reduction of capital k, towards the new value of the steady state , at an exponential rate given by the stabile root of the system.

Substituting (55) in (49), we can calculate the derived of the consumption trajectory as function of τ_r:

$$\frac{dc_t(\tau_r)}{d\tau_r} = p_k(k^*, \tau_r)(1 - e^{\lambda_1 t})\frac{p_{\tau_r}(k^*, \tau_r)}{\lambda_1} + + p_{\tau_r}(k^*, \tau_r) = p_{\tau_r}\left((1 - e^{\lambda_1 t})\frac{p_k}{\lambda_1} + 1\right). \quad (56)$$

from the equation(45) we obtain:

$$\frac{dc_0(\tau_r)}{d\tau_r} = p_{\tau_r} > 0. \quad (57)$$

Using (47), (39) and (45) in (56) we obtain:

$$\frac{dc_\infty(\tau_r)}{d\tau_r} = p_{\tau_r}\left(\frac{p_k}{\lambda_1} + 1\right) = p_{\tau_r}\left(\frac{p_k + \lambda_1}{\lambda_1}\right) = = \frac{p_{\tau_r}(f'(k^*) - n - \delta)}{\lambda_1} = = \frac{p_{\tau_r}}{\lambda_1}\left(\frac{\rho}{1 - \tau_r} - n\right) < 0. \quad (58)$$

Considering the relation (57) we obtain that the immediate effects of fiscal policy consist of the growth of consumption per capita, witch determines the shift of the optimal trajectory upwards. Considering the relation (58) we obtain that the long term effects consist in the reduction of consumption per capita until the reach of the new steady state value.

The relations (57)-(58) show that an increase in the tax rate of the interest income τ_r will discourage the accumulation due to the reduction of the net income rate, witch will consequently encourage the short-term consumption, thus, on the long-term, the capital will be reduced, a reduction witch will determine a reduction of consumption on the long-term

Next, we shall analyze the effects of the change of income rate of capital τ_r over social welfare.

Differentiating with respect to τ_r

$$V(\tau_r) = \frac{1}{\theta}\int_0^\infty e^{-(\rho - n)t}\left(1 - e^{-\theta c_t(\tau_k)}\right) dt \quad (59)$$

evaluated in the steady state and using (56), we obtain

$$
\begin{aligned}
V'(\tau_k) &= e^{-\theta c^*}\int_0^\infty e^{-(\rho-n)t}\frac{dc_t(\tau_r)}{d\tau_r}dt = \qquad (60)\\
&= e^{-\theta c^*}\int_0^\infty e^{-(\rho-n)t}p_{\tau_r}\left((1-e^{\lambda_1 t})\frac{p_k}{\lambda_1}+1\right)dt=\\
&= e^{-\theta c^*}p_{\tau_r}\int_0^\infty e^{-(\rho-n)t}\left(\frac{p_k+\lambda_1}{\lambda_1}-e^{\lambda_1 t}\frac{p_k}{\lambda_1}\right)dt=\\
&= e^{-\theta c^*}p_{\tau_r}\frac{p_k+\lambda_1}{\lambda_1}\int_0^\infty e^{-(\rho-n)t}dt-\\
&- e^{-\theta c^*}p_{\tau_r}\frac{p_k}{\lambda_1}\int_0^\infty e^{-(\rho-n)t}e^{\lambda_1 t}dt=\\
&= e^{-\theta c^*}\frac{p_{\tau_r}}{\lambda_1}\left(\frac{p_k+\lambda_1}{\rho-n}-\frac{p_k}{\rho-n-\lambda_1}\right)=\\
&= e^{-\theta c^*}p_{\tau_r}\frac{\rho-n-p_k-\lambda_1}{(\rho-n)(\rho-n-\lambda_1)}.
\end{aligned}
$$

Using (47) and (39) in (60), we obtain

$$
\begin{aligned}
V'(\tau_r) = e^{-\theta c^*}p_{\tau_r}\frac{\rho-n-\dfrac{\rho}{1-\tau_r}}{(\rho-n)(\rho-n-\lambda_1)} =\\
= e^{-\theta c^*}p_{\tau_r}\frac{1}{\rho-n-\lambda_1}\frac{-\tau_r}{1-\tau_r}. \qquad (61)
\end{aligned}
$$

Because $p_{\tau_r} > 0$, $\lambda_1 < 0$ and $\rho > n$, we have $V'(\tau_r) < 0$. Therefore, the welfare is strictly decreasing function as function of the tax rate on interest income τ_r .

Hence, we have that the optimal policy is that the tax rate on interest income τ_r be zero.

7 Conclusion

In this paper we have analyzed an economic growth model with taxes in which the utility is given by exponential function. This economic growth model leads us to an optimal control problem. We have determined the necessary and sufficient conditions for optimality. Using these conditions, we have shown that the optimal trajectory for the optimal control problem is the solution for a differential equations system. We have proven the existence, uniqueness and stability of the study state for a differential equations system. We have shown that the optimum level of capital is not influenced by labor force and if the labor force would grow in time then the optimum level of consumption per capita would shrink in time as well. We have investigated the effects of the fiscal policy changes on the steady-state values of the capital and consumption, and on welfare. We have shown that a decrease in tax rate on interest income τ_r encourages accumulation by growing the net return on saving. On the long run, therefore, the capital is a decreasing function of τ_r. Also, we have shown that a decrease in tax rate on interest income τ_r will grow the welfare.

References

[1] R.–J. Barro, X. Sala-i-Martin, *Economic Growth,* McGraw-H ill, New York, 1995.

[2] O. Bundău, An economical growth model with taxes and exponential utility, *Sci. Bull. of the ”Politehnica” University of Timisoara, Fasc. 2,* 2008.

[3] E. Burmeister, A.–R. Dobell, *Matematical theories of economic growth,* Great Britan, 1970.

[4] J. Coletsos, B. Kokkinis, Optimization Methods for Nonlinear Eliptic Optimal Control Problems with State Constraints, *Wseas Transaction on Mathematics, issue 11, vol. 5, pp. 1169-1182,* 2006.

[5] Ş. Cruceanu and C. Vârsan, *Elemente de control optimal şi Aplicaţii în economie,* Ed.Tehnică, Bucharest, 1978.

[6] Z. Man, X. Yu, Modelling and Stability Analysis of Nonlinear TCP Dynamics with RED, *Wseas Transaction on Mathematics, issue 8, vol. 6, pp. 796-802*, 2006.

[7] A. Seierstad, K. Sydsæter, *Optimal Control Theory with Economic Applications,* North-Holland, Amsterdam, 1987.

[8] T. Tamai, Optimal fiscal policy in an endogenous growth model with public capital: a note, *Journal of Economics, Vol 93, nr.1, pp. 81-93,* 2008.

[9] C. Udrişte, M. Ferrara, Multitime Models of Optimal Growth, *Wseas Transaction on Mathematics, issue 1, vol. 7,* 2008.