Numerical Simulations for Dynamic Stochastic and Hybrid Models of Internet Networks

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Abstract: - In this paper, we consider Internet models, which respond to a congestion signal from the network described by a stochastic and hybrid differential equation. We consider Internet networks with one source and \( r \) access links, as well as with \( r \) sources and one access link. We analyze the conditions for the existence of a solution and the algorithms needed to determine the solution. We carry out numerical simulations for certain parameter values.

Key-Words: Internet models, networks, dynamic stochastic models, stochastic delay differential equation, Euler method, numerical simulations

1 Introduction

Congestion control mechanisms and active queue management schemes (AQM) for the Internet have been extensively studied since the work of Kelly et all [5]. In [10], the Hopf bifurcation has been studied for the model of an Internet network with \( r > 1 \) link and single source, which can be formulated as:

\[
\begin{aligned}
\dot{x}_i(t) &= k(v - af(x_i(t - \tau))) - \\
&- b \sum_{j=1, j \neq i}^{r} x_j(t - \tau) f(x_j(t - \tau)), \quad i = 1, \ldots, r
\end{aligned}
\]  

where \( x_i(t) \) is the sending rate of the source \( i \) at time \( t \), \( k \) is a positive gain parameter, \( \tau \) is the sum of forward and return delays, \( v \) is a target (set-point), \( a, b \) are positive real parameters and the congestion indication function \( f(x) \) is increasing, nonnegative, which characterizes the congestion. The discretizing model of (1) has been analyzed in [9], and the value of \( k \), for which the Neimark-Sacker bifurcation takes place, has been determined.

The dynamic model with a single link and \( r \) sources can be described by:

\[
\begin{aligned}
\dot{x}_i(t) &= k_i x_i(t - \tau_i) \left( \frac{a_i}{x_i(t)} - b_i x_i(t)p(t) \right) \\
p(t) &= k_{r+1} p(t) \left( \sum_{i=1}^{r} x_i(t - \tau_i) - c \right), \quad i = 1, \ldots, r
\end{aligned}
\]  

where, \( x_i(t) \) is the rate at which the source \( i \) transmits data at the time \( t \), \( a_i \) and \( b_i \) are positive real numbers, \( p(t) \) is the loss probability function, \( \tau_i \) is round-trip delay for source \( i \), \( c \) is the capacity, \( k_i, k \) are gain parameters. The discretized model of (2) has been analyzed in [11].

By employing the dynamic delayed feedback control, we can consider the controlled congestion control system with communication delay as follows:
The models (1), (2), and (3) lead to dynamic models described by stochastic delay differential equations, obtained through the randomization of one of the parameters. Thus, in section 2 we will consider stochastic delay differential equations, described by stochastic delay differential equations, obtained through the randomization of parameter $a$ in the case of the model (1), respectively, of parameters $a_i$, $c_i$, $i=1,...,r$, in the case of the model (2), and the parameters $a_1,a_2,a_4,a_5$, in the case of the model (3).

For certain functions which describe the stochastic delay differential equations (SDDE), the existence and uniqueness of the solution is justified. In section 3, we describe the algorithm which approximates the solution of the equations for the stochastic system with $r=2$ links and a source with a single link and $r=2$ source, respectively. In section 3 we describe the algorithm numeric for these cases. In section 4, we associate the determinist models (1),(2),(3), hybrid system, by adding a term that comprizes an Ito integral.

For certain parameter values, we carry out numerical simulation with the Maple13 software for the simulation of Wiener and Liu processes.

### 2 SDDE Models for Internet networks

Let $(\Omega,F,\Pr)$ be a complete probability space with a filtration $(F_t)$ satisfying the usual conditions; that is the filtration $(F_t)_{t\geq0}$ is right -continuous and each $(F_t)$, where $t \geq 0$ contains all $\Pr$-null sets in $F$. For general theory we refer to [12]. With $E(x) = \int_{\Omega} x d\Pr$ we say for $1 \leq p \leq \infty$ that $x \in L^p = L^p(\Omega,F,\Pr)$ if $E(|x|^p) < \infty$ and we define $\|x\|_p = (E(|x|^p)^{\frac{1}{p}}$. Here, $E$ denotes the expectation.

Let $w(t) = (w_1(t),...,w_r(t))^T$ be a $r$-dimensional Wiener process given on the filtered probability space $(\Omega,F,\Pr)$.

The stochastic delay differential equation (SDDE) with one fixed lag, $(0<T<\infty)$ is given by:

$$
\begin{align*}
\dot{x}_1(t) &= x_1(t - \tau) \left(1 - \frac{x_2(t)}{\tau^2 x_1(t)} - a_1x_1(t)x_2(t)\right) \\
\dot{x}_2(t) &= \frac{a_2}{a_3} x_2(t)(x_1(t - \tau) - a_3) + x_3(t) \\
\dot{x}_3(t) &= a_4(x_1(t) - x_1(t - \tau)) + a_5x_3(t)
\end{align*}
$$

where $a_3 > 0$ is the washout filter time constant, which guarantees the stability of controller, $a_4 > 0$ is the feedback gain parameter. This model was analyzed in [4].

The models (1), (2), and (3) lead to dynamic models described by stochastic delay differential equations, by randomizing one of the parameters. Thus, in section 2 we will consider stochastic delay differential equations, obtained through the randomization of parameter $a$ in the case of the model (1), respectively, of parameters $a_i$, $c_i$, $i=1,...,r$, in the case of the model (2), and the parameters $a_1,a_2,a_4,a_5$, in the case of the model (3).

For certain functions which describe the stochastic delay differential equations (SDDE), the existence and uniqueness of the solution is justified. In section 3, we describe the algorithm which approximates the solution of the equations for the stochastic system with $r=2$ links and a source with a single link and $r=2$ source, respectively. In section 3 we describe the algorithm numeric for these cases. In section 4, we associate the determinist models (1),(2),(3), hybrid system, by adding a term that comprizes an Ito integral.

For certain parameter values, we carry out numerical simulation with the Maple13 software for the simulation of Wiener and Liu processes.
The stochastic model for the Internet network with a single link and sources can be described by:

\[ x_i(t) = x_i(0) + \int_0^t k_i(s)ds + \int_0^t \sum_{j=1, j \neq i}^r k_{ij} f(x_j(s - \tau))ds + \int_0^t \alpha_i \int_0^t \psi_i(s)d\tau x_i(s)ds, \quad t \in [0, T], \quad i = 1, ..., r \]

where \( \psi_i(s) = \psi_i(s - \tau) \) and \( \tau \) is the delay.

3 Numerical simulation for the SDDE equations (6), (7) and (8)

The problem of solving an SDDE is reduced to one of solving a sequence of systems of SDDE of increasing dimension on successive intervals \([n, n+1]\). Using the Euler method of first order from [1], [12], [13] for (6), we obtain:

\[ x_i(k+1) = x_i(k) + h k_i s(t) f(x_i(s - \tau)) - b_i \int_0^t k_{ij} f(x_j(s - \tau))ds, \quad i = 1, ..., r \]

where \( G(h) \) is a random number generator, \( k, m \in N, h \in (0,1) \).

Using results from [1], [12], [13] we have that the algorithms given by (9) is convergent.

For \( r = 2 \), \( k_1 = 0.2 \), \( a_1 = 18 \), \( b = 2 \), \( v = 100 \), \( \alpha_1 = 0.5 \), \( \alpha_2 = 0.5 \), \( m = 3 \), \( n = 1300 \), \( h = \frac{1}{1300} \),

\[ f(x) = \frac{x^2}{20 - 3x}, \]

with Maple 13, we obtain in Fig. 1, the orbit \((k, x_1(k))\), in Fig. 2 the orbit \((k, x_2(k))\) and in Fig. 3 the orbit \((x_1(k), x_2(k))\).
Using the Euler method of first order for (7) we obtain:

\[
x_i(k+1) = x_i(k) + h k x_i(k-m) (a_i (x_i(k)-b_i p(k)x_i(k))) +
+k_i \alpha_i (x_i(k-m)) g_i(k) (\alpha_i h), \quad i=1,...,r
\]

\[
p(k+1) = p(k) + h k_{r+1} p(k) \left( \sum_{i=1}^{r} x_i(k-m) - c \right) -
-k_{r+1} \alpha_{r+1} p(k) g_i(h).
\]

(10)

For \( r = 2, \quad k = 1000, \quad h = \frac{1}{1000}, \quad m = 3, \quad k_1 = 1, \quad k_2 = 2, \)

\( k_3 = 0.005, \quad c = 65, \quad a_1 = \frac{2}{3}, \quad a_2 = 0.9, \quad b_1 = 0.5, \)

\( b_2 = 0.5, \quad a_1 = 0.2, \quad a_2 = 0.2, \quad a_3 = 0.4, \quad g(x) = \frac{1}{x}, \) with

Maple 13, we obtain the orbits: in Fig. 4, the orbit \((k, x_1(k))\), in Fig. 5 the orbit \((k, x_2(k))\), in Fig. 6 the orbit \((k, p(k))\), in Fig. 7 the orbit \((x_1(k), x_2(k))\), in

Fig. 8 the orbit \((x_1(k), p(k))\) and in Fig. 9 the orbit \((x_2(k), p(k))\).

Fig. 2 The orbit \((k, x_2(k))\)

Fig. 3 The orbit \((x_1(k), x_2(k))\)

Fig. 4 The orbit \((k, x_1(k))\)

Fig. 5 The orbit \((k, x_2(k))\)

Fig. 6 The orbit \((k, p(k))\)
Using the Euler method of first order for (8) we obtain:

\[
x_1(k+1) = x_1(k) + h x_3(k-m) \left( \frac{1-p(k)}{m^2 x(k)} - a_3 x_3(k) \right)
- a_1 x_1(k-m) x_3(k) G(h),
\]

\[
x_2(k+1) = x_2(k) + h \left( \frac{\alpha_2}{\alpha_3} x_2(k) (x_1(k-m) - a_3) + x_3(k) \right)
+ \frac{\alpha_2}{\alpha_3} x_2(k) x_3(k-m) G(h),
\]

\[
x_3(k+1) = x_3(k) + h (a_4 (x_1(k) - x_1(k-m)) - a_5 x_3(k))
- a_5 x_3(k) G(h).
\]

For \( a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = 0.2, \quad a_4 = 0.5, \quad a_5 = 0.2, \quad \alpha_1 = 0.5, \quad \alpha_2 = 6, \quad \alpha_3 = 0.4, \quad m = 4, \quad k = 1300, \quad h = \frac{1}{1300}, \) with Maple 13, we obtain the orbits:

In Fig.10 the orbit \((k, x_1(k))\), in Fig.11 the orbit \((k, x_2(k))\), in Fig.12 the orbit \((k, x_3(k))\), in Fig.13 the orbit \((x_1(k), x_2(k))\), in Fig.14 the orbit \((x_1(k), x_3(k))\) and in Fig.15 the orbit \((x_2(k), x_3(k))\).
Randomness in a basic type of objective uncertainty and probability theory is a branch of mathematics for studying the behavior of random phenomena. The concept of fuzzy set was initiated by Zadeh [14] via the membership function in 1965. In order to measure a fuzzy event, Liu B. [6] introduced the concept of credibility measure. The credibility theory was founded as a branch of mathematics for studying the behavior of fuzzy phenomena. Fuzziness and randomness are two basic types of uncertainty. In many cases, fuzziness and randomness simultaneously appear in a system.

### 4 Hybrid differential equations associated for the SDDE equations (1), (2) and (3)

Randomness in a basic type of objective uncertainty and probability theory is a branch of mathematics for studying the behavior of random phenomena. The concept of fuzzy set was initiated by Zadeh [14] via the membership function in 1965. In order to measure a fuzzy event, Liu B. [6] introduced the concept of credibility measure. The credibility theory was founded as a branch of mathematics for studying the behavior of fuzzy phenomena. Fuzziness and randomness are two basic types of uncertainty. In many cases, fuzziness and randomness simultaneously appear in a system.
Let $\Theta$ be a nonempty set, and let $P$ be the power of $\Theta$ (i.e., all subsets of $\Theta$). Each element in $P$ is called an event. In order to present an axiomatic definition of credibility, we accept the following four axioms:

1. $Cr(\emptyset) = 1$;
2. $Cr(A) \leq Cr(B)$, whenever $A \subseteq B$;
3. $Cr(A) + Cr(A^c) = 1$, for any $A \in P$;
4. $Cr\left(\bigcup_i A_i\right) = \sup_i Cr(A_i)$, for any events $\{A_i\}$ with $\sup_i Cr(A_i) < 0.5$.

The set function $Cr$ is called a credibility measure and $(\Theta, P, Cr)$ is a credibility space. A fuzzy variable is a function from the credibility space $(\Theta, P, Cr)$ to the set of real numbers. If a fuzzy variable $\xi$ is defined as a function on a credibility space, then we may get its membership function via:

$$\mu(x) = (2Cr(\xi = x)) \wedge 1, \ x \in R.$$  \hspace{1cm} (12)

Suppose that $(\Theta, P, Cr)$ is a credibility space and $(\Omega, F, Pr)$ is a probability space. The product $(\Theta, P, Cr) \times (\Omega, F, Pr)$ is called a chance space. A hybrid variable is a measurable function, from a chance space $(\Theta, P, Cr) \times (\Omega, F, Pr)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set $\{(\theta, \omega) \in \Theta \times \Omega | \xi(\theta, \omega) \in B\}$ is an event.

Let $T$ be an index set and let $(\Theta, P, Cr)$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, P, Cr)$ to the set of real numbers. A fuzzy process is a function from $T \times (\Theta, P, Cr) \times (\Omega, F, Pr)$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set $B$ of real numbers, the set $\{(\theta, \omega) \in \Theta \times \Omega | \xi(t, \theta, \omega) \in B\}$ is an event.

Suppose $w_t$ is standard Brownian motion, $C_t$ is a standard $C$ process, and $f, g, h$ are some given functions. Then:

$$dx(t) = f(t, x(t))dt + g(t, x(t))dw(t) + h(t, x(t))dC_t$$ \hspace{1cm} (14)

is called a hybrid differential equation. A solution is a hybrid process $x(t)$ that satisfies (14) identically in $t$.

Hybrid differential equation associated to (1) is:

$$d\xi = k_i(v-a)f(x_i(t-\tau)) - b\sum_{j=1}^r x_j(t-\tau)f(x_j(t-\tau))dt + \alpha_i(x_i-x_c)d\xi(t) + \beta_i dC_t, \ i=1..r$$ \hspace{1cm} (15)

$$x_i(t)=\xi_i(t), t\in[-\tau,0], \alpha_i > 0, \beta_i \in [0,1], \ i=1..r$$

where $x_c$ is the solution of equation $v = af(x)$.

The numerical simulation of (15) is given by:

$$x_i(k+1) = x_i(k) + h(k)(v-a)f(x_i(k-m)) + \sum_{j=1}^r x_j(k-m)f(x_j(k-m))$$ \hspace{1cm} (16)

$$+ \alpha_i(x_i-x_c)\xi_i(k) + \beta_i L_i(k),$$

where $m \in \mathbb{N}$, $h \in (0,1]$, $\alpha_i > 0$, $\beta_i \in [0,1]$, $z_i > 0$, $i=1..r$,

$$G(h) = \text{random}[\text{normald}[0, \sqrt{h}]],$$

$$L(k, z_i) = 2\left(1 + \exp\left(\frac{\pi x_i}{h \sqrt{6} \sigma_i^2(k)}\right)\right)^{-1},$$ \hspace{1cm} (17)

$$S_i(k) = \sum_{j=0}^{k-1} x_i(j).$$

For $r = 2$, $k_1 = 0.2$, $a = 18$, $b = 2$, $v = 100$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\beta_1 = 1$, $i = 1, 2$, $m = 3$, $n = 1500$, $h = \frac{1}{1500}$, $f(x) = \frac{x^2}{20-3x}$, with Maple 13, we obtain in Fig. 16, the orbit $(k, x_i(k))$, in
The hybrid differential equation associated to (2) is:

\[
\begin{align*}
    d\chi &= k\chi(t)\left(1 - \frac{a_1}{\chi(t)} - b_1\chi(t)p(t)\right)dt + \\
    &+ \alpha_1(\chi(t) - \chi_{eq})d\omega(t) + \beta_1\chi(t)d\zeta(t), \quad i = 1, \ldots, r \\
    dp &= k_{pq}p(t)\left(\sum_{i=1}^{\omega(t)}\chi(t) - c\right)dt + \\
    &+ \alpha_{pq}(p(t) - p_{eq})d\omega(t) + \beta_{pq}p(t)d\zeta(t) \\
    \alpha_i &> 0, \quad \beta_{i,q} > 0, \quad \beta_i, \beta_{i,q} \in [0, 1], \quad i = 1, \ldots, r
\end{align*}
\]

where \((x_{eq}, p_{eq})\) is the equilibrium point of the (2).

The numerical simulation of (17) is given by:

\[
\begin{align*}
    x_i(k+1) &= x_i(k) + h x_i(k)\left(1 - \frac{a_1}{x_i(k)} - b_1 p(k)x_i(k)\right) + \\
    &+ \alpha(x_i(k) - x_{eq}) d\omega(k) + \beta x_i(k) d\zeta(k), \quad i = 1, \ldots, r \\
    p(k+1) &= p(k) + h_{pq}p(k)\left(\sum_{j=1}^{\omega(k)}x_j(k) - c\right) + \alpha_{pq}(p(k) - p_{eq}) d\omega(k) + \\
    &+ \beta_{pq}p(k) d\zeta(k) \\
    k, m &\in N, \quad h \in (0, 1), \quad \alpha_i > 0, \quad \beta_i, \beta_{i,q} \in [0, 1], \quad z_i > 0, \quad i = 1, \ldots, r
\end{align*}
\]

where \(G(h), L(k, z_i)\) are given by (17).

For \(r = 2, \quad n = 1500, \quad h = \frac{1}{1500}, \quad a_1 = \frac{2}{3}, \quad a_2 = 0.9, \quad b_1 = 0.5, \quad b_2 = 0.5, \quad c = 65, \quad k_1 = 1, \quad k_2 = 1, \quad k_3 = 0.005, \quad \alpha_1 = 0.5, \quad \alpha_2 = 0.3, \quad \alpha_3 = 0.4, \quad \beta_1 = 1, \quad i = 1, 2, \quad z_1 = 0.5, \quad z_2 = 0.3, \quad z_3 = 0.4, \) with Maple 13, we obtain in Fig.19 the orbit \((k, x_1(k))\), in Fig.20 the orbit \((k, x_2(k))\) and in Fig.21 the orbit \((k, p(k))\).
The hybrid differential equation associated to (3) is:

\[
x_1(k+1) = x_1(k) + h \alpha_1 (x_1(t) - x_{1e}) dt + \alpha_1(x_1(t) - x_{1e}) dx_1(t) + \beta_1 x_1(t) dC_1(t),
\]

\[
x_2(k+1) = x_2(k) + h \alpha_2 (x_2(t) - x_{2e}) dt + \alpha_2(x_2(t) - x_{2e}) dx_2(t) + \beta_2 x_2(t) dC_2(t),
\]

\[
x_3(k+1) = x_3(k) + h \alpha_3 (x_3(t) - x_{3e}) dt + \alpha_3(x_3(t) - x_{3e}) dx_3(t) + \beta_3 x_3(t) dC_3(t).
\]

For, \( \alpha_1 = 1, \alpha_2 = 0.5, \alpha_3 = 0.2, \alpha_4 = 0.5, \alpha_5 = 0.5, \alpha_i = 2, \alpha_2 = 3, \alpha_3 = 4, \beta_1 = 2, \beta_2 = 8, \beta_3 = 5, \beta_i = 1, i = 1, 2, 3, m = 4, h = \frac{1}{1000}, n = 1000, \) with Maple 13, we obtain in Fig. 22 the orbit \((k, x_1(k))\), in Fig.23 the orbit \((k, x_2(k))\) and in Fig.24 the orbit \((x_1(k), x_2(k))\).
5 Conclusion

This paper has introduced SDDE for dynamic stochastic and hybrid models of Internet Networks. The paper has shown that these equations belong to the category of equations that accept a unique solution.

We have described a numerical algorithm in order to determine the approximate solution. The solutions have been visualized with the help of a program in Maple 13, using the Box-Muller method for the simulation of Wiener and Liu processes. A similar study will be conducted for cases in which other confidences will be randomized and fuzziness. Also, we will analyze the stability similarly to what is analyzed in [7]. The models from this paper can be extended considering the fractional integral [3], [14].

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