ON DIFFERENT TYPES OF NON-ADDITIVE SET MULTIFUNCTIONS

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Abstract: In this paper, we study different types of non-additive set multifunctions (such as: uniformly autocontinuous, null-additive, null-null-additive), presenting relationships among them and some of their properties regarding atoms and pseudo-atoms. We also study non-atomicity and non-pseudo-atomicity of regular null-additive set multifunctions defined on the Baire (Borel respectively) δ-ring of a Hausdorff locally compact space and taking values in the family of non-empty closed subsets of a real normed space.

Key–words: uniformly autocontinuous, null-null-additive, (pseudo)-atom, non-(pseudo)-atomic, extension, regular, Darboux property.

1 Introduction

The theory of fuzziness has many applications in probabilities (e.g. Dempster [3], Shafer [30]), computer and systems sciences, artificial intelligence (e.g. Mastorakis [21]), physics, biology, medicine (e.g. Pham, Brandl, Nguyen N.D. and Nguyen T.V. [27] in prediction of osteoporotic fractures), theory of probabilities, economic mathematics, human decision making (e.g. Liginlal on Ow [20]).

In the last years, many authors (e.g. Choquet [2], Denneberg [4], Dobrakov [5], Li [19], Pap [24, 25, 26], Precupanu [28], Sugeno [31], Suzuki [32]) investigated the non-additive field of measure theory due to its applications in mathematical economics, statistics or theory of games (see e.g. Aumann and Shapley [1]). In non-additive measure theory, some continuity conditions are used to prove important results with respect to non-additive measures (for example, Theorem of Egoroff in Li [19]). Many concepts and results of classical measure theory (such as: regularity, extension, decomposition, integral) have been studied in the set-valued case. In [11-15] and [22] we extended and studied the concepts of atom, pseudo-atom, Darboux property, semi-convexity to the case of set-valued set functions.

In this paper, we study different types of non-additive set multifunctions (such as: uniformly autocontinuous, null-additive, null-null-additive), presenting relationships among them and some of their properties regarding atoms and pseudo-atoms. We also study non-atomicity and non-pseudo-atomicity of regular null-additive set multifunctions defined on the Baire (Borel respectively) δ-ring of a Hausdorff locally compact space and taking values in $\mathcal{P}_f(X)$, the family of non-empty closed subsets of a real normed space X. We also improve in this paper several results of [11,12,13,14] established for multisubmeasures.

2 Preliminaries

Let T be an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{C}$ a ring of subsets of T. The usage of different types of the domain $\mathcal{C}$ will be adequate to the results that will be proved and also with respect to the references.

By $i = \overline{1,n}$ we mean $i \in \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{N}^* = \mathbb{N} \backslash \{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$, $\overline{\mathbb{R}}_+ = [0, +\infty]$ and $\overline{\mathbb{R}} = [-\infty, \infty]$. We make the convention $\infty - \infty = 0$.

Definition 2.1. A set function $\nu : \mathcal{C} \to \overline{\mathbb{R}}_+$ is said to be:

(i) *monotone* if $\nu(A) \le \nu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

(ii) *null-monotone* if for every $A, B \in \mathcal{C}$, $A \subseteq B$ and $\nu(B) = 0 \Rightarrow \nu(A) = 0$.

(iii) a *submeasure* (in the sense of Drewnowski [6]) if $\nu(\emptyset) = 0$, ν is monotone and *subadditive*, that is, $\nu(A \cup B) \le \nu(A) + \nu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

(iv) *finitely additive* if $\nu(\emptyset) = 0$ and $\nu(A \cup B) = \nu(A) + \nu(B)$, for every $A, B \in \mathcal{C}$, so that $A \cap B = \emptyset$.

(v) *exhaustive* if $\lim_{n\to\infty} \nu(A_n) = 0$, for every sequence of pairwise disjoint sets $(A_n) \subset \mathcal{C}$.

(vi) *increasing convergent* if $\lim_{n\to\infty} \nu(A_n) = \nu(A)$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \nearrow A$ (that is, $A_n \subseteq A_{n+1}$, for every $n \in \mathbb{N}^*$ and $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$).

(vii) *decreasing convergent* if $\lim_{n\to\infty} \nu(A_n) = \nu(A)$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow A$ (that is, $A_n \supseteq A_{n+1}$, for every $n \in \mathbb{N}^*$ and $A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{C}$).

(viii) *order-continuous* (shortly *o-continuous*) if $\lim_{n\to\infty} \nu(A_n) = 0$, for every sequence of sets $(A_n) \subset \mathcal{C}$, so that $A_n \searrow \emptyset$.

(ix) *autocontinuous from above* if for every $A \in \mathcal{C}$ and every $(B_n) \subseteq \mathcal{C}$, so that $\lim_{n\to\infty} \nu(B_n) = 0$, we have $\lim_{n\to\infty} \nu(A \cup B_n) = \nu(A)$.

(x) *uniformly autocontinuous* if for every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$, so that for every $A \in \mathcal{C}$ and every $B \in \mathcal{C}$, with $\nu(B) < \delta$, we have $\nu(A \cup B) < \nu(A) + \varepsilon$.

(xi) *null-additive* if $\nu(A \cup B) = \nu(A)$, whenever $A, B \in \mathcal{C}$ and $\nu(B) = 0$.

(xii) *null-null-additive* if $\nu(A \cup B) = 0$, whenever $A, B \in \mathcal{C}$ and $\nu(A) = \nu(B) = 0$.

Definition 2.2. Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be a set function, with $\nu(\emptyset) = 0$.

(i) A set $A \in \mathcal{C}$ is said to be an *atom* of ν if $\nu(A) > 0$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\nu(B) = 0$ or $\nu(A \backslash B) = 0$.

(ii) A set $A \in \mathcal{C}$ is called a *pseudo-atom* of ν if $\nu(A) > 0$ and $B \in \mathcal{C}$, $B \subseteq A$ implies $\nu(B) = 0$ or $\nu(B) = \nu(A)$.

(iii) ν is said to be *non-atomic* (*non-pseudo-atomic* respectively) if it has no atoms (no pseudo-atoms respectively).

Now, let (X, d) be a metric space. $\mathcal{P}_0(X)$ is the family of all non-empty subsets of X, $\mathcal{P}_f(X)$ the family of non-empty closed subsets of X and $\mathcal{P}_{bf}(X)$ the family of non-empty closed bounded subsets of X.

For every $M, N \in \mathcal{P}_0(X)$, we denote $h(M, N) = \max\{e(M, N), e(N, M)\}$, where $e(M, N) = \sup_{x\in M} d(x, N)$ is the excess of M over N and $d(x, N)$ is the distance from x to N. It is known that h becomes an extended metric on $\mathcal{P}_f(X)$ (i.e. is a metric which can also take the value $+\infty$) and h becomes a metric (called Hausdorff) on $\mathcal{P}_{bf}(X)$ (Hu and Papageorgiou [16]).

In the sequel, $(X, \|\cdot\|)$ will be a real normed space, with the distance d induced by its norm. On $\mathcal{P}_0(X)$ we consider the Minkowski addition "$\overset{\bullet}{+}$", defined by:

$$M \overset{\bullet}{+} N = \overline{M + N}, \text{ for every } M, N \in \mathcal{P}_0(X),$$

where $M + N = \{x + y | x \in M, y \in N\}$ and $\overline{M + N}$ is the closure of $M + N$ with respect to the topology induced by the norm of X.

We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_0(X)$, where 0 is the origin of X. We have $|M| = \sup_{x\in M} \|x\|$, for every $M \in \mathcal{P}_0(X)$.

Definition 2.3. I. If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a set multifunction, then μ is said to be:

(i) *monotone* if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

(ii) *null-monotone* if for every $A, B \in \mathcal{C}$, $A \subseteq B$ and $\mu(B) = \{0\} \Rightarrow \mu(A) = \{0\}$.

(iii) a *multisubmeasure* if it is monotone, $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$ (or, equivalently, for every $A, B \in \mathcal{C}$).

(iv) a *multimeasure* if $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

(v) *autocontinuous from above* if for every $A \in \mathcal{C}$ and every $(B_n) \subset \mathcal{C}$ so that $\lim_{n\to\infty} |\mu(B_n)| = 0$, we have $\lim_{n\to\infty} h(\mu(A \cup B_n), \mu(A)) = 0$.

(vi) *uniformly autocontinuous* if for every $\varepsilon > 0$, there is $\delta(\varepsilon) = \delta > 0$, so that for every $A \in \mathcal{C}$ and every $B \in \mathcal{C}$, with $|\mu(B)| < \delta$, we have $h(\mu(A \cup B), \mu(A)) < \varepsilon$.

(vii) *null-additive* if for every $A, B \in \mathcal{C}$, $\mu(B) = \{0\} \Rightarrow \mu(A \cup B) = \mu(A)$.

(viii) *null-null-additive* if for every $A, B \in \mathcal{C}$, so that $\mu(A) = \mu(B) = \{0\}$, we have $\mu(A \cup B) = \{0\}$.

Remark 2.4. I. All the concepts of Definition 2.3 may also be defined in the case $X = \overline{\mathbb{R}}$ (for (iii) and (iv) we must suppose, moreover, that $\mu(A) + \mu(B)$ is well defined for every $A, B \in \mathcal{C}$).

II. If μ is $\mathcal{P}_f(X)$-valued, then in Definition 2.3-(iii) and (iv) it usually appears " $\dot{+}$ " instead of " $+$ ", because the sum of two closed sets is not always closed.

III. In some of our following results, we shall assume μ to be $\mathcal{P}_f(X)$-valued, when we need h to be an extended metric.

IV. Every monotone set multifunction is null-monotone.

V. Every monotone multimeasure is a multisubmeasure.

VI. For any multivalued set function $\mu : \mathcal{C} \to \mathcal{P}_0(X)$, we consider the set function $\overline{\mu} : \mathcal{P}(T) \to \overline{\mathbb{R}}_+$, called *the variation of* μ, defined for every $A \in \mathcal{P}(T)$ by:

$$\overline{\mu}(A) = \sup\{\sum_{i=1}^{n} |\mu(B_i)|; B_i \subset A, B_i \in \mathcal{C}, \forall i \in \{1, \ldots, n\}, B_i \cap B_j = \emptyset, \forall i \neq j\}.$$

For every $A \in \mathcal{C}$, we have $|\mu(A)| \leq \overline{\mu}(A)$. So, if $\overline{\mu}(A) = 0$, then $\mu(A) = \{0\}$. If μ is null-monotone, then $\overline{\mu}(A) = 0$ if and only if $\mu(A) = \{0\}$, for every $A \in \mathcal{C}$. If μ is a multisubmeasure, then $\overline{\mu}$ is finitely additive (Gavriluţ [7]).

Suppose $T \in \mathcal{C}$ and μ is a multisubmeasure, so that $\overline{\mu}$ is countably additive and $\overline{\mu}(T) > 0$. Then we can generate a system of upper and lower probabilities (with applications in statistical inference - see Dempster [3]) in the following way:

Let $\mathcal{A} = \{E \subset X | \mu^{-1}(E), \mu^{+1}(E) \in \mathcal{C}\}$, where for every $E \subset X$,

$$\mu^{-1}(E) = \{t \in T | \mu(\{t\}) \cap E \neq \emptyset\}$$

and $\mu^{+1}(E) = \{t \in T | \mu(\{t\}) \subset E\}$. For every $E \in \mathcal{A}$, we define *the upper probability* of E to be

$$P^*(E) = \frac{\overline{\mu}(\mu^{-1}(E))}{\overline{\mu}(T)}$$

and *the lower probability* of E to be

$$P_*(E) = \frac{\overline{\mu}(\mu^{+1}(E))}{\overline{\mu}(T)}.$$

We remark that P^*, $P_* : \mathcal{A} \to [0, 1]$ and $P_*(E) \leq P^*(E)$, for every $E \in \mathcal{A}$.

One may regard $\overline{\mu}(\mu^{-1}(E))$ as the largest possible amount of probability from the measure $\overline{\mu}$ that can be transferred to outcomes $x \in E$ and $\overline{\mu}(\mu^{+1}(E))$ as the minimal amount of probability that can be transferred to outcomes $x \in E$.

Remark 2.5. Definitions 2.3 generalize those of Definition 2.1 in two directions.

I. Let $\nu : \mathcal{C} \to \overline{\mathbb{R}}_+$ be a set function and $\mu : \mathcal{C} \to \mathcal{P}_f(\overline{\mathbb{R}}_+)$ defined by $\mu(A) = \{\nu(A)\}$, for every $A \in \mathcal{C}$. Then the following statements hold:

(i) μ is null-monotone (null-additive, null-null-additive, autocontinuous from above respectively) if and only if the same is ν.

(ii) μ is a multimeasure if and only if ν is finitely additive.

(iii) μ is monotone if and only if ν is constant, $\nu(A) = \alpha \in [0, +\infty]$, for every $A \in \mathcal{C}$. In this case, $\mu(A) = \{\alpha\}$, for every $A \in \mathcal{C}$. So, the monotonicity becomes interesting in set-valued case, when the set multifunction is not single-valued.

(iv) If μ is uniformly autocontinuous, then ν is uniformly autocontinuous too. Indeed, let $\varepsilon > 0$. Since μ is uniformly autocontinuous, there is $\delta(\varepsilon) = \delta > 0$ such that

$$(1) \qquad \begin{array}{l} \forall A \in \mathcal{C}, \forall B \in \mathcal{C},\ |\mu(B)| < \delta \\ \Rightarrow h(\mu(A \cup B), \mu(A)) < \varepsilon. \end{array}$$

Let $A \in \mathcal{C}$ and $B \in \mathcal{C}$ so that $\nu(B) = |\mu(B)| < \delta$. From (1), it follows $h(\mu(A \cup B), \mu(A)) = |\nu(A \cup B) - \nu(A)| < \varepsilon$, which implies $\nu(A \cup B) < \nu(A) + \varepsilon$. So ν is uniformly autocontinuous.

The converse is not valid. For example, let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$, $\nu(T) = 1, \nu(\{a\}) = 0, \nu(\{b\}) = \nu(\emptyset) = 2$ and $\mu(A) = \{\nu(A)\}$, for every $A \in \mathcal{C}$.

We prove that ν is uniformly autocontinuous: for every $\varepsilon > 0$, let $\delta = \frac{1}{2} > 0$. Then $\nu(B) < \frac{1}{2} \Rightarrow B = \{a\}$. We now have $\nu(A \cup B) < \nu(A) + \varepsilon$, for every $A \in \mathcal{C}$. So ν is uniformly autocontinuous.

But μ is not uniformly autocontinuous. Indeed, there exists $\varepsilon = 1$ such that for every $\delta > 0$, there exist $A = \{b\}$ and $B = \{a\}$ with $|\mu(B)| = 0 < \delta$, so that $h(\mu(A \cup B), \mu(A)) = 1 = \varepsilon$.

(v) If ν is monotone and uniformly autocontinuous, then μ is also uniformly autocontinuous. This results from the following equality: $h(\mu(A \cup B), \mu(A)) = |\nu(A \cup B) - \nu(A)| = \nu(A \cup B) - \nu(A)$, for every $A, B \in \mathcal{C}$, since ν is monotone.

II. Let $\nu : \mathcal{C} \to \overline{\mathbb{R}}_+$ be a set function and $\mu : \mathcal{C} \to \mathcal{P}_f(\overline{\mathbb{R}}_+)$ defined by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. Then the following statements hold:

(i) μ is monotone (null-monotone, autocontinuous from above, null-additive, null-null-additive respectively) if and only if the same is ν.

(ii) μ is a multisubmeasure (a multimeasure respectively) if and only if ν is a submeasure (finitely additive respectively).

(iii) If μ is uniformly autocontinuous, then ν is also uniformly autocontinuous. (One reasons like in I-(iv) from above). To see that the converse is not valid, we consider ν defined like in I-(iv) and $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. Thus, ν is uniformly autocontinuous, but μ is not uniformly autocontinuous.

(iv) If ν is monotone and uniformly autocontinuous, then μ is uniformly autocontinuous. (The proof follows like in I-(v) from above).

Theorem 2.6. *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction. Then the following statements hold:*

I. If μ is a multisubmeasure, then μ is uniformly autocontinuous.

II. If μ is a multisubmeasure, then μ is null-additive.

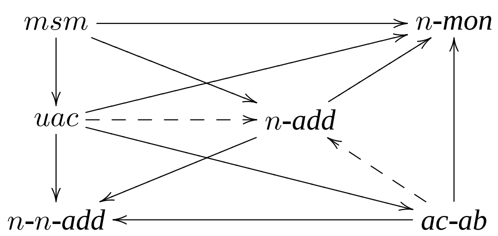
III. If μ is uniformly autocontinuous, then μ is autocontinuous from above and null-null-additive.

IV. If μ is autocontinuous from above, then μ is null-monotone and null-null-additive.

V. If μ is null-additive, then μ is null-null-additive and null-monotone.

VI. Suppose $\mu : \mathcal{C} \to \mathcal{P}_f(X)$. If μ is autocontinuous from above, then μ is null-additive.

These relationships are synthetized in the following schema:



" $- - \to$ " means the hypothesis "$\mu : \mathcal{C} \to \mathcal{P}_f$"
msm=multisubmeasure
n-mon=null-monotone
uac=uniformly autocontinuous
n-add=null-additive
n-n-add=null-null-additive
ac-ab=autocontinuous from above

Proof. I. Let $A \in \mathcal{C}$, $\varepsilon > 0$ and $B \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$. Since μ is monotone, it results $\mu(A) \subseteq \mu(A \cup B)$ which implies $e(\mu(A), \mu(A \cup B)) = 0$.

Since $\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, it follows:

$$e(\mu(A \cup B), \mu(A)) \leq h(\mu(A) + \mu(B), \mu(A)) \leq |\mu(B)| < \varepsilon.$$

So, $h(\mu(A \cup B), \mu(A)) < \varepsilon$, which proves that μ is uniformly autocontinuous.

II. Let $A, B \in \mathcal{C}$, so that $\mu(B) = \{0\}$. Since μ is monotone, we have $\mu(A) \subseteq \mu(A \cup B)$. Since μ is a multisubmeasure, we have $\mu(A \cup B) \subseteq \mu(A) + \mu(B) = \mu(A)$. So $\mu(A \cup B) = \mu(A)$, which proves that μ is null-additive.

III. First, we prove that μ is autocontinuous from above. Let $A \in \mathcal{C}$ and $(B_n) \subset \mathcal{C}$, so that $|\mu(B_n)| \to 0$. Since μ is uniformly autocontinuous, for every $\varepsilon > 0$, there is $\delta(\varepsilon) = \delta > 0$, so that for every $A \in \mathcal{C}$ and every $B \in \mathcal{C}$, with $|\mu(B)| < \delta$, we have

$$h(\mu(A \cup B), \mu(A)) < \varepsilon. \tag{2}$$

Since $|\mu(B_n)| \to 0$, there is $n_0 \in \mathbb{N}$, such that $|\mu(B_n)| < \delta$, for every $n \in \mathbb{N}$, $n \geq n_0$. From (2) it follows $h(\mu(A \cup B_n), \mu(A)) < \varepsilon$, for every natural $n \geq n_0$, which implies that $\lim_{n\to\infty} h(\mu(A \cup B_n), \mu(A)) = 0$. So μ is autocontinuous from above. We now prove that μ is null-null-additive. Let $A, B \in \mathcal{C}$, such that $\mu(A) = \mu(B) = \{0\}$. So, $|\mu(B)| = 0 < \delta$ and, since μ is uniformly autocontinuous, it results $|\mu(A \cup B)| < \varepsilon$, for every $\varepsilon > 0$. This implies $\mu(A \cup B) = \{0\}$. So μ is null-null-additive.

IV. First, we prove that μ is null-monotone. Let $A, B \in \mathcal{C}$, so that $A \subseteq B$ and $\mu(B) = \{0\}$. Let $B_n = B$, for every $n \in \mathbb{N}$. So $|\mu(B_n)| \to 0$. Since μ is autocontinuous from above, we obtain $|\mu(A)| = h(\mu(A \cup B_n), \mu(A)) \to 0$. This implies $|\mu(A)| = 0$ and so, $\mu(A) = \{0\}$, which shows that μ is null-monotone. We now show that μ is null-null-additive. Let $A, B \in \mathcal{C}$, such that $\mu(A) = \mu(B) = \{0\}$ and let $B_n = B$, for every $n \in \mathbb{N}$. Then $|\mu(B_n)| \to 0$. Since μ is autocontinuous from above, we have $\lim_{n\to\infty} h(\mu(A \cup B_n), \mu(A)) = 0$. This implies $|\mu(A \cup B)| = 0$, so $\mu(A \cup B) = \{0\}$ and thus μ is null-null-additive.

V. It results straightforward from definitions.

VI. Let $A, B \in \mathcal{C}$ so that $\mu(B) = \{0\}$. We consider $B_n = B$, for every $n \in \mathbb{N}$, so $|\mu(B_n)| \to 0$. By the autocontinuity from above, it follows $h(\mu(A \cup B), \mu(A)) = 0$. Since μ is $\mathcal{P}_f(X)$-valued, it results $\mu(A \cup B) = \mu(A)$, which proves that μ is null-additive. □

In the following examples we observe that the converses of the statements of Theorem 2.6 are not valid.

Examples 2.7

I. Let $T = \mathbb{N}$, $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined for every $A \in \mathcal{C}$ by $\mu(A) = \{0\}$ if A is finite and $\mu(A) = [1, \infty)$, if A is countable. Then μ is uniformly autocontinuous and it is not a multisubmeasure.

II. Let $T = \{a, b\}, \mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(T) = [0, 2]$, $\mu(\{a\}) = \mu(\{b\}) =$

$[0, \frac{1}{2}]$ and $\mu(\emptyset) = \{0\}$. Then μ is null-additive, but it is not a multisubmeasure.

III. Let $T = [0,1]$, $\mathcal{C}$ the Borel σ-algebra on T, $\lambda : \mathcal{C} \to \mathbb{R}_+$ the Lebesgue measure and $\mu : \mathcal{C} \to \mathcal{P}_f(\overline{\mathbb{R}}_+)$ defined by $\mu(A) = \{\nu(A)\}$, where $\nu(A) = \mathrm{tg}(\frac{\pi}{2}\lambda(A))$, for every $A \in \mathcal{C}$.

According to Example 4-[17], ν is autocontinuous from above. From Remark 2.5-I-(i), it results that μ is autocontinuous from above.

According to Example 4-[17], ν is not uniformly autocontinuous. Now, from Remark 2.5-I-(iv), it follows that μ is not uniformly autocontinuous.

IV. Let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(T) = [0,2]$, $\mu(\{b\}) = [0,1]$ and $\mu(\{a\}) = \mu(\emptyset) = \{0\}$. Then μ is null-monotone and null-null-additive, but it is not a multisubmeasure, not null-additive and, since μ is $\mathcal{P}_f$-valued, not uniformly autocontinuous and not autocontinuous from above.

V. Let $T = \{a, b\}, \mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(A) = \{1,2\}$ if $A = T$ and $\mu(A) = \{0\}$ otherwise. Then μ is null-monotone, but μ is not null-null-additive and not null-additive.

VI. Let $T = \{a, b\}, \mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(A) = \{1,2\}$ if $A = \{a\}$ or $A = \{b\}$, $\mu(\emptyset) = \{3\}$ and $\mu(\{a,b\}) = \{0\}$. Then μ is null-null-additive, but not null-monotone.

VII. Let $T = [0, +\infty)$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(\emptyset) = \{0\}$, $\mu(A) = A$ if $\mathrm{card}A = 1$, $\mu(A) = [0, \delta(A)]$ if A is bounded with $\mathrm{card}A \geq 2$ and $\mu(A) = [0, \infty)$ if A is not bounded. Here, $\mathrm{card}A$ is the cardinal of A and $\delta(A) = \sup\{\|t - s\|; t, s \in A\}$ is the diameter of A. Then μ is null-additive, but not autocontinuous from above. Indeed, there exist $A = \{1\}$ and $B_n = [0, \frac{1}{n}]$, for every $n \in \mathbb{N}^*$, such that $|\mu(B_n)| = \frac{1}{n} \to 0$, but $h(\mu(A \cup B_n), \mu(A)) = h([0,1], \{1\}) = 1 \nrightarrow 0$.

Remark 2.8. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction and the set function $|\mu| : \mathcal{C} \to \overline{\mathbb{R}}_+$ defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{C}$. Then the following statements hold:

I. μ is null-monotone (null-null-additive respectively) if and only if the same is $|\mu|$.

II. If μ is monotone, then $|\mu|$ is also monotone. The converse is not true. Indeed, let $T = \{a, b\}, \mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(T) = \{1\}$, $\mu(\{a\}) = \mu(\{b\}) = [0,1]$ and $\mu(\emptyset) = \{0\}$. We have $|\mu(A)| = 1$ if $A \neq \emptyset$ and $|\mu(\emptyset)| = 0$. Then $|\mu|$ is monotone, but μ is not monotone.

III. If μ is null-additive, then $|\mu|$ is null-additive. The converse is not valid. Indeed, let $T = \{a, b\}, \mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(T) = [0,1]$, $\mu(\{a\}) = \{1\}$ and $\mu(\{b\}) = \mu(\emptyset) = \{0\}$. We have $|\mu(A)| = 1$ if $A = T$ or $A = \{a\}$ and $|\mu(A)| = 0$ if $A = \{b\}$ or $A = \emptyset$. Then $|\mu|$ is null-additive, but μ is not null-additive.

IV. If μ is autocontinuous from above, then $|\mu|$ is autocontinuous from above and this results from the inequality:

$$\big||\mu(A \cup B)| - |\mu(A)|\big| \leq h(\mu(A \cup B), \mu(A)), \forall A, B \in \mathcal{C}.$$

The converse is not true. Indeed, let $T = [0,1]$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(\emptyset) = \mu(\{0\}) = \{0\}$, $\mu(A) = A$ if $A = \left[0, \frac{1}{n}\right]$, $n \in \mathbb{N}^*$, $\mu(A) = [0,1]$ if $A = \left[0, \frac{1}{n}\right] \cup \{1\}$, $n \in \mathbb{N}^*$ and $\mu(A) = \{1\}$ otherwise. Then $|\mu(A)| = 0$ if $A = \emptyset$ or $A = \{0\}$, $|\mu(A)| = \frac{1}{n}$ if $A = \left[0, \frac{1}{n}\right]$, $n \in \mathbb{N}^*$ and $|\mu(A)| = 1$ otherwise.

Let us prove that $|\mu|$ is autocontinuous from above. Consider $A \in \mathcal{C}$ and $(B_n) \subset \mathcal{C}$ so that $|\mu(B_n)| \to 0$. Then we may suppose, without any loss of generality, that $B_n \in \{\emptyset, \{0\}, [0, \frac{1}{n}]\}$, for every $n \in \mathbb{N}^*$. It follows $|\mu(A \cup B_n)| \to |\mu(A)|$, which proves that $|\mu|$ is autocontinuous from above.

We now show that μ is not autocontinuous from above. Indeed, there exist $A = \{1\}$ and $B_n = [0, \frac{1}{n}]$, for every $n \in \mathbb{N}^*$, such that $|\mu(B_n)| = \frac{1}{n} \to 0$ and $h(\mu(A \cup B_n), \mu(A)) = h([0,1], \{1\}) = 1 \nrightarrow 0$. So μ is not autocontinuous from above.

V. If μ is uniformly autocontinuous, then the same is $|\mu|$ and this results like in IV. The converse is not valid. Indeed, we consider μ as in IV. Since μ is not autocontinuous from above, according to Theorem 2.6-III, it results that μ is not uniformly autocontinuous. We prove that $|\mu|$ is uniformly autocontinuous. Let $\varepsilon > 0$ and $\delta = \varepsilon$. Also, let $B \in \mathcal{C}$, so that $|\mu(B)| < \delta = \varepsilon$. Then $|\mu(A \cup B)| < |\mu(A)| + \varepsilon$, for every $A \in \mathcal{C}$, which proves that $|\mu|$ is uniformly autocontinuous.

Definition 2.9. (Gavriluţ [7-10]) A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is said to be:

(i) *exhaustive* if $\lim\limits_{n \to \infty} |\mu(A_n)| = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$.

(ii) *order continuous* (shortly, *o-continuous*) $\lim\limits_{n \to \infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$, such that $A_n \searrow \emptyset$.

(iii) *increasing convergent* if $\mu(\bigcup\limits_{n=1}^{\infty} A_n) = \lim\limits_{n \to \infty} \mu(A_n)$ with respect to h, for every increasing sequence of sets $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$, such that $A_n \nearrow A$, where $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

(iv) *decreasing convergent* if $\mu(\bigcap\limits_{n=1}^{\infty} A_n) = \lim\limits_{n \to \infty} \mu(A_n)$ with respect to h, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, such that $A_n \searrow A$, where $A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{C}$.

(v) *fuzzy* if $\mu(\emptyset) = \{0\}$ and μ is monotone, increasing convergent and decreasing convergent.

Remark 2.10. (Gavriluţ [7-10]) I. If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is exhaustive and increasing convergent, then μ is o-continuous.

II. Suppose $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is monotone and o-continuous. Then μ is exhaustive.

III. Suppose $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is uniformly autocontinuous, with $\mu(\emptyset) = \{0\}$. Then the following statements hold:

(i) If μ is o-continuous, then μ is increasing convergent.

(ii) μ is o-continuous if and only if μ is decreasing convergent.

(iii) If μ is monotone, then μ is o-continuous if and only if it is fuzzy.

IV. If $\nu : \mathcal{C} \to \mathbb{R}_+$ is a set function and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ is defined by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$, then μ is exhaustive (o-continuous, increasing convergent, decreasing convergent, fuzzy respectively) if and only if the same is ν.

V. If $\mathcal{C}$ is finite, then any set multifunction, with $\mu(\emptyset) = \{0\}$ is exhaustive, o-continuous, increasing convergent and decreasing convergent.

3 Atoms and pseudo-atoms

In this section, we present some properties of atoms and pseudo-atoms for different types of set multifunctions.

Definition 3.1. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

(i) A set $A \in \mathcal{C}$ is said to be an *atom* of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

(ii) A set $A \in \mathcal{C}$ is called a *pseudo-atom* of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(B) = \mu(A)$.

(iii) μ is said to be *non-atomic* (*non-pseudo-atomic* respectively) if it has no atoms (no pseudo-atoms respectively).

(iv) μ has *the Darboux property* if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$ and every $p \in (0, 1)$, there is $B \in \mathcal{C}$ so that $B \subseteq A$ and $\mu(B) = p\,\mu(A)$.

Remark 3.2. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

I. If μ is monotone, then μ is non-atomic (non-pseudo-atomic respectively) if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there is $B \in \mathcal{C}$ so that $B \subseteq A$, $\mu(B) \supsetneq \{0\}$ and $\mu(A \backslash B) \supsetneq \{0\}$ ($\mu(A) \supsetneq \mu(B)$ respectively).

II. If μ is null-monotone, then $A \in \mathcal{C}$ is an atom of μ if and only if A is an atom of $\overline{\mu}$.

III. If μ is null-additive, then every atom of μ is a pseudo-atom of μ (as we shall see in Examples 3.5-I, the converse is not valid). Consequently, any non-pseudo-atomic monotone null-additive set multifunction is non-atomic.

Definition 3.3. Let $\mu_1, \mu_2 : \mathcal{C} \to \mathcal{P}_0(X)$ be set multifunctions. One says that μ_1 is *absolutely continuous with respect to* μ_2 (denoted by $\mu_1 \ll \mu_2$) if for every $A \in \mathcal{C}$, $\mu_2(A) = \{0\} \Rightarrow \mu_1(A) = \{0\}$.

Remark 3.4.

I. Let $\mu_1, \mu_2 : \mathcal{C} \to \mathcal{P}_0(X)$ be monotone set multifunctions so that $\mu_1(\emptyset) = \mu_2(\emptyset) = \{0\}$ and $\mu_1 \ll \mu_2$. Let $A \in \mathcal{C}$, with $\mu_1(A) \supsetneq \{0\}$. If A is an atom of μ_2, then A is an atom of μ_1 too.

II. Suppose $\mu_1, \mu_2 : \mathcal{C} \to \mathcal{P}_0(X)$ are monotone set multifunctions so that $\mu_1(\emptyset) = \mu_2(\emptyset) = \{0\}$, $\mu_1 \ll \mu_2$ and $\mu_1(A) \supsetneq \{0\}$, for every $A \in \mathcal{C} \backslash \{\emptyset\}$. If μ_1 is non-atomic, then μ_2 is also non-atomic.

Example 3.5. I. Let $T = \{a, b, c\}, \mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined by $\mu(A) = [0, 1]$ if $A \neq \emptyset$ and $\mu(A) = \{0\}$ if $A = \emptyset$. Then μ is null-additive, $A = \{a, b\}$ is a pseudo-atom of μ, but not an atom of μ.

II. Let $T = 2\mathbb{N} = \{0, 2, 4, \ldots\}$, $\mathcal{C} = \mathcal{P}(T)$ and for every $A \in \mathcal{C}$:

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A = \emptyset \\ \frac{1}{2}A \cup \{0\}, & \text{if } A \neq \emptyset \end{cases}$$

where $\frac{1}{2}A = \{\frac{x}{2} |\ x \in A\}$. μ is a multisubmeasure.

If $A \in \mathcal{C}$, with card$A = 1$ and $A \neq \{0\}$ or $A \in \mathcal{C}, A = \{0, 2n\}$, $n \in \mathbb{N}^*$, then A is an atom of μ (and a pseudo-atom of μ too, according to Remark 3.2-III and Theorem 2.6-II). By card$A$ we mean the cardinal of A.

If $A \in \mathcal{C}$, with card$A \geq 2$ and there exist $a, b \in A$ such that $a \neq b$ and $ab \neq 0$, then A is not a pseudo-atom of μ (and not an atom of μ, according to Remark 3.2-III).

III. Let $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A \text{ is finite} \\ \{0\} \cup [n_A, +\infty), & \text{if } A \text{ is infinite and} \\ & n_A = \min A. \end{cases}$$

Then μ is monotone and non-pseudo-atomic.

Remark 3.6. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

I. If $A \in \mathcal{C}$ is a pseudo-atom of μ and $B \in \mathcal{C}, B \subseteq A$ such that $\mu(B) \supsetneq \{0\}$, then B is a pseudo-atom of μ and $\mu(B) = \mu(A)$.

II. Suppose μ is null-monotone and $\mu(\emptyset) = \{0\}$. If $A \in \mathcal{C}$ is an atom of μ and $B \in \mathcal{C}, B \subseteq A$ such that $\mu(B) \supsetneq \{0\}$, then B is an atom of μ and $\mu(A \backslash B) = \{0\}$.

Theorem 3.7. *Suppose $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is monotone, so that $\mu(\emptyset) = \{0\}$ and $A, B \in \mathcal{C}$ are pseudo-atoms of μ. Then the following statements hold:*

I. $\mu(A) \neq \mu(B) \Rightarrow \mu(A \cap B) = \{0\}$.

II. Suppose μ is null-null-additive. If $\mu(A \cap B) = \{0\}$, then $A \backslash B$ and $B \backslash A$ are pseudo-atoms of μ and $\mu(A \backslash B) = \mu(A)$, $\mu(B \backslash A) = \mu(B)$.

Proof. I) Suppose $\mu(A \cap B) \supsetneq \{0\}$. According to Remark 3.6-I, we have $\mu(A \cap B) = \mu(A) = \mu(B)$, which is false.

II. Let us prove that $\mu(A \backslash B) \supsetneq \{0\}$. Suppose on the contrary that $\mu(A \backslash B) = \{0\}$. Since μ is null-null-additive, we have $\mu(A) = \mu((A \backslash B) \cup (A \cap B)) = \{0\}$, which is false. So, $\mu(A \backslash B) \supsetneq \{0\}$ and from Remark 3.6-I, it results that $A \backslash B$ is a pseudo-atom of μ and $\mu(A \backslash B) = \mu(A)$. Analogously, $B \backslash A$ is a pseudo-atom of μ and $\mu(B \backslash A) = \mu(B)$. □

Theorem 3.8. *Suppose $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is monotone and null-null-additive, so that $\mu(\emptyset) = \{0\}$ and $A, B \in \mathcal{C}$ are pseudo-atoms of μ. Then there exist pairwise disjoint sets $E_1, E_2, E_3 \in \mathcal{C}$, with $A \cup B = E_1 \cup E_2 \cup E_3$, such that, for every $i \in \{1, 2, 3\}$, either E_i is a pseudo-atom of μ, or $\mu(E_i) = \{0\}$.*

Proof. Let $E_1 = A \cap B, E_2 = A \backslash B, E_3 = B \backslash A$. We have the following cases:

(i) $\mu(E_1) = \{0\}$. According to Theorem 3.7-II, E_2 and E_3 are pseudo-atoms of μ and $\mu(E_2) = \mu(A), \mu(E_3) = \mu(B)$.

(ii) $\mu(E_1) \supsetneq \{0\}, \mu(E_2) \supsetneq \{0\}, \mu(E_3) \supsetneq \{0\}$. By Remark 3.6-I, E_1 is a pseudo-atom of μ and $\mu(E_1) = \mu(A) = \mu(B)$. Analogously, E_2 and E_3 are pseudo-atoms of μ.

(iii) $\mu(E_1) \supsetneq \{0\}, \mu(E_2) = \{0\}, \mu(E_3) \supsetneq \{0\}$. From Remark 3.6-I, it results that E_1 is a pseudo-atom of μ and $\mu(E_1) = \mu(A) = \mu(B)$. Analogously, E_3 is a pseudo-atom of μ and $\mu(E_3) = \mu(B)$.

The last two cases are similar to (iii).

(iv) $\mu(E_1) \supsetneq \{0\}, \mu(E_2) \supsetneq \{0\}, \mu(E_3) = \{0\}$.

(v) $\mu(E_1) \supsetneq \{0\}, \mu(E_2) = \mu(E_3) = \{0\}$. □

Remark 3.9. By induction, the same result of Theorem 3.8 can be obtained for every finite family $\{A_i\}_{i=1}^n$ of pseudo-atoms of μ. Consequently, we can write $\bigcup\limits_{i=1}^{n} A_i = (\bigcup\limits_{j=1}^{m} B_j) \cup E$, where $\{B_j\}_{j=1}^m$, E are pairwise disjoint sets of $\mathcal{C}$, such that $\{B_j\}_{j=1}^m$ are pseudo-atoms of μ and $\mu(E) = \{0\}$.

Theorem 3.10. *Suppose $\mathcal{C}$ is a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is fuzzy, null-null-additive and exhaustive. Then there exists a sequence $(B_n)_{n \in \mathbb{N}^*}$ of pairwise disjoint pseudo-atoms of μ satisfying the conditions:*

(i) $|\mu(B_n)| \geq |\mu(B_{n+1})|, \forall n \in \mathbb{N}^*$,

(ii) $\lim\limits_{n\to\infty} |\mu(B_n)| = 0$,

(iii) $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^*$, such that $|\mu(\bigcup_{k=n_0}^{\infty} B_k)| < \varepsilon$.

Proof. Let $\mathcal{A}_m = \{E \in \mathcal{C} | E$ is a pseudo-atom of μ and $\frac{1}{m} \leq |\mu(E)| < \frac{1}{m+1}\}$, for every $m \in \mathbb{N}^*$. Then $\mathcal{A}_m$ contains at most finite pairwise disjoint sets. Suppose, on the contrary, there are infinite pairwise disjoint sets $(E_n)_{n \in \mathbb{N}^*} \subset \mathcal{A}_m$. So, we have $|\mu(E_n)| \geq \frac{1}{m}$, for every $n \in \mathbb{N}^*$. Since μ is exhaustive, it follows $\lim\limits_{n\to\infty} |\mu(E_n)| = 0$, which is false. Hence, there exist at most finite pairwise disjoint pseudo-atoms in $\mathcal{A}_m$, for every $m \in \mathbb{N}^*$ and denote all of them by $\{B_n\}_{n=1}^{\infty}$. Now, (i) is evidently satisfied. Since (B_n) are pairwise disjoint and μ is exhaustive, it results (ii). We remark that $\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} B_k = \emptyset$. If we denote $A_n = \bigcup\limits_{k=n}^{\infty} B_k$, for every $n \in \mathbb{N}^*$, then we have $A_n \searrow \emptyset$. Since μ is o-continuous (according to Remark 2.10-I), it follows $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$. Consequently, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}^*$, such that $|\mu(A_{n_0})| < \varepsilon$, that is $|\mu(\bigcup\limits_{k=n_0}^{\infty} B_k)| < \varepsilon$, which proves (iii). □

In the end of this section, we establish the following result which will be useful in section 4.

Proposition 3.11. *Suppose $\mathcal{C}_1$, $\mathcal{C}_2$ are two rings so that $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and $\mathcal{C}_1$ is dense in $\mathcal{C}_2$ with respect to a monotone null-additive set multifunction $\mu : \mathcal{C}_2 \to \mathcal{P}_f(X)$ (that is, for every $\varepsilon > 0$ and every $A \in \mathcal{C}_2$, there is $B \in \mathcal{C}_1$ so that $B \subseteq A$ and $|\mu(A \backslash B)| < \varepsilon$), with $\mu(\emptyset) = \{0\}$. If μ is non-atomic (non-pseudo-atomic respectively) on $\mathcal{C}_2$, then μ is also non-atomic (non-pseudo-atomic respectively) on $\mathcal{C}_1$.*

Proof. Suppose that, on the contrary, there is an atom (pseudo-atom respectively) $A \in \mathcal{C}_1$ for $\mu_{/\mathcal{C}_1}$. Then $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}_1$ with $B \subseteq A$ we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$ ($\mu(A) = \mu(B)$ respectively).

Because $A \in \mathcal{C}_2, \mu(A) \supsetneq \{0\}$ and μ is non-atomic (non-pseudo-atomic respectively) on $\mathcal{C}_2$, there is $B_0 \in \mathcal{C}_2$ so that $B_0 \subseteq A$, $\mu(B_0) \supsetneq \{0\}$ and $\mu(A \backslash B_0) \supsetneq \{0\}$ ($\mu(A) \supsetneq \mu(B_0)$ respectively). Then $|\mu(B_0)| > 0$ and, since $\mathcal{C}_1$ is dense in $\mathcal{C}_2$, for $\varepsilon_0 = |\mu(B_0)|$, there exists $C_0 \in \mathcal{C}_1$ so that $C_0 \subseteq B_0$ and $|\mu(B_0 \backslash C_0)| < \varepsilon_0$.

Now, because $C_0 \in \mathcal{C}_1$ and $C_0 \subseteq A$, by the assumption made we get $\mu(C_0) = \{0\}$ or $\mu(A \backslash C_0) = \{0\}$ ($\mu(A) = \mu(C_0)$ respectively).

I. If $\mu(C_0) = \{0\}$, then, by the null-additivity of μ, $|\mu(B_0)| = |\mu((B_0 \backslash C_0) \cup C_0)| = |\mu(B_0 \backslash C_0)| < |\mu(B_0)|$, which is false.

II. If $\mu(A \backslash C_0) = \{0\}$ (respectively, $\mu(A) = \mu(C_0)$), then, in both cases, by the null-additivity of μ, $\mu(A) = \mu(C_0) \supsetneq \mu(B_0)$, which is false because $C_0 \subseteq B_0$, so $\mu(C_0) \subseteq \mu(B_0)$.

Consequently, μ is non-atomic (non-pseudo-atomic respectively) on $\mathcal{C}_1$. □

4 Extension theorem by preserving non-atomicity (non-pseudo-atomicity respectively)

In this section, X is a Banach space and $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ is an exhaustive set multifunction. In Gavriluţ and Croitoru [13] the following result is established:

Lemma 4.1. *For every $\varepsilon > 0$ and every $A \subseteq T$, there exists $K \in \mathcal{C}$ such that $K \subseteq A$ and $|\mu(B \backslash K)| < \varepsilon$, for every $B \in \mathcal{C}$, with $K \subseteq B \subseteq A$.*

Using Lemma 4.1, we obtain the following results which improve those of [13].

Theorem 4.2. *Let $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ be an exhaustive multisubmeasure. Then μ extends (ie. $\mu^*(A) = \mu(A)$, for every $A \in \mathcal{C}$) to an exhaustive monotone set multifunction $\mu^* : \mathcal{P}(T) \to \mathcal{P}_{bf}(X)$. If μ is non-atomic (non-pseudo-atomic respectively), then the same is μ^*.*

Proof. According to [13], it only remains to establish the non-pseudo-atomicity part. Suppose μ is non-pseudo-atomic and, on the contrary, there is a pseudo-atom A_0 for μ^*. Then $\mu^*(A_0) \supsetneq \{0\}$ and for every $B \subseteq T$, with $B \subseteq A_0$, we have $\mu^*(B) = \{0\}$ or $\mu^*(A_0) = \mu^*(B)$. Because $\mu^*(A_0) \supsetneq \{0\}$, by the definition of μ^*, there exists $C_0 \in \mathcal{C}$ so that $C_0 \subseteq A_0$ and $\mu(C_0) \supsetneq \{0\}$.

Since μ is non-pseudo-atomic, there is $D_0 \in \mathcal{C}$ so that $D_0 \subseteq C_0$, $\mu(D_0) \supsetneq \{0\}$ and $\mu(C_0) \supsetneq \mu(D_0)$. For D_0, $\mu^*(D_0) = \{0\}$ or $\mu^*(A_0) = \mu^*(D_0)$.

If $\mu^*(D_0) = \{0\}$, then $\mu(D_0) = \mu^*(D_0) = \{0\}$, which is false.

If $\mu^*(A_0) = \mu^*(D_0)$, then $\mu^*(D_0) = \mu(D_0) \subsetneq \mu(C_0) = \mu^*(C_0) \subseteq \mu^*(A_0) = \mu^*(D_0)$, a contradiction. So, μ^* is non–pseudo-atomic. □

From now on, suppose, moreover, that $\mathcal{C}$ is an algebra of subsets of T.

Consider $\mathcal{C}_\mu = \{A \subseteq T$; for every $\varepsilon > 0$, there exist $K, D \in \mathcal{C}$ such that $K \subseteq A \subseteq D$ and $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, with $B \subseteq D \backslash K\}$. We immediately observe that, because of the monotonicity of μ, we also have $\mathcal{C}_\mu = \{A \subseteq T$; for every $\varepsilon > 0$, there exist $K, D \in \mathcal{C}$ such that $K \subseteq A \subseteq D$ and $|\mu(D \backslash K)| < \varepsilon\}$.

One can easily check that $\mathcal{C} \subseteq \mathcal{C}_\mu$ and $\mathcal{C}_\mu$ is an algebra. Also, $\mathcal{C}$ is dense in $\mathcal{C}_\mu$ with respect to μ^*. Indeed, for every $\varepsilon > 0$ and every $A \in \mathcal{C}_\mu$, there exist $B, D \in \mathcal{C}$ so that $B \subseteq A \subseteq D$ and $|\mu(D \backslash B)| < \varepsilon$. Then $|\mu^*(A \backslash B)| \le |\mu^*(D \backslash B)| = |\mu(D \backslash B)| < \varepsilon$.

Theorem 4.3. *Let $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ be an exhaustive multisubmeasure. If μ is non-atomic (non-pseudo-atomic respectively), then the same is $\mu^*_{/\mathcal{C}_\mu}$ and it uniquely extends μ.*

Proof. According to [13] and also the same as in the proof of Theorem 4.2, we get that $\mu^*_{/\mathcal{C}_\mu}$ is non-atomic (non-pseudo-atomic respectively).

We now prove that the extension μ^* is unique. Suppose, on the contrary, there is another set multifunction $\varphi : \mathcal{C}_\mu \to \mathcal{P}_{bf}(X)$ having the properties of $\mu^*_{/\mathcal{C}_\mu}$, which extends μ. Let $A \in \mathcal{C}_\mu$ be arbitrarily. By the definition of $\mathcal{C}_\mu$, there are $K, D \in \mathcal{C}$ so that $K \subseteq A \subseteq D$ and $|\mu(D \backslash K)| < \varepsilon$. Then for every $\varepsilon > 0$, we have:

$$\begin{aligned} & e(\mu^*(A), \varphi(A)) \le e(\mu^*(A), \mu^*(D)) + \\ & + e(\mu^*(D), \varphi(A)) = e(\mu(D), \varphi(A)) \le \\ & \le e(\mu(D), \mu(K)) + e(\mu(K), \varphi(A)) = \\ & e(\mu(D), \mu(K)) \le \\ & \le |\mu(D \backslash K)| < \varepsilon, \end{aligned}$$

hence $\mu^*(A) \subseteq \varphi(A)$. On the other hand,

$$\begin{aligned} & e(\varphi(A), \mu^*(A)) \\ & \le e(\varphi(A), \varphi(D)) + e(\varphi(D), \mu^*(A)) = \\ & = e(\varphi(D), \mu^*(A)) = e(\mu(D), \mu^*(A)) \le \\ & \le e(\mu(D), \mu(K)) + e(\mu(K), \mu^*(A)) \le \\ & \le |\mu(D \backslash K)| + e(\mu^*(K), \mu^*(A)) = \\ & |\mu(D \backslash K)| < \varepsilon \end{aligned}$$

and the conclusion follows. □

Corollary 4.4. Let $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ be an exhaustive multisubmeasure. Then μ is non-atomic (non-pseudo-atomic respectively) on $\mathcal{C}$ if and only if $\mu^*_{/\mathcal{C}_\mu}$ is non-atomic (non-pseudo-atomic respectively).

Proof. The "*if part*" follows by Theorem 4.3 and the "*only if part*" follows by Proposition 3.11, since $\mathcal{C}$ is dense in $\mathcal{C}_\mu$. □

5 Regular non-atomic (non-pseudo atomic respectively) set multifunctions

In this section, we establish some results concerning non-atomicity and non-pseudo-atomicity for null-additive regular set multifunctions defined on the Baire (Borel respectively) δ-ring $\mathcal{B}_0$ ($\mathcal{B}$ respectively) of a Hausdorff locally compact space and taking values in $\mathcal{P}_f(X)$.

From now on, let T be a Hausdorff locally compact space, $\mathcal{C}$ a ring of subsets of T, $\mathcal{B}_0$ the Baire δ-ring generated by the G_δ-compact subsets of T (that is, compact sets which are countable intersections of open sets) and $\mathcal{B}$ the Borel δ-ring generated by the compact subsets of T.

Definition 5.1. (Gavriluţ [11]) I. Let $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ be a monotone set multifunction, with $\mu(\emptyset) = \{0\}$.

I. A set $A \in \mathcal{C}$ is said to be (with respect to μ):

(i) R - *regular* if for every $\varepsilon > 0$, there exist a compact set $K \subseteq A$, $K \in \mathcal{C}$ and an open set $D \supset A$, $D \in \mathcal{C}$ such that $e(\mu(D), \mu(K)) < \varepsilon$.

(ii) R_l - *regular* if for every $\varepsilon > 0$, there is a compact set $K \subseteq A$, $K \in \mathcal{C}$ such that $e(\mu(A), \mu(K)) < \varepsilon$.

(iii) R_r - *regular* if for every $\varepsilon > 0$, there exists an open set $D \supset A, D \in \mathcal{C}$ such that $e(\mu(D), \mu(A)) < \varepsilon$;

II. μ is said to be R - *regular* (R_l - *regular*, R_r - *regular* respectively) if every $A \in \mathcal{C}$ is R - regular (R_l - regular, R_r - regular respectively).

Theorem 5.2. *Suppose $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is monotone, null-additive and $\mu(\emptyset) = \{0\}$. Let $A \in \mathcal{B}$ with $\mu(A) \supsetneq \{0\}$. Then the following statements hold:*

I. If A is an atom of μ, then there is a compact set $K_0 \in \mathcal{B}$ so that $K_0 \subseteq A$ and $\mu(A \backslash K_0) = \{0\}$.

II. A is an atom of μ if and only if

$$(3) \qquad \exists! a \in A \text{ so that } \mu(A \backslash \{a\}) = \{0\}.$$

III. μ is non-atomic if and only if μ is diffused, that is

$$(4) \qquad \mu(\{t\}) = \{0\}, \text{ for every } t \in T.$$

Proof. I. Let $A \in \mathcal{B}$ be an atom of μ and $\mathcal{K}_A = \{K \subseteq A; K \text{ is a compact set and } \mu(A\backslash K) = \{0\}\} \subset \mathcal{B}$.

First, we prove that every $K \in \mathcal{K}_A$ is an atom of μ. Indeed, if $K \in \mathcal{K}_A$, then, by the null-additivity of μ, we have $\mu(A) = \mu((A\backslash K) \cup K) = \mu(K) \supsetneq \{0\}$. Also, for every $B \in \mathcal{B}$, with $B \subseteq K$, since $K \subseteq A$ and A is an atom of μ, we get $\mu(B) = \{0\}$ or $\mu(A\backslash B) = \{0\}$.

If $\mu(A\backslash B) = \{0\}$, then $\{0\} \subseteq \mu(K\backslash B) \subseteq \mu(A\backslash B) = \{0\}$, so $\mu(K\backslash B) = \{0\}$.

Consequently, $K \in \mathcal{K}_A$ is an atom of μ.

We now prove that $K_1 \cap K_2 \in \mathcal{K}_A$, for every $K_1, K_2 \in \mathcal{K}_A$. Indeed, if $K_1, K_2 \in \mathcal{K}_A$, then $K_1 \cap K_2$ is a compact set of T and $\mu(A\backslash(K_1 \cap K_2)) = \mu((A\backslash K_1) \cup (A\backslash K_2)) = \{0\}$.

We prove that $\bigcap_{K \in \mathcal{K}_A} K$, denoted by K_0, is a non-void set. Suppose that, on the contrary, $K_0 = \emptyset$. There are $K_1, K_2, ..., K_{n_0} \in \mathcal{K}_A$ so that $\bigcap_{i=1}^{n_0} K_i = \emptyset$, hence $\mu(\bigcap_{i=1}^{n_0} K_i) = \{0\}$. But $\bigcap_{i=1}^{n_0} K_i \in \mathcal{K}_A$, which implies $\mu(\bigcap_{i=1}^{n_0} K_i) \supsetneq \{0\}$, a contradiction.

Now, we prove that $K_0 \in \mathcal{K}_A$. Obviously, K_0 is a compact set. Let be $K \in \mathcal{K}_A$. Then $\mu(A\backslash K) = \{0\}$.

If $K = K_0$, then $K_0 \in \mathcal{K}_A$.

If $K \neq K_0$, then $K_0 \subsetneq K$.

Because $\mu(A\backslash K_0) = \mu((A\backslash K) \cup (K\backslash K_0)) = \mu(K\backslash K_0)$, it remains to prove that $\mu(K\backslash K_0) = \{0\}$. Suppose, on the contrary, that $\mu(K\backslash K_0) \supsetneq \{0\}$. Consider $B \in \mathcal{B}$, with $B \subseteq K\backslash K_0$. Then $B \subseteq K$ and, since K is an atom of μ, then $\mu(B) = \{0\}$ or $\mu(K\backslash B) = \{0\}$. If $\mu(K\backslash B) = \{0\}$, then $\mu((K\backslash K_0)\backslash B) = \{0\}$. So, $K\backslash K_0$ is an atom of μ. Because A is an atom of μ and $\mu(K\backslash K_0) \supsetneq \{0\}$, then $\mu(A\backslash(K\backslash K_0)) = \{0\}$.

Consequently, $\mathcal{K}_A = \{B \subseteq A; B \text{ is a compact set and } \mu(A\backslash B) = \{0\}\}$ and $\mathcal{K}_{K\backslash K_0} = \{C \subseteq K\backslash K_0; C \text{ is a compact set and } \mu((K\backslash K_0)\backslash C) = \{0\}\}$.

Let be $C \in \mathcal{K}_{K\backslash K_0}$. Then $\mu((K\backslash K_0)\backslash C) = \{0\}$ and, since $\mu(A\backslash(K\backslash K_0)) = \{0\}$, we get that $\mu(A\backslash C) = \{0\}$, which implies $C \in \mathcal{K}_A$. Therefore, $K_0 \subseteq C$, but $C \subseteq K\backslash K_0$, a contradiction. Consequently, $\mu(K\backslash K_0) = \{0\}$.

So, if $A \in \mathcal{B}$ is an atom of μ, there is a compact set $K_0 \in \mathcal{B}$ so that $K_0 \subseteq A$ and $\mu(A\backslash K_0) = \{0\}$.

II. The "*if part*". Let $A \in \mathcal{B}$ be an atom of μ. We show that the set K_0 from the proof of I is a singleton

$\{a\}$. Suppose, on the contrary, that there exist $a, b \in A$, with $a \neq b$ and $K_0 \supseteq \{a, b\}$.

Since T is a Hausdorff locally compact space, there exists an open neighbourhood V of a so that $b \notin \overline{V}$. Obviously, $K_0 = (K_0 \backslash V) \cup (K_0 \cap \overline{V})$ and $K_0 \backslash V$, $K_0 \cap \overline{V}$ are nonvoid, compact subsets of A.

We prove that $K_0 \backslash V \in \mathcal{K}_A$ or $K_0 \cap \overline{V} \in \mathcal{K}_A$. Indeed, if $K_0 \backslash V \notin \mathcal{K}_A$ and $K_0 \cap \overline{V} \notin \mathcal{K}_A$, then $\mu(A \backslash (K_0 \backslash V)) \supsetneq \{0\}$ and $\mu(A \backslash (K_0 \cap \overline{V})) \supsetneq \{0\}$. Since A is an atom of μ, then $\mu(K_0 \backslash V) = \{0\}$ and $\mu(K_0 \cap \overline{V}) = \{0\}$. Then $\mu(K_0) = \{0\}$ and since $\mu(A \backslash K_0) = \{0\}$, we have $\{0\} \subsetneq \mu(A) = \{0\}$, a contradiction. Consequently, $K_0 \backslash V \in \mathcal{K}_A$ or $K_0 \cap \overline{V} \in \mathcal{K}_A$. Because $K_0 \subseteq K$, for every $K \in \mathcal{K}_A$, we get that $K_0 \subseteq K_0 \backslash V$ or $K_0 \subseteq K_0 \cap \overline{V}$, which is impossible. So, $\exists a \in A$ so that $\mu(A \backslash \{a\}) = \{0\}$.

For the uniqueness: suppose, on the contrary, that there are $a, b \in A$, with $a \neq b$, $\mu(A \backslash \{a\}) = \{0\}$ and $\mu(A \backslash \{b\}) = \{0\}$. Then $\{0\} \subseteq \mu(\{a\}) \subseteq \mu(A \backslash \{b\}) = \{0\}$, so $\mu(\{a\}) = \{0\}$ and this implies $\mu(A) = \{0\}$, which is a contradiction.

The "only if part". Consider $A \in \mathcal{B}$, with $\mu(A) \supsetneq \{0\}$ having the property (3) and let $B \in \mathcal{B}$, with $B \subseteq A$. If $a \notin B$, then $B \subseteq A \backslash \{a\}$. Because $\mu(A \backslash \{a\}) = \{0\}$, then $\mu(B) = \{0\}$. If $a \in B$, then $A \backslash B \subseteq A \backslash \{a\}$, hence $\mu(A \backslash B) = \{0\}$. Consequently, A is an atom of μ.

III. *The "only if part"*. Suppose that, on the contrary, there is an atom $A_0 \in \mathcal{C}$ of μ. By II, $\exists! a \in A_0$ so that $\mu(A_0 \backslash \{a\}) = \{0\}$. On the other hand, $\mu(\{a\}) = \{0\}$, so $\mu(A_0) = \{0\}$, a contradiction. Consequently, μ is non-atomic.

The "if part". Suppose that, on the contrary, there is $t_0 \in T$ so that $\mu(\{t_0\}) \supsetneq \{0\}$. Because μ is non-atomic, there is a set $B \in \mathcal{B}$ such that $B \subseteq \{t_0\}$, $\mu(B) \supsetneq \{0\}$ and $\mu(\{t_0\} \backslash B) \supsetneq \{0\}$. Consequently, $B = \emptyset$ or $B = \{t_0\}$, which is false. □

Remark 5.3.

I. If $\mathcal{C} = \mathcal{B}_0$ (or $\mathcal{B}$), then the condition

(5) $\forall t \in T, \exists A_t \in \mathcal{C}$ s.t. $t \in A_t$ and $\mu(A_t) = \{0\}$

implies the condition

(6) $\forall B \in \mathcal{C}$, with $\mu(B) \supsetneq \{0\}, \forall t \in T$, $\exists A_t \in \mathcal{C}$ s.t. $t \in A_t$ and $e(\mu(B), \mu(A_t)) > 0$.

II. If $\mathcal{C} = B$, then (5) is equivalent to (4).

Theorem 5.4. *Let $\mathcal{C} = \mathcal{B}_0$ (or $\mathcal{B}$) and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ monotone, null-null-additive, with $\mu(\emptyset) = \{0\}$. If μ is R-regular and if it has the property* (6), *then it is non-pseudo-atomic. If, moreover, μ is null-additive, then μ is also non-atomic.*

Proof. Supose that, on the contrary, there is a pseudo-atom $B \in \mathcal{C}$ of μ.

Because μ is R-regular then, according to [10], it is R_l-regular. Consequently, there is a compact set $K \in \mathcal{C}$ so that $K \subseteq B$ and $h(\mu(B), \mu(K)) < |\mu(B)|$.

We observe that $\mu(K) \supsetneq \{0\}$. Indeed, if $\mu(K) = \{0\}$, then $|\mu(B)| < |\mu(B)|$, which is false.

According to (6), for every $t \in K$, there exists $A_t \in \mathcal{C}$ so that $t \in A_t$ and $e(\mu(B), \mu(A_t)) > 0$.

Because μ is R-regular then, by [10], it is R_r-regular. Then, for every $t \in K$, for A_t there is an open set $D_t \in \mathcal{C}$ so that $A_t \subseteq D_t$ and

$$e(\mu(D_t), \mu(A_t)) \leq h(\mu(D_t), \mu(A_t)) < \\ < e(\mu(B), \mu(A_t)).$$

Since $t \in A_t$ and $A_t \subseteq D_t$, then $K \subseteq \bigcup_{t \in K} D_t$. Consequently, there exists $p \in \mathbb{N}^*$ so that $K \subseteq \bigcup_{i=1}^{p} D_{t_i}$, with $t_i \in K$, for every $i = \overline{1, p}$.

Since $\{0\} \subsetneq \mu(K) = \mu(\bigcup_{i=1}^{p} (D_{t_i} \cap K))$, by the null-null-additivity of μ one can easily check there is $s = \overline{1, p}$ such that $\mu(D_{t_s} \cap K) \supsetneq \{0\}$. Consequently,

$$\{0\} \subsetneq \mu(D_{t_s} \cap K) \subseteq \mu(K) \subseteq \mu(B).$$

Obviously, we also have $\mu(D_{t_s}) \supsetneq \{0\}$.

Since B is a pseudo-atom of μ, $\mu(B) \supsetneq \{0\}$ and $\mu(D_{t_s}) \supsetneq \{0\}$, then $\mu(B) = \mu(B \cap D_{t_s})$.

On the other hand, because $e(\mu(D_{t_s}), \mu(A_{t_s})) < e(\mu(B), \mu(A_{t_s}))$, then

$$e(\mu(B \cap D_{t_s}), \mu(A_{t_s})) \leq \\ \leq e(\mu(B \cap D_{t_s}), \mu(D_{t_s})) + e(\mu(D_{t_s}), \mu(A_{t_s})) \\ = e(\mu(D_{t_s}), \mu(A_{t_s})) < e(\mu(B), \mu(A_{t_s})).$$

But $\mu(B) = \mu(D_{t_s} \cap B)$, a contradiction. So, μ is non-pseudo-atomic, as claimed. If, moreover, μ is null-additive, then, by Remark 3.2-III, it is also non-atomic. □

Concluding remarks.

In this paper, we have presented the relationships among different types of set multifunctions (such as: multisubmeasures, uniformly autocontinuous, autocontinuous from above, null-additive, null-null-additive) and some of their properties regarding atoms, pseudo-atoms, non-atomicity, non-pseudo-atomicity and extensions by preserving non-atomicity (non-pseudo-atomicity respectively).

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