On the Solution of p-Laplacian for non-Newtonian fluid flow

NIKOS E. MASTORAKIS
Technical University of Sofia,
Industrial Engineering Department
Sofia 1000, Bulgaria
mastor@wseas.org

HASSAN FATHABADI
Azad University (South Tehran Branch)
Electrical Engineering Department
Tehran, Iran
h4477@hotmail.com

Abstract: - The p-Laplacian, or the p-Laplace operator, is a quasilinear elliptic partial differential operator of 2nd order. In this paper, we examine several numerican schemes and we investigate their solution of non-linear problems in fluid mechanics.

Key-Words: - Numerical schemes, *p*-Laplacian, non-Newtonian fluid flow, nonlinear diffusion, nonlinear partial differential equation

1 Introduction

Many nonlinear problems in physics and mechanics are formulated in equations that contain the p-Laplacian, (i.e. the p-Laplace operator), where the p-Laplacian operator is defined as follows

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

In a recent paper, [35], Bognar presented a very interesting numerical and analytic investigation of problems of fluid mechanics that are described with PDEs containing the p-Laplacian operator. Previous publications (also reported in [35]) include reaction-diffusion problems [5], non-Newtonian fluid flows [1], fluid flows through certain types of porous media ([4], [34], the Lane-Emden equations for equilibrium configurations of spherically symmetric gaseous stellar objects [28], singular solutions for the Emden-Fowler equation [10] and the Einstein-Yang-Mills equations [7], the existence and nonexistence of black hole solutions, nonlinear elasticity [27], glaciology [29] and petroleum extraction [30]. It is clear that for $p=2$: $\Delta_p = \Delta$. The study of the p-Laplacian equation started more than thirty years ago (see [21], [23], and [24]). In recent years, rapid development has been achieved for the study of equation involving operator Δ_p and a vast literature has appeared on the theory of quasilinear differential equations.). In [36] Strikwerda summarized many Finite Difference Schemes for PDEs while in [37] Mickens presented Nonstandard Finite Difference Models for PDEs while some similar books are given in[38],[39],[40],[41],[42],[43].

In [35], Bognar studied the equation of turbulent filtration in porous media

$$\theta\frac{\partial\rho}{\partial t} = c^{\alpha}\lambda\operatorname{div}\left(\left|\nabla\rho^{n}\right|^{p-2}\nabla\rho^{n}\right), \qquad (1)$$

where $\theta>0$ and the constants $n>0$ and $p>1$ satisfy $np>1$. If we scale out the constants in (1), we derive

$$\frac{\partial u}{\partial t} = \Delta_p\left(u^n\right) \qquad (2)$$

where a particular case of (2) is the non-Newtonian filtration equation

$$\frac{\partial u}{\partial t} = \Delta_p u \qquad (3)$$

which is also called evolution p-Laplacian equation. The equation (7) is degenerate if $p>2$ and singular if $1<p<2$. The case $p>1+\frac{1}{n}$ is called the slow diffusion and the case $p<1+\frac{1}{n}$, the fast diffusion. Since the equation (7) possesses degeneracy in the slow diffusion case and possesses singularity in the fast diffusion case, it does not admit classical solutions in general. Also in the paper [35], Bognar studied the equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda u^q, \tag{4}$$

where $p>1$, $q>0$ and λ are some constants, in which the nonlinear term λu^q describes the nonlinear source in the diffusion process, called "heat source" if $\lambda>0$ and "cold source" if $\lambda<0$. Just as the Newtonian equation ($p=2$), the appearance of nonlinear sources will exert a great influence to the properties of solutions and the influence of "heat source" and "cold source" is completely different.

In this paper, we try to solve the equations (2), (3) and (4) using numerical schemes.

2 Numerical Schemes for (3)

Let's start from (3). Actually (3) is special case of (2), if in (2) one considers n=1.

The descritization in (3) leads from $u(t,x,y,z)$ to $u^i_{j,k,l}$ where *i* represents the (discrete) time and *j,k,l* the spatial coordinates. The steps of the descritization are h, h_1, h_2, h_3 with respect to *t,x,y,z*.

So, we have introduced a grid of points in the R^4 space of (*t,x,y,z*).

So:

$$\nabla u = \left(\frac{u^i_{j+1,k,l}-u^i_{j,k,l}}{h_1}, \frac{u^i_{j,k+1,l},-u^i_{j,k,l}}{h_2}, \frac{u^i_{j,k,l+1},-u^i_{j,k,l}}{h_3}\right) \tag{5}$$

We introduce the notation

$$(q^i_{j,k,l}, r^i_{j,k,l}, s^i_{j,k,l}) = |\nabla u|^{p-2}\nabla u \tag{6}$$

or

$$(q^i_{j,k,l}, r^i_{j,k,l}, s^i_{j,k,l}) = \left(\sqrt{\left(\frac{u^i_{j+1,k,l}-u^i_{j,k,l}}{h_1}\right)^2 + \left(\frac{u^i_{j,k+1,l},-u^i_{j,k,l}}{h_2}\right)^2 + \left(\frac{u^i_{j,k,l+1},-u^i_{j,k,l}}{h_3}\right)^2}\right)^{p-2} \cdot \left(\frac{u^i_{j+1,k,l}-u^i_{j,k,l}}{h_1}, \frac{u^i_{j,k+1,l},-u^i_{j,k,l}}{h_2}, \frac{u^i_{j,k,l+1},-u^i_{j,k,l}}{h_3}\right) \tag{7}$$

Then as a **first numerical scheme** we can propose:

$$\frac{u^{i+1}_{j,k,l}-u^i_{j,k,l}}{h} = \frac{q^i_{j,k,l}-q^i_{j-1,k,l}}{h_1} + \frac{r^i_{j,k,l}-r^i_{j,k-1,l}}{h_2} + \frac{s^i_{j,k,l}-s^i_{j,k,l-1}}{h_3} \tag{S.1}$$

(backward Euler)

and as a **second numerical scheme** we can propose:

$$\frac{u^{i+1}_{j,k,l}-u^i_{j,k,l}}{h} = \frac{q^{i+1}_{j,k,l}-q^{i+1}_{j-1,k,l}}{h_1} + \frac{r^{i+1}_{j,k,l}-r^{i+1}_{j,k-1,l}}{h_2} + \frac{s^{i+1}_{j,k,l}-s^{i+1}_{j,k,l-1}}{h_3} \tag{S.2}$$

(forward Euler)

where because in (5) we took the differences *j+1 – j, k+1 – k, l+1 – l* , we take now *j – (j-1), k – (k-1), l – (l-1).*

Assume that initial condition is $u(0,x,y,z)=x-2y+z$ and also boundary conditions are

$u(t,x,y,z\le 0)=2t+x-2y$,
$u(t,x,y,z\ge 1)=2t+x-2y+1$,
$u(t,x,y\le 0,z)=2t+x+z$,
$u(t,x,y\ge 1,z)=2t+x-2+z$,
$u(t,x\le 0,y,z)=2t-2y+z$,
$u(t,x\ge 1,y,z)=2t+1-2y+z$.

Fig.1 shows the results of the equation solution by implementing the schemes (S.1) and (S.2) in Matlab with using above conditions and p=3.

A third scheme is the Crank-Nicolson scheme which has as righ-hand side the average of the two righ-hand sides of (S.1) and (S.2).

That is:

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{1}{2}\left(\frac{q_{j,k,l}^{i}-q_{j-1,k,l}^{i}}{h_1}+\frac{r_{j,k,l}^{i}-r_{j,k-1,l}^{i}}{h_2}+\frac{s_{j,k,l}^{i}-s_{j,k,l-1}^{i}}{h_3}\right)$$
$$+\frac{1}{2}\left(\frac{q_{j,k,l}^{i+1}-q_{j-1,k,l}^{i+1}}{h_1}+\frac{r_{j,k,l}^{i+1}-r_{j,k-1,l}^{i+1}}{h_2}+\frac{s_{j,k,l}^{i+1}-s_{j,k,l-1}^{i+1}}{h_3}\right)$$

(S.3)

(Crank-Nicolson)

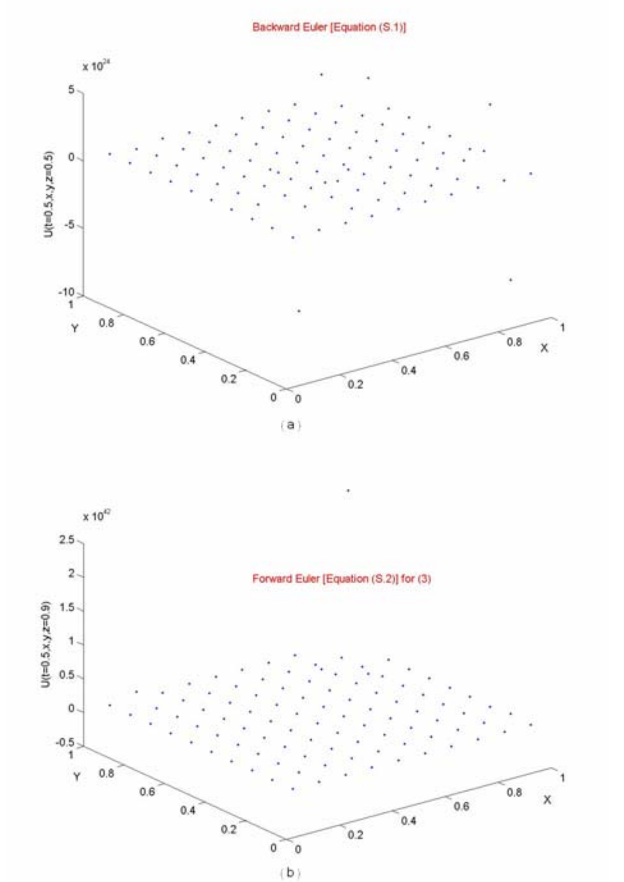

Fig. 1 a) Backward Euler (S.1) b) Forward Euler (S.2).

A fourth scheme is

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{q_{j+1,k,l}^{i}-q_{j-1,k,l}^{i}}{2h_1}+\frac{r_{j,k+1,l}^{i}-r_{j,k-1,l}^{i}}{2h_2}+\frac{s_{j,k,l+1}^{i}-s_{j,k,l-1}^{i}}{2h_3}$$

(S.4)

(backward Euler- central differences)

A fifth scheme is

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{q_{j+1,k,l}^{i+1}-q_{j-1,k,l}^{i+1}}{2h_1}+\frac{r_{j,k+1,l}^{i+1}-r_{j,k-1,l}^{i+1}}{2h_2}+\frac{s_{j,k,l+1}^{i+1}-s_{j,k,l-1}^{i+1}}{2h_3}$$

(S.5)

(forward Euler- central differences)

The results of the schemes (S.4) and (S.5) have been shown in Fig.2 with using $u(1,x,y,z)=2+x-2y+z$ as the limited time condition.

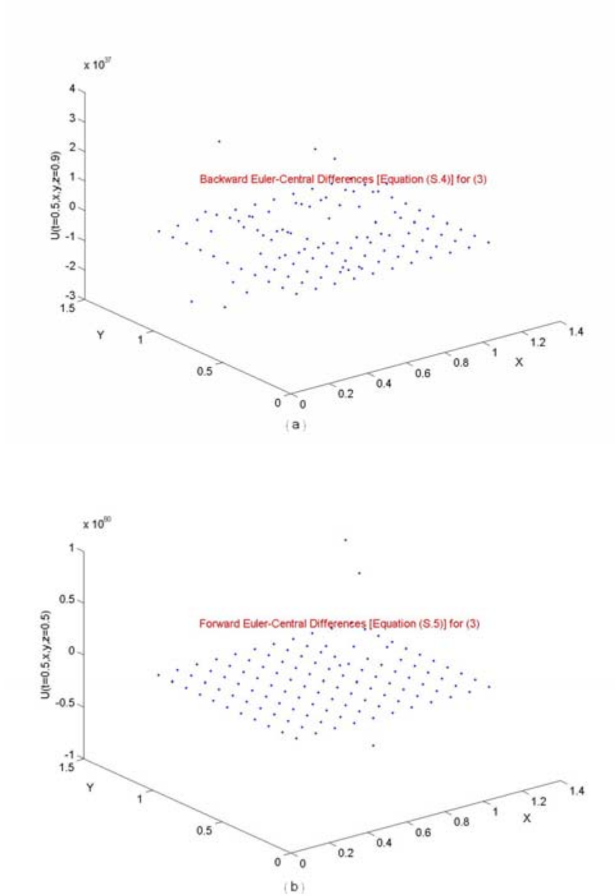

Fig. 2 a) Backward Euler- central differences (S.4)
b) Forward Euler- central differences (S.5).

A sixth scheme is

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{1}{2}\left(\frac{q_{j+1,k,l}^{i}-q_{j-1,k,l}^{i}}{2h_1}+\frac{r_{j,k+1,l}^{i}-r_{j,k-1,l}^{i}}{2h_2}+\frac{s_{j,k,l+1}^{i}-s_{j,k,l-1}^{i}}{2h_3}\right)$$
$$+\frac{1}{2}\left(\frac{q_{j+1,k,l}^{i+1}-q_{j-1,k,l}^{i+1}}{2h_1}+\frac{r_{j,k+1,l}^{i+1}-r_{j,k-1,l}^{i+1}}{2h_2}+\frac{s_{j,k,l+1}^{i+1}-s_{j,k,l-1}^{i+1}}{2h_3}\right)$$

(S.6)

(Crank-Nicolson - central differences)

A seventh scheme could be

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{3q_{j+2,k,l}^{i}-4q_{j+1,k,l}^{i}+q_{j,k,l}^{i}}{2h_1}+\frac{3r_{j,k+2,l}^{i}-4r_{j,k+1,l}^{i}+r_{j,k,l}^{i}}{2h_2}$$
$$+\frac{3s_{j,k,l+2}^{i}-4s_{j,k,l+1}^{i}+s_{j,k,l}^{i}}{2h_3}$$

(S.7)

(backward Euler- 2nd order derivatives)

An eighth scheme could be

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{3q_{j+2,k,l}^{i+1}-4q_{j+1,k,l}^{i+1}+q_{j,k,l}^{i+1}}{2h_1}+\frac{3r_{j,k+2,l}^{i+1}-4r_{j,k+1,l}^{i+1}+r_{j,k,l}^{i+1}}{2h_2}+\frac{3s_{j,k,l+2}^{i+1}-4s_{j,k,l+1}^{i+1}+s_{j,k,l}^{i+1}}{2h_3} \quad \text{(S.8)}$$

(forward Euler- 2nd order derivatives)

The results of the schemes (S.7) and (S.8) have been shown in Fig.3 with using the same conditions.

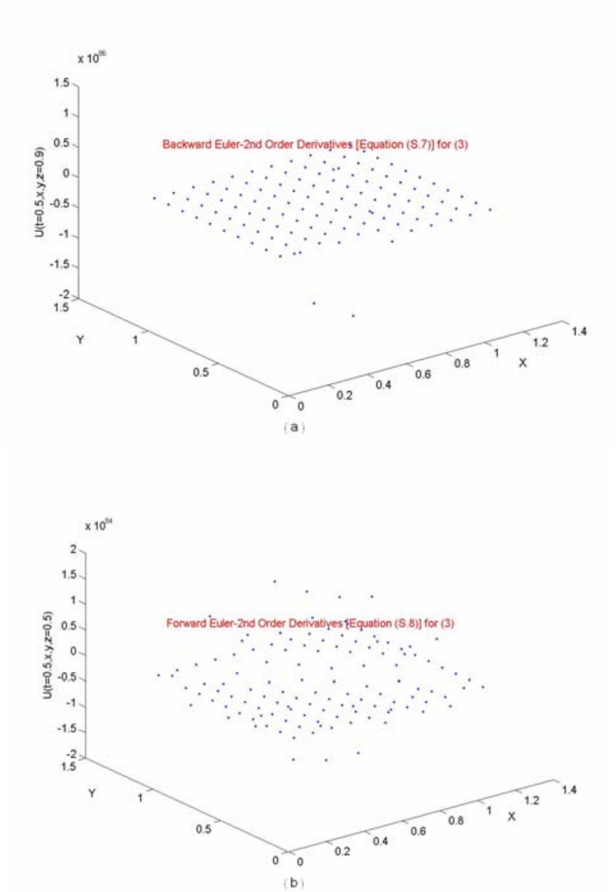

Fig. 3 a) Backward Euler- 2nd order derivatives (S.7)
b) Forward Euler- 2nd order derivatives (S.8).

A ninth scheme is

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{1}{2}\left(\frac{3q_{j+2,k,l}^{i}-4q_{j+1,k,l}^{i}+q_{j,k,l}^{i}}{2h_1}+\frac{3r_{j,k+2,l}^{i}-4r_{j,k+1,l}^{i}+r_{j,k,l}^{i}}{2h_2}+\frac{3s_{j,k,l+2}^{i}-4s_{j,k,l+1}^{i}+s_{j,k,l}^{i}}{2h_3}\right)+\frac{1}{2}\left(\frac{3q_{j+2,k,l}^{i+1}-4q_{j+1,k,l}^{i+1}+q_{j,k,l}^{i+1}}{2h_1}+\frac{3r_{j,k+2,l}^{i+1}-4r_{j,k+1,l}^{i+1}+r_{j,k,l}^{i+1}}{2h_2}+\frac{3s_{j,k,l+2}^{i+1}-4s_{j,k,l+1}^{i+1}+s_{j,k,l}^{i+1}}{2h_3}\right) \quad \text{(S.9)}$$

(Crank Nicolson- 2nd order derivatives)

Several other schemes can also be derived by appropriate shifts of *j,k,l* and several different combinations of these shifts in the righ-hand side of the schemes (instead of *j*+2,*j*+1,*j* indices we could have *j*+1,*j*,*j*-1, or *j*,*j*-1,*j*-2 -- instead of *k*+2, *k* +1, *k* indices we could have *k* +1, *k*, *k* -1, or *k*, *k* -1, *k* -2 -- instead of *l*+2, *l* +1, *l* indices we could have *l* +1, *l*, *l* -1, or *l*, *l* -1, *l* -2

So, we find 3*3*3 combinations, i.e. 27 combinations and multiplying it by 3 – forward, backward, Crank-Nicolson –, we have totally 81 additional numerical schemes, i.e. (S.7), (S.10), (S.8), (S.9), (S.10), (S.11), …(S.87).

3 Numerical Schemes for (2)

Several numerical schemes can be proposed now for (2) i.e. $\frac{\partial u}{\partial t}=\Delta_p\left(u^n\right)$.

It is denoted $w=u^n$ introducing again the notation

$$(q_{j,k,l}^{i},r_{j,k,l}^{i},s_{j,k,l}^{i})=\left|\nabla w\right|^{p-2}\nabla w=\left|\nabla u^n\right|^{p-2}\nabla u^n \quad (6)$$

or

$$(q_{j,k,l}^{i},r_{j,k,l}^{i},s_{j,k,l}^{i})=\left(\sqrt{\left(\frac{w_{j+1,k,l}^{i}-w_{j,k,l}^{i}}{h_1}\right)^2+\left(\frac{w_{j,k+1,l}^{i},-w_{j,k,l}^{i}}{h_2}\right)^2+\left(\frac{w_{j,k,l+1}^{i},-w_{j,k,l}^{i}}{h_3}\right)^2}\right)^{p-2}\cdot\left(\frac{w_{j+1,k,l}^{i}-w_{j,k,l}^{i}}{h_1},\frac{w_{j,k+1,l}^{i},-w_{j,k,l}^{i}}{h_2},\frac{w_{j,k,l+1}^{i},-w_{j,k,l}^{i}}{h_3}\right)$$

Then as a **first numerical scheme** we can propose:

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{q_{j,k,l}^{i}-q_{j-1,k,l}^{i}}{h_1}+\frac{r_{j,k,l}^{i}-r_{j,k-1,l}^{i}}{h_2}+\frac{s_{j,k,l}^{i}-s_{j,k,l-1}^{i}}{h_3} \quad \text{(S.1)}$$

(backward Euler)

and as a **second numerical scheme** we can propose:

$$\frac{u_{j,k,l}^{i+1}-u_{j,k,l}^{i}}{h}=\frac{q_{j,k,l}^{i+1}-q_{j-1,k,l}^{i+1}}{h_1}+\frac{r_{j,k,l}^{i+1}-r_{j,k-1,l}^{i+1}}{h_2}+\frac{s_{j,k,l}^{i+1}-s_{j,k,l-1}^{i+1}}{h_3} \quad \text{(S.2)}$$

(forward Euler)

The results of the schemes (S.1) and (S.2) have been shown in Fig.4 with using the same conditions and n=4.

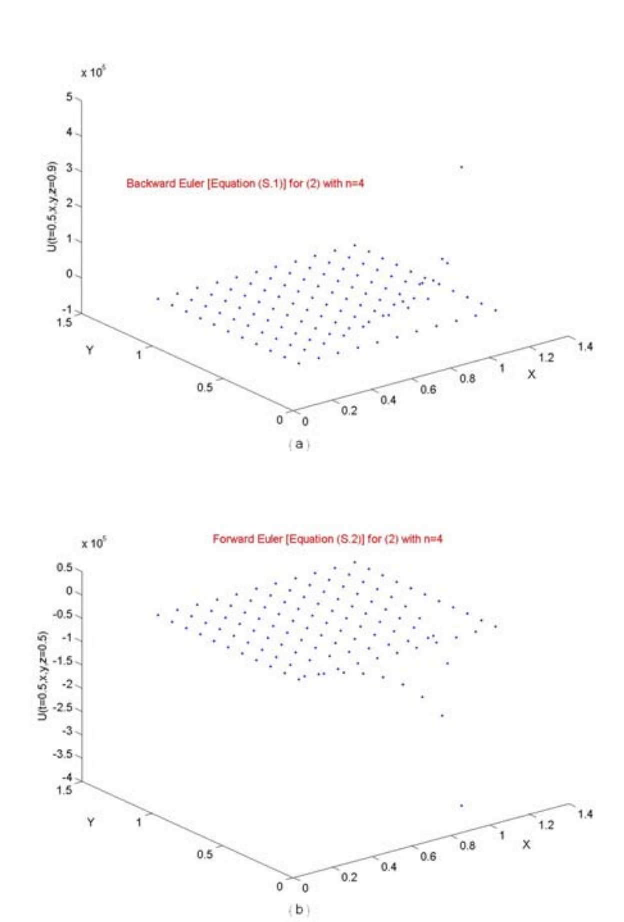

Fig. 4 a) Backward Euler (S.1) for (2) with n=4
b) Forward Euler (S.2) for (2) with n=4.

A third scheme is the Crank-Nicolson scheme which has as righ-hand side the average of the two righ-hand sides of (S.1) and (S.2)
That is:

$$\frac{u^{i+1}_{j,k,l}-u^{i}_{j,k,l}}{h}=\frac{1}{2}\left(\frac{q^{i}_{j,k,l}-q^{i}_{j-1,k,l}}{h_1}+\frac{r^{i}_{j,k,l}-r^{i}_{j,k-1,l}}{h_2}+\frac{s^{i}_{j,k,l}-s^{i}_{j,k,l-1}}{h_3}\right)$$
$$+\frac{1}{2}\left(\frac{q^{i+1}_{j,k,l}-q^{i+1}_{j-1,k,l}}{h_1}+\frac{r^{i+1}_{j,k,l}-r^{i+1}_{j,k-1,l}}{h_2}+\frac{s^{i+1}_{j,k,l}-s^{i+1}_{j,k,l-1}}{h_3}\right)$$

(S.3)
(Crank-Nicolson)

Quite similarly, we can derive the numerical schemes (S.4), (S.5), (S.6), (S.7), (S.8), (S.9), (S.10), (S.11), …,(S.87)

4 Numerical Schemes for (4)

We can introduce now some numerical schemes for (4): $\frac{\partial u}{\partial t}=\operatorname{div}\left(\left|\nabla u\right|^{p-2}\nabla u\right)+\lambda u^{q}$,

$$\frac{u^{i+1}_{j,k,l}-u^{i}_{j,k,l}}{h}=\frac{q^{i}_{j,k,l}-q^{i}_{j-1,k,l}}{h_1}+\frac{r^{i}_{j,k,l}-r^{i}_{j,k-1,l}}{h_2}+\frac{s^{i}_{j,k,l}-s^{i}_{j,k,l-1}}{h_3}+\lambda(u^{i}_{j,k,l})^{q}$$

(S.1)
(backward Euler)

$$\frac{u^{i+1}_{j,k,l}-u^{i}_{j,k,l}}{h}=\frac{q^{i+1}_{j,k,l}-q^{i+1}_{j-1,k,l}}{h_1}+\frac{r^{i+1}_{j,k,l}-r^{i+1}_{j,k-1,l}}{h_2}+\frac{s^{i+1}_{j,k,l}-s^{i+1}_{j,k,l-1}}{h_3}+\lambda(u^{i}_{j,k,l})^{q}$$

(S.2)
(forward Euler)

and

$$\frac{u^{i+1}_{j,k,l}-u^{i}_{j,k,l}}{h}=\frac{1}{2}\left(\frac{q^{i}_{j,k,l}-q^{i}_{j-1,k,l}}{h_1}+\frac{r^{i}_{j,k,l}-r^{i}_{j,k-1,l}}{h_2}+\frac{s^{i}_{j,k,l}-s^{i}_{j,k,l-1}}{h_3}\right)$$
$$+\frac{1}{2}\left(\frac{q^{i+1}_{j,k,l}-q^{i+1}_{j-1,k,l}}{h_1}+\frac{r^{i+1}_{j,k,l}-r^{i+1}_{j,k-1,l}}{h_2}+\frac{s^{i+1}_{j,k,l}-s^{i+1}_{j,k,l-1}}{h_3}\right)+\lambda(u^{i}_{j,k,l})^{q}$$

(S.3)
(Crank-Nicolson)

The results of the schemes (S.1) and (S.2) have been shown in Fig.5 with using the same conditions, q=2 and $\lambda=2\times10^{-6}$.

Quite similarly, we can derive the numerical schemes (S.4), (S.5), (S.6), (S.7), (S.8), (S.9), (S.10), (S.11), …,(S.87).

Also, in this case, we can propose more numerical schemes by changing the term $\lambda(u^{i}_{j,k,l})^{q}$ as $\lambda(u^{i+1}_{j,k,l})^{q}$ or as $\lambda[\frac{1}{2}(u^{i}_{j,k,l}+u^{i+1}_{j,k,l})]^{q}$.

5 Conclusion

In this paper, several numerical schemes of some nonlinear partial differential equations are proposed involving the so called p-Laplacian which have applications in the engineering practice. The investigated nonlinear problems appear in the mathematical modeling of many problems in fluid mechanics or in continuum mechanics in general.

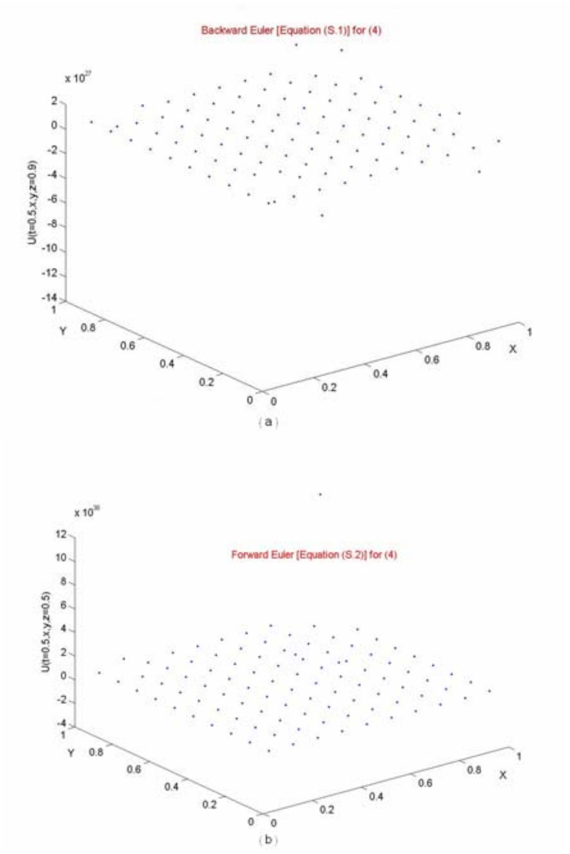

Fig. 5 a) Backward Euler (S.1) for (4)
b) Forward Euler (S.2) for (4).

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