CONVERGENCE THEOREMS FOR TOTALLY-MEASURABLE FUNCTIONS

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Abstract: We establish some convergence theorems for sequences of totally-measurable functions with respect to a submeasure of finite variation. We also present relationships among different types of convergences such as convergence in submeasure, almost uniformly convergence, convergence in $\mathcal{L}^p$ spaces.

Key–words: totally-measurable, convergence, submeasure, $\mathcal{L}^p$ space.

1 Introduction

Recently, non-additivity was investigated by many authors (e.g. Dobrakov [4], Drewnowski [5], Jiang and Suzuki [17], Li [18], Pap [20], Precupanu [22], Sugeno [26], Suzuki [27], Zadeh [28]) due to its applications in mathematical economics, statistics, theory of games, probabilities, biology, physics, medicine, human decision making. Dempster [3] and Shafer [25] have founded the theory of evidence based on two dual non-additive measures: belief measures (Bel) and plausibility measures (Pl) with applications in multi-criteria decision making.

Different notions and theorems of non-additive measure theory (such as: continuity, regularity, extensions, decompositions, measures, integrals, atoms) were studied and extended to the set-valued case (see, for example, [1], [2], [7-14], [19], [23,24]).

In this paper, we study different types of convergences for sequences of totally-measurable functions with respect to a submeasure of finite variation. We also establish some relationships among these different types of convergences, such as, for instance, convergence in submeasure, convergence in variation, almost uniformly convergence, uniform convergence and convergence in $\mathcal{L}^p$ spaces.

2 Terminology and basic results

In what follows, without any special assumptions, we suppose T is an abstract space, $\mathcal{P}(T)$ is the family of all subsets of T, $\mathcal{A}$ is an algebra of subsets of T and $\nu : \mathcal{A} \to [0, +\infty)$ is a set function.

For every $A \subseteq T$, we denote $T \backslash A$ by cA.

By $i = \overline{1, n}$ we mean $i \in \{1, 2, \dots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{N}^* = \mathbb{N} \backslash \{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

We recall the following notions and results:

Definition 2.1.
ν is said to be:
(i) *monotone* if for every $A, B \in \mathcal{A}$ we have:

$$A \subseteq B \Rightarrow \nu(A) \le \nu(B).$$

(ii) a *submeasure* (in the sense of Drewnowski [5]) if $\nu(\emptyset) = 0$, ν is monotone and

$$\nu(A \cup B) \le \nu(A) + \nu(B),$$

for every $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$.

(iii) *finitely additive* if $\nu(\emptyset) = 0$ and

$$\nu(A \cup B) = \nu(A) + \nu(B),$$

for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$.

(iv) *order-continuous* (briefly, *o-continuous*) if $\lim_{n\to\infty} \nu(A_n) = 0$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow \emptyset$ (that is, $A_n \supseteq A_{n+1}$, for every $n \in \mathbb{N}^*$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$).

(v) *subadditive* if

$$\nu(A \cup B) \leq \nu(A) + \nu(B),$$

for every $A, B \in \mathcal{A}$.

(vi) *σ-subadditive* if

$$\nu(A) \leq \sum_{n=1}^{\infty} \nu(A_n),$$

for every sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

One can easily check the following results:

Example 2.2.

I) If μ is the real Lebesgue measure on $[0, 1]$, then the set functions $\nu_1, \nu_2 : \mathcal{A} \to [0, +\infty)$ defined for every $A \in \mathcal{A}$ by

$$\nu_1(A) = \sqrt{\mu(A)} \text{ and } \nu_2(A) = \frac{\mu(A)}{1 + \mu(A)}$$

are submeasures.

II) If $\nu_1, \nu_2 : \mathcal{A} \to [0, +\infty)$ are finitely additive, then the set function $\nu : \mathcal{A} \to [0, +\infty)$ defined for every $A \in \mathcal{A}$ by

$$\nu(A) = \max\{\nu_1(A), \nu_2(A)\}$$

is a submeasure.

Definition 2.3.

(i) A *partition* of T is a finite family $P = \{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup_{i=1}^{n} A_i = T$.

(ii) Let $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ be two partitions of T.

P' is said to be *finer than* P, denoted by $P \leq P'$ or $P' \geq P$, if for every $j = \overline{1,m}$, there exists $i_j = \overline{1,n}$ so that $B_j \subseteq A_{i_j}$.

(iii) The *common refinement* of two partitions $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ is the partition $P \wedge P' = \{A_i \cap B_j\}_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$.

We denote by $\mathcal{P}$ the class of all partitions of T and if $A \in \mathcal{A}$ is fixed, by $\mathcal{P}_A$, the class of all partitions of A.

We consider the following non-negative extended real-valued set functions associated to ν:

(i) $\overline{\nu} : \mathcal{P}(T) \to \overline{\mathbb{R}}_+$ defined by

$$\overline{\nu}(A) = \sup\{\sum_{i=1}^{n} \nu(A_i)\},$$

for every $A \subseteq T$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$, with $A_i \subseteq A$, for every $i = \overline{1,n}$.

$\overline{\nu}$ is called *the variation* of ν.

ν is said to be *of finite variation* on $\mathcal{A}$ if $\overline{\nu}(A) < \infty$, for every $A \in \mathcal{A}$.

(ii) $\widetilde{\nu} : \mathcal{P}(T) \to \overline{\mathbb{R}}_+$ defined by

$$\widetilde{\nu}(A) = \inf\{\overline{\nu}(B); A \subseteq B, B \in \mathcal{A}\},$$

for every $A \subseteq T$.

Proposition 2.4.

Let $\nu : \mathcal{A} \to [0, +\infty)$ be an arbitrary set function. Then the following statements hold:

(i) $\overline{\nu}$ and $\widetilde{\nu}$ are monotone.

(ii) $\overline{\nu} \leq \widetilde{\nu}$.

Proof.

(i) First, we prove that $\overline{\nu}$ is monotone.

Let $A, B \in \mathcal{P}(T)$ so that $A \subseteq B$ and let $\{E_i\}_{i=1}^n$, $n \in \mathbb{N}^*$, be an arbitrary family of disjoint subsets $E_i \in \mathcal{A}$, so that $E_i \subseteq A$, for every $i \in \{1, 2, \ldots, n\}$. So, we have $E_i \subseteq B$, for each $i \in \{1, 2, \ldots, n\}$ and by the definition of $\overline{\nu}$, it results

$$\sum_{i=1}^{n} \nu(E_i) \leq \overline{\nu}(B). \tag{1}$$

Taking in (1) the supremmum over $\{E_i\}_{i=1}^n$, we obtain $\overline{\nu}(A) \leq \overline{\nu}(B)$, which shows that $\overline{\nu}$ is monotone,

Now, we prove that $\widetilde{\nu}$ is monotone.

Let $A, B \in \mathcal{P}(T)$ so that $A \subseteq B$ and let $E \in \mathcal{A}$ such that $B \subseteq E$. So, we have $A \subseteq E$ and by the definition of $\widetilde{\nu}$, it follows

$$\widetilde{\nu}(A) \leq \overline{\nu}(E). \tag{2}$$

Taking in (2) the infimmum over $E \in \mathcal{A}$ with $B \subseteq E$, we obtain $\widetilde{\nu}(A) \leq \widetilde{\nu}(B)$, which proves that $\widetilde{\nu}$ is monotone.

(ii) Let $A \in \mathcal{P}(T)$ be fixed and let $B \in \mathcal{A}$ be arbitrarily so that $A \subseteq B$.

Since $\overline{\nu}$ is monotone, it follows

$$\overline{\nu}(A) \leq \overline{\nu}(B). \tag{3}$$

Taking in (3) the infimmum over $B \in \mathcal{A}$ with $A \subseteq B$, we obtain $\overline{\nu}(A) \leq \widetilde{\nu}(A)$, as claimed. □

From now on, $\nu : \mathcal{A} \to \mathbb{R}_+$ will be a submeasure of finite variation.

Remark 2.5.
I) The following statements are equivalent:

(i) ν is σ-subadditive;

(ii) ν is order-continuous;

(iii) $\overline{\nu}$ is σ-additive on $\mathcal{A}$.

II) (i) $\widetilde{\nu}$ is a submeasure on $\mathcal{P}(T)$.

(ii) If, moreover, ν is σ-subadditive, then $\widetilde{\nu}$ is σ-subadditive on $\mathcal{P}(T)$.

III) $\overline{\nu}$ is finitely additive on $\mathcal{A}$, $\nu(A) \leq \overline{\nu}(A)$ and $\widetilde{\nu}(A) = \overline{\nu}(A)$, for every $A \in \mathcal{A}$.

IV) For every $A \in \mathcal{A}$, $\nu(A) = 0$ if and only if $\overline{\nu}(A) = 0$.

In what follows, $f : T \to \mathbb{R}$ will be a real valued bounded function.

Definition 2.6.
I) f is said to be *totally-measurable on* $(T, \mathcal{A}, \nu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

$$(*) \quad \begin{cases} a)\ \widetilde{\nu}(A_0) < \varepsilon \text{ and} \\ b)\ \sup\limits_{t,s\in A_i} |f(t) - f(s)| < \varepsilon, \forall i = \overline{1,n}. \end{cases}$$

II) f is said to be *totally-measurable on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is totally measurable on $(B, \mathcal{A}_B, \nu_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $\nu_B = \nu|_{\mathcal{A}_B}$.

Remark 2.7.
If f is totally-measurable on T, then f is totally-measurable on every $A \in \mathcal{A}$.

Now, we present some properties of totally-measurable functions.

Proposition 2.8.
Let $f : T \to \mathbb{R}$ be a bounded function and $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$. Then f is totally-measurable on $A \cup B$ if and only if it is totally-measurable on A and totally-measurable on B.

Proof.
The "if part" is straightforward. For the "only if part", by the totally-measurability of f on A and B, there are $P_\varepsilon^A = \{A_i\}_{i=\overline{0,n}} \in \mathcal{P}_A$ and $P_\varepsilon^B = \{B_j\}_{i=\overline{0,q}} \in \mathcal{P}_B$ satisfying the condition $(*)$. Since $\overline{\nu}$ is additive on $\mathcal{A}$, then $P_\varepsilon^{A\cup B} = \{A_0 \cup B_0, A_1, ..., A_n, B_1, ..., B_q\} \in \mathcal{P}_{A\cup B}$ also satisfies condition $(*)$, so f is totally-measurable on $A \cup B$. □

Remark 2.9.
I) In the above proposition, A and B need not to be disjoint. Indeed, if we take arbitrary $A, B \in \mathcal{A}$, since $A \cup B = (A \backslash B) \cup B$ and totally-measurability is hereditary, the statement follows.

II) Under the assumptions of the above proposition, let $\{A_i\}_{i=\overline{1,p}} \subset \mathcal{A}$. Then f is totally-measurable on $\bigcup\limits_{i=1}^{p} A_i$ if and only if the same is f on every $A_i, i = \overline{1,p}$.

Proposition 2.10.
Suppose $\mathcal{A}$ is a σ-algebra and $\widetilde{\nu}$ is o-continuous on $\mathcal{A}$. Let $f : T \to \mathbb{R}$ be a bounded function and $(A_n)_n$ a sequence of pairwise disjoint sets of $\mathcal{A}$. Then f is totally-measurable on every $A_n, n \in \mathbb{N}$ if and only if the same is f on $A = \bigcup\limits_{n=1}^{\infty} A_n$.

Proof.
The "only if part" immediately follows.
The "if part".

We observe that $A \backslash \bigcup\limits_{k=1}^{n} A_k \searrow \emptyset$. Since $\widetilde{\nu}$ is o-continuous, for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$, with $\widetilde{\nu}(A \backslash \bigcup\limits_{k=1}^{n_0} A_k) < \varepsilon$.

Since for every $l = \overline{1, n_0}$, f is totally-measurable on A_l, let $\{B_j^1\}_{j=\overline{0,p_1}}, \{B_j^2\}_{j=\overline{0,p_2}}, ..., \{B_j^{p_{n_0}}\}_{j=\overline{0,p_{n_0}}}$ be the corresponding partitions satisfying $(*)$.

The partition $P_\varepsilon^A = \{(A \backslash \bigcup\limits_{k=1}^{n_0} A_k), \{B_j^1\}_{j=\overline{1,p_1}}, \{B_j^2\}_{j=\overline{1,p_2}}, ..., \{B_j^{p_{n_0}}\}_{j=\overline{1,p_{n_0}}}\} \in \mathcal{P}_A$ satisfies $(*)$, so f is totally-measurable on $A = \bigcup\limits_{n=1}^{\infty} A_n$. □

Definition 2.11.
We say that *a property (P) holds ν-almost everywhere (*shortly, *ν-ae*) if there is $A \subseteq T$, with $\widetilde{\nu}(A) = 0$, so that the property (P) is valid on $T \backslash A$.

In the sequel, we introduce different types of convergences, that will be studied throughout the paper.

Definition 2.12.
Let $f, f_n : T \to \mathbb{R}$ be real functions for every $n \in \mathbb{N}$.

One says that the sequence (f_n):

(i) *converges in variation to* f (denoted by $f_n \xrightarrow{\overline{\nu}} f$) if for every $\delta > 0$, $\lim_{n\to\infty} \overline{\nu}(B_n(\delta)) = 0$, where

$$B_n(\delta) = \{t \in T; |f_n(t) - f(t)| \geq \delta\}.$$

(ii) *converges in submeasure to* f (denoted by $f_n \xrightarrow{\nu} f$) if for every $\delta > 0$, $\lim_{n\to\infty} \widetilde{\nu}(B_n(\delta)) = 0$, where $B_n(\delta)$ is above defined.

(iii) *is fundamental (or Cauchy) in submeasure* if for every $\delta > 0$, it holds

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \widetilde{\nu}(\{t \in T; |f_n(t) - f_m(t)| \geq \delta\}) = 0.$$

(iv) *converges* $\widetilde{\nu}$*-almost everywhere to* f (denoted by $f_n \xrightarrow{\widetilde{\nu}-ae} f$) if there is $A \in \mathcal{P}(T)$ so that $\widetilde{\nu}(A) = 0$ and (f_n) pointwise converges to f on $T \backslash A$.

(v) *converges* $\overline{\nu}$*-almost everywhere to* f (denoted by $f_n \xrightarrow{\overline{\nu}-ae} f$) if there is $A \in \mathcal{P}(T)$ so that $\overline{\nu}(A) = 0$ and (f_n) is pointwise convergent to f on $T \backslash A$.

(vi) *converges almost uniformly to* f (denoted by $f_n \xrightarrow{au} f$) if for every $\varepsilon > 0$, there is $A_\varepsilon \subseteq T$ so that $\widetilde{\nu}(A_\varepsilon) < \varepsilon$ and $f_n \underset{T\backslash A_\varepsilon}{\xrightarrow{u}} f$ (where $f_n \xrightarrow{u} f$ denotes the uniform convergence).

One can easily verify the following statements:

Remark 2.13.
I) If $f_n \xrightarrow{u} f$, then $f_n \xrightarrow{au} f$.

II) According to Proposition 2.4-(ii), the following implications hold:

$$f_n \xrightarrow{\widetilde{\nu}} f \Rightarrow f_n \xrightarrow{\overline{\nu}} f,$$
$$f_n \xrightarrow{\widetilde{\nu}-ae} f \Rightarrow f_n \xrightarrow{\overline{\nu}-ae} f.$$

We now recall some results concerning a Gould [16] type integral with respect to a submeasure (according to Gavriluţ and Petcu [15]).

Definition 2.14.
Let $f : T \to \mathbb{R}$ be a real bounded function and let

$$\sigma(P) = \sum_{i=1}^{n} f(t_i)\nu(A_i),$$

for any partition $P = \{A_i\}_{i=\overline{1,n}}$ of T and every $t_i \in A_i, i = \overline{1,n}, n \in \mathbb{N}^*$.

(i) f is said to be ν*-integrable on* T if the net $(\sigma(P))_{P\in(\mathcal{P},\leq)}$ is convergent in $\mathbb{R}$, where $\mathcal{P}$, the set of all partitions of T, is ordered by the relation " $\leq$ " given in Definition 2.3-(ii).

If $(\sigma(P))_{P\in(\mathcal{P},\leq)}$ is convergent, then its limit is called *the integral of* f *on* T *with respect to* ν, denoted by $\int_T f d\nu$.

(ii) If $B \in \mathcal{A}, f$ is said to be ν*-integrable on* B if the restriction $f|_B$ of f to B is ν-integrable on $(B, \mathcal{A}_B, \nu_B)$.

Remark 2.15.
f is ν-integrable on T if and only if there is $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a partition P_ε of T, so that for every other partition of T, $P = \{A_i\}_{i=\overline{1,n}}$, with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have

$$|\sigma(P) - I| < \varepsilon.$$

Proposition 2.16. [15]
If A *is an arbitrary set of* $\mathcal{A}$*, then* f *is* ν*-integrable on* A *if and only if* f *is totally-measurable on* A.

Let $p \in [1, +\infty)$.
In what follows, we recall some results of [13,14].

Theorem 2.17. **(Minkowski Inequality)**
Suppose $f, g : T \to \mathbb{R}$ *are bounded totally-measurable on* T*. Then* $|f|^p, |g|^p$ *and* $|f + g|^p$ *are* ν*-integrable on* T *and, moreover,*

$$(\int_T |f + g|^p d\nu)^{\frac{1}{p}} \leq (\int_T |f|^p d\nu)^{\frac{1}{p}} + (\int_T |g|^p d\nu)^{\frac{1}{p}}.$$

We consider $\mathcal{L}^p = \{f : T \to \mathbb{R}; f$ is bounded on T and $|f|^p$ is ν-integrable on $T\}$.
It is easy to verify that $\mathcal{L}^p$ is a linear space.

From Theorem 2.17, we immediately obtain the following result:

Corollary 2.18.
The function $||\cdot|| : \mathcal{L}^p \to \mathbb{R}_+$*, defined for every* $f \in \mathcal{L}^p$ *by*

$$||f|| = (\int_T |f|^p d\nu)^{\frac{1}{p}},$$

is a semi-norm.

In the following, we introduce another type of convergence:

Definition 2.19.
Let $f \in \mathcal{L}^p$ and $f_n \in \mathcal{L}^p$, for every $n \in \mathbb{N}$.
The sequence (f_n) is said to be *semi-norm convergent* to f (denoted by $f_n \stackrel{sn}{\longrightarrow} f$) if $\lim\limits_{n\to\infty} \|f_n\|_p = \|f\|_p$.

3 Convergence theorems

In this section, we present some relationships among different types of convergences introduced in Definition 2.8.

Following the classical proofs (see, for example, Precupanu [21]) we obtain the next results.

Theorem 3.1.
Let $\nu : \mathcal{A} \to [0, +\infty)$ be a submeasure of finite variation and let us consider $f, g, f_n : T \to \mathbb{R}$.

(i) If $f_n \stackrel{\nu}{\to} f$ and $f = g$ $\nu-$ae, then $f_n \stackrel{\nu}{\to} g$.

(ii) If $\widetilde{\nu}$ is σ-subadditive on $\mathcal{P}(T)$, $f_n \stackrel{\nu}{\to} f$ and $f_n \stackrel{\nu}{\to} g$, then $f = g$ $\nu-$ae.

(iii) If (f_n) converges in submeasure, then it is fundamental in submeasure.

(iv) If $f_n \stackrel{\nu}{\longrightarrow} f$, then every subsequence $f_{n_k} \stackrel{\nu}{\longrightarrow} f$.

Proof.
(i) Since $f_n \stackrel{\nu}{\longrightarrow} f$, for every $\delta > 0$, it holds:

$$(4) \qquad \lim_{n\to\infty} \widetilde{\nu}(\{t \in T; |f_n(t) - f(t)| > \delta\}) = 0.$$

Since $f = g$ ν-ae, it results:

$$(5) \qquad \widetilde{\nu}(\{t \in T; f(t) \neq g(t)\}) = 0.$$

Now, for every $\delta > 0$, we have

$$(6) \qquad \begin{aligned} &\widetilde{\nu}(\{t \in T; |f_n(t) - g(t)| > \delta\}) \leq \\ &\leq \widetilde{\nu}(\{t \in T; |f_n(t) - f(t)| > \delta\}) + \\ &+\widetilde{\nu}(\{t \in T; f(t) \neq g(t)\}) = \\ &= \widetilde{\nu}(\{t \in T; |f_n(t) - f(t)| > \delta\}). \end{aligned}$$

Thus, from (4), (5) and (6) we obtain $\lim\limits_{n\to\infty} \widetilde{\nu}(\{t \in T; |f_n(t) - g(t)| > \delta\}) = 0$, which proves that $f_n \stackrel{\nu}{\longrightarrow} g$.

(ii) Since $f_n \stackrel{\nu}{\longrightarrow} f$ and $f_n \stackrel{\nu}{\longrightarrow} g$, for every $\delta > 0$ it holds:

$$(7) \qquad \begin{aligned} &\lim_{n\to\infty} \widetilde{\nu}(\{t \in T; |f_n(t) - f(t)| > \delta\}) = \\ &= \lim_{n\to\infty} \widetilde{\nu}(\{t \in T; |f_n(t) - g(t)| > \delta\}) = 0. \end{aligned}$$

Now, for $\delta > 0$, we have

$$(8) \qquad \begin{aligned} &\widetilde{\nu}(\{t \in T; |f(t) - g(t)| \geq \delta\}) \leq \\ &\leq \widetilde{\nu}(\{t \in T; |f_n(t) - f(t)| \geq \tfrac{\delta}{2}\}) + \\ &+\widetilde{\nu}(\{t \in T; |f_n(t) - g(t)| \geq \tfrac{\delta}{2}\}). \end{aligned}$$

From (7) and (8) it follows that $\widetilde{\nu}(\{t \in T; |f(t) - g(t)| \geq \delta\}) = 0$, for each $\delta > 0$.

Since $\{t \in T; f(t) \neq g(t)\} = \bigcup\limits_{n=1}^{\infty} \{t \in T; |f(t) - g(t)| \geq \frac{1}{n}\}$ and $\widetilde{\nu}$ is σ-subadditive, it results $\widetilde{\nu}(\{t \in T; f(t) \neq g(t)\}) = 0$. Therefore, $f = g$ ν-ae.

(iii) Suppose $f_n \stackrel{\nu}{\longrightarrow} f$. Since

$$|f_n - f_m| \leq |f_n - f| + |f - f_m|, \ \ \forall n, m \in \mathbb{N}^*,$$

we have for every $\delta > 0$:

$$\begin{aligned} &\{t \in T; |f_n(t) - f_m(t)| \geq \delta\} \subset \\ &\subset \{t \in T; |f_n(t) - f(t)| \geq \tfrac{\delta}{2}\} \cup \\ &\cup \{t \in T; |f_m(t) - f(t)| \geq \tfrac{\delta}{2}\}, \end{aligned}$$

which implies that

$$\begin{aligned} &\widetilde{\nu}\{t \in T; |f_n(t) - f_m(t)| \geq \delta\} \leq \\ &\leq \widetilde{\nu}\{t \in T; |f_n(t) - f(t)| \geq \tfrac{\delta}{2}\} + \\ &+\widetilde{\nu}\{t \in T; |f_m(t) - f(t)| \geq \tfrac{\delta}{2}\}. \end{aligned}$$

Since $f_n \stackrel{\nu}{\longrightarrow} f$, it follows $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} \widetilde{\nu}(\{t \in T; |f_n(t) - f_m(t)| \geq \delta\}) = 0$, for every $\delta > 0$, i.e. the sequence (f_n) is fundamental in submeasure.

(iv) As in the proof of (iii), since

$$|f_{n_k} - f| \leq |f_{n_k} - f_m| + |f_m - f|, \ \ \forall k, m \in \mathbb{N}^*,$$

we have for every $\delta > 0$:

$$(9) \qquad \begin{aligned} &\widetilde{\nu}\{t \in T; |f_{n_k}(t) - f(t)| \geq \delta\} \leq \\ &\leq \widetilde{\nu}\{t \in T; |f_{n_k}(t) - f_m(t)| \geq \tfrac{\delta}{2}\} + \\ &+\widetilde{\nu}\{t \in T; |f_m(t) - f(t)| \geq \tfrac{\delta}{2}\}. \end{aligned}$$

Since $f_n \stackrel{\nu}{\longrightarrow} f$, we have $\lim\limits_{m\to\infty} (\{t \in T; |f_m(t) - f(t)| \geq \frac{\delta}{2}\}) = 0$ and by (iii), because (f_n) is fundamental in submeasure, it results that

$$\lim_{\substack{k\to\infty \\ m\to\infty}} \widetilde{\nu}(\{t \in T; |f_{n_k} - f_m(t)| \geq \frac{\delta}{2}\}) = 0.$$

Now, from (9) it follows $\lim_{k\to\infty} \widetilde{\nu}(\{t \in T; |f_{n_k}(t) - f(t)| \geq \delta\}) = 0$, which proves that $f_{n_k} \xrightarrow{\nu} f$. □

Theorem 3.2.

Let $\nu : \mathcal{A} \to [0, +\infty)$ be a submeasure of finite variation, so that $\widetilde{\nu}$ is subadditive on $\mathcal{P}(T)$ and $f, g, f_n : T \to \mathbb{R}$ are real functions so that $f_n \xrightarrow{\widetilde{\nu}-ae} f$ and $f_n \xrightarrow{\widetilde{\nu}-ae} g$. Then $f \overset{ae}{=} g$.

Proof.

Since $f_n \xrightarrow{\widetilde{\nu}-ae} f$ and $f_n \xrightarrow{\widetilde{\nu}-ae} g$, there are $A, B \in \mathcal{P}(T)$ so that $\widetilde{\nu}(A) = \widetilde{\nu}(B) = 0$, $f_n \xrightarrow[T\backslash A]{p} f$ and $f_n \xrightarrow[T\backslash B]{p} g$ (where $f_n \xrightarrow{p} f$ denotes the pointwise convergence).

It follows that $f(t) = g(t)$ for every $t \in T\backslash(A \cup B)$.

Since $\widetilde{\nu}$ is subadditive on $\mathcal{P}(T)$, we have

$$0 \leq \widetilde{\nu}(A \cup B) \leq \widetilde{\nu}(A) + \widetilde{\nu}(B) = 0.$$

So $f \overset{ae}{=} g$, as claimed. □

One may immediately prove the following:

Theorem 3.3.

Let $f, f_n : T \to \mathbb{R}$, $n \in \mathbb{N}$, be real functions. If $f_n \xrightarrow{au} f$, then $f_n \xrightarrow{\nu} f$.

Proof.

Let $\varepsilon > 0$. Since $f_n \xrightarrow{au} f$, it results that there is $A \in \mathcal{P}(T)$ so that $\widetilde{\nu}(A) < \varepsilon$ and $f_n \xrightarrow[T\backslash A]{u} f$.

For every $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(t) - f(t)| < \delta,$$

for every $n \in \mathbb{N}, n \geq n_0$ and $t \in T\backslash A$. But $\{t \in T; |f_n(t) - f(t)| \geq \delta\} \subseteq A$ and so,

$$\widetilde{\nu}(\{t \in T; |f_n(t) - f(t)| \geq \delta\}) \leq \widetilde{\nu}(A) < \varepsilon,$$

for every $n \in \mathbb{N}$, $n \geq n_0$, which proves that $f_n \xrightarrow{\nu} f$. □

In the following theorem, we establish that, under some assumptions, convergence in submeasure preserves totally-measurability.

Theorem 3.4.

If $\overline{\nu}$ is subadditive on $\mathcal{A}$ and for every $n \in \mathbb{N}$, $f_n : T \to \mathbb{R}$ is totally-measurable and (f_n) is convergent in submeasure to $f : T \to \mathbb{R}$, then f is totally-measurable.

Proof.

Since for every $n \in \mathbb{N}$, f_n is totally-measurable, then for every $\varepsilon > 0$ there exists $P^n_\varepsilon = \{A^n_i\}_{i=\overline{0,m_n}} \in \mathcal{P}$ so that $\widetilde{\nu}(A^n_0) < \frac{\varepsilon}{2^n}$ and $\sup_{t,s\in A^n_i} |f_n(t) - f_n(s)| < \frac{\varepsilon}{3\cdot 2^n}$, for every $i = \overline{1, m_n}$.

Since $\lim_{n\to\infty} \widetilde{\nu}(B_n(\delta)) = 0$, for every $\delta > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $\widetilde{\nu}(B_n(\delta)) < \frac{\varepsilon}{2}$, for every $n \geq n_0$. Let, particularly, $\delta = \frac{\varepsilon}{3}$.

Then for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ so that $\widetilde{\nu}(B_{n_0}\left(\frac{\varepsilon}{3}\right)) < \frac{\varepsilon}{2}$.

By the definition of $\widetilde{\nu}$ we find a set $C_{n_0} \in \mathcal{A}$ so that $B_{n_0}(\frac{\varepsilon}{3}) \subseteq C_{n_0}$ and $\overline{\nu}(C_{n_0}) = \widetilde{\nu}(C_{n_0}) < \frac{\varepsilon}{2}$.

Consider $P_\varepsilon = \{C_{n_0} \cup A^{n_0}_0, A^{n_0}_1 \cap cC_{n_0}, A^{n_0}_2 \cap cC_{n_0}, ..., A^{n_0}_{m_{n_0}} \cap cC_{n_0}\} \in \mathcal{P}$.

Since $\overline{\nu}$ is subadditive, it holds:

$$\widetilde{\nu}(C_{n_0} \cup A^{n_0}_0) = \overline{\nu}(C_{n_0} \cup A^{n_0}_0)$$

$$\leq \overline{\nu}(C_{n_0}) + \overline{\nu}(A^{n_0}_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{n_0}} \leq \varepsilon.$$

Now, it only remains to prove that

$$\sup_{t,s\in A^{n_0}_i \cap cC_{n_0}} |f(t) - f(s)| < \varepsilon,$$

for every $i = \overline{1, m_{n_0}}$.

Indeed, we have:

$$\sup_{t,s\in A^{n_0}_i \cap cC_{n_0}} |f(t) - f(s)| \leq \sup_{t\in cC_{n_0}} |f(t) - f_{n_0}(t)| +$$

$$+ \sup_{t,s\in A^{n_0}_i} |f_{n_0}(t) - f_{n_0}(s)| + \sup_{s\in cC_{n_0}} |f_{n_0}(s) - f(s)|$$

$$< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3\cdot 2^{n_0}} + \tfrac{\varepsilon}{3} < \varepsilon, \text{ for every } i = \overline{1, m_{n_0}}.$$ □

Theorem 3.5.

Let $f : T \to \mathbb{R}$ and $f_n : T \to \mathbb{R}$, for each $n \in \mathbb{N}^*$. If $f_n \xrightarrow{au} f$, then $f_n \xrightarrow{\widetilde{\nu}-ae} f$.

Proof.

Since $f_n \xrightarrow{au} f$, for $\varepsilon = \frac{1}{m}$ $(m \in \mathbb{N}^*)$, there is $A_m \in \mathcal{P}(T)$ so that $\widetilde{\nu}(A_m) < \frac{1}{m}$ and $f_n \xrightarrow[T\backslash A_m]{u} f$.

Consider $B = \bigcap_{m=1}^{\infty} A_m$. Since $\widetilde{\nu}$ is monotone, we have $0 \leq \widetilde{\nu}(B) \leq \widetilde{\nu}(A_m) < \frac{1}{m}$, for every $m \in \mathbb{N}^*$, which yields $\widetilde{\nu}(B) = 0$.

Now, if $t \in T\backslash B = T\backslash \bigcap_{m=1}^{\infty} A_m = \bigcup_{m=1}^{\infty} (T\backslash A_m)$, then there exists $m_0 \in \mathbb{N}^*$ such that $t \in T\backslash A_{m_0}$ and so, $f_n(t) \to f(t)$. This shows that $f_n \xrightarrow[T\backslash B]{p} f$, which proves that $f_n \xrightarrow{\widetilde{\nu}-ae} f$. □

The next result is a Riesz type theorem.

Theorem 3.6.

Suppose $\widetilde{\nu}$ is σ-subadditive on $\mathcal{P}(T)$ and let $f_n : T \to \mathbb{R}$, for any $n \in \mathbb{N}$. If the sequence (f_n) is fundamental in submeasure, then there exists a subsequence of (f_n) that converges almost uniformly.

Proof.

Since (f_n) is fundamental in submeasure, we have for every $\delta > 0$:

$$(10)\quad \lim_{n\to\infty} \widetilde{\nu}(\{t \in T; |f_{n+m}(t) - f_n(t)| \geq \delta\}) = 0,$$

for each $m \in \mathbb{N}$.

From (10), taking $\delta = 1$, there exists $n_1 \in \mathbb{N}$ such that

$$\widetilde{\nu}(\{t \in T; |f_{n_1+m}(t) - f_{n_1}(t)| \geq 1\}) < 1.$$

Now, for $\delta = \frac{1}{2}$, there exists $n_2 \in \mathbb{N}$, $n_2 > n_1$ so that

$$\widetilde{\nu}(\{t \in T; |f_{n_2+m}(t) - f_{n_2}(t)| \geq \frac{1}{2}\}) < \frac{1}{2}.$$

Recurrently, there exists a sequence $(n_p)_{p\in\mathbb{N}} \subset \mathbb{N}$, so that $n_1 < n_2 < \ldots$, satisfying for every $p \in \mathbb{N}$:

$$(11)\quad \begin{aligned} &\widetilde{\nu}(\{t \in T; |f_{n_p+m}(t) - f_{n_p}(t)| \geq \\ &\geq \frac{1}{2^{p-1}}\}) < \frac{1}{2^{p-1}}, \end{aligned}$$

for every $m \in \mathbb{N}$.

Now we prove that $(f_{n_p})_{p\in\mathbb{N}^*}$ is almost uniformly convergent.

Let $E_p = \{t \in T; |f_{n_p+m}(t) - f_{n_p}(t)| \geq \frac{1}{2^{p-1}}\}$ for every $p \in \mathbb{N}^*$ and $\delta > 0$. Then there exists $n_0(\delta) = n_0 \in \mathbb{N}^*$ such that

$$(12)\quad \frac{1}{2^{n_0-2}} < \delta.$$

Denoting $E = \bigcup_{p=n_0}^{\infty} E_p$, from (11) and (12) it follows:

$$\begin{aligned} &\widetilde{\nu}(E) = \widetilde{\nu}(\bigcup_{p=n_0}^{\infty} E_p) \leq \sum_{p=n_0}^{\infty} \widetilde{\nu}(E_p) \leq \\ &\leq \sum_{p=n_0}^{\infty} \frac{1}{2^{p-1}} = \frac{1}{2^{n_0-2}} < \delta. \end{aligned}$$

Now, for each $t \in T\backslash E$ and every $i, j \in \mathbb{N}$, so that $i > j > n_0$, we have:

$$|f_{n_i}(t) - f_{n_j}(t)| \leq \sum_{k=j}^{\infty} |f_{n_k}(t) - f_{n_{k+1}}(t)| < \frac{1}{2^{j-2}},$$

which shows that (f_{n_p}) is uniformly convergent on $T\backslash E$. Thus, the subsequence (f_{n_p}) is almost uniformly convergent. □

From Theorems 3.5 and 3.6, the following corollary holds:

Corollary 3.7.

Suppose $\widetilde{\nu}$ is σ-subadditive on $\mathcal{P}(T)$ and let $f_n : T \to \mathbb{R}$, for every $n \in \mathbb{N}$. If (f_n) is fundamental in submeasure, then there exists a subsequence of (f_n) that is $\widetilde{\nu}$-almost everywhere convergent.

Proof.

Since (f_n) is fundamental in submeasure, according to Theorem 3.6, there exists (f_{n_p}) a subsequence of (f_n), such that $f_{n_p} \stackrel{au}{\longrightarrow} f$. By Theorem 3.5, it follows that $f_{n_p} \stackrel{\widetilde{\nu}-ae}{\longrightarrow} f$. □

Theorem 3.8.

Suppose $\widetilde{\nu}$ is σ-subadditive on $\mathcal{P}(T)$ and let $f_n : T \to \mathbb{R}$, for every $n \in \mathbb{N}$. If (f_n) is fundamental in submeasure, then (f_n) is convergent in submeasure.

Proof.

According to Corollary 3.7, there exists (f_{n_p}) a subsequence of (f_n), such that $f_{n_p} \stackrel{\widetilde{\nu}-ae}{\longrightarrow} f$, where $f : T \to \mathbb{R}$ is a real function.

Now, for every $\delta > 0$ it holds:

$$(13)\quad \begin{aligned} &\{t \in T; |f_n(t) - f(t)| \geq \delta\} \subseteq \\ &\subseteq \{t \in T; |f_n(t) - f_{n_p}(t)| \geq \frac{\delta}{2}\} \cup \\ &\cup \{t \in T; |f_{n_p}(t) - f(t)| \geq \tfrac{\delta}{2}\}. \end{aligned}$$

Let $\varepsilon > 0$. Since (f_n) is fundamental in submeasure, there exists $n_1 \in \mathbb{N}$, so that for every $p \in \mathbb{N}$ with $n_p \geq n_1$, we have:

$$(14)\quad \widetilde{\nu}(\{t \in T; |f_n(t) - f_{n_p}(t)| \geq \frac{\delta}{2}\}) < \frac{\varepsilon}{2}.$$

Since $f_{n_p} \stackrel{\widetilde{\nu}-ae}{\longrightarrow} f$, it results there is $n_2 \in \mathbb{N}$ such that for every $p \in \mathbb{N}$, with $n_p \geq n_2$, it holds:

$$(15)\quad \widetilde{\nu}(\{t \in T; |f_{n_p}(t) - f(t)| \geq \frac{\delta}{2}\}) < \frac{\varepsilon}{2}.$$

Now let $n_0 = \max\{n_1, n_2\}$ and $n \in \mathbb{N}$, $n \geq n_0$. From (13), (14) and (15) we obtain:

$$\begin{aligned} &\widetilde{\nu}(\{t \in T; |f_n(t) - f(t)| \geq \delta\}) \leq \\ &\leq \widetilde{\nu}(\{t \in T; |f_n(t) - f_{n_p}(t)| \geq \tfrac{\delta}{2}\}) + \\ &+ \widetilde{\nu}(\{t \in T; |f_{n_p}(t) - f(t)| \geq \frac{\delta}{2}\}) < \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which proves that $f_n \xrightarrow{\nu} f$. □

In the sequel, consider an arbitrary $p \in [1, +\infty)$.

Theorem 3.9.
Let $f \in \mathcal{L}^p$ and $f_n \in \mathcal{L}^p$, for every $n \in \mathbb{N}$. Then the following statements hold:

(i) $f_n \xrightarrow{u} f \Rightarrow f_n \xrightarrow{\mathcal{L}^p} f$.

(ii) $f_n \xrightarrow{\mathcal{L}^p} f \Rightarrow f_n \xrightarrow{\overline{\nu}} f$.

(iii) $f_n \xrightarrow{\mathcal{L}^p} f \Rightarrow f_n \xrightarrow{sn} f$.

Proof.
(i) Since $f_n \xrightarrow{u} f$, then for every $\varepsilon > 0$, there is $n_0(\varepsilon) = n_0 \in \mathbb{N}$ so that for every $n \geq n_0, |f_n(t) - f(t)| < \varepsilon$, for every $t \in T$.

Then, using the properties of the integral (see [14]) for every $n \geq n_0$, we have:

$$||f_n - f||_p = \left(\int_T |f_n - f|^p d\nu\right)^{\frac{1}{p}} \leq \left(\int_T \varepsilon^p d\nu\right)^{\frac{1}{p}}$$
$$= \left(\int_T \varepsilon^p d\overline{\nu}\right)^{\frac{1}{p}} = \varepsilon(\overline{\nu}(T))^{\frac{1}{p}},$$

so $\lim\limits_{n\to\infty} ||f_n - f||_p = 0$.

(ii) For every $\delta > 0$ and every $n \in \mathbb{N}$, consider

$$B_n(\delta) = \{t \in T; |f_n(t) - f(t)| \geq \delta\} \in \mathcal{P}(T).$$

We have $f_n \xrightarrow{\overline{\nu}} f$ if and only if for every $\delta > 0$, $\lim\limits_{n\to\infty} \overline{\nu}(B_n(\delta)) = 0$.

Let $\delta > 0$ be arbitrary. We have:

$$\overline{\nu}(B_n(\delta)) = \sup\{\sum_{i=1}^{l_n} \nu(A_i^n); (A_i^n)_{i=\overline{1,l_n}} \subset \mathcal{A}$$

pairwise disjoint, for every $i = \overline{1, l_n}$, $A_i^n \subseteq B_n(\delta)\}$.

Consider an arbitrary sequence of pairwise disjoint sets $(A_i^n)_{i=\overline{1,l_n}} \subset \mathcal{A}$, where for every $i = \overline{1, l_n}$, $A_i^n \subseteq B_n(\delta)$.

Then, for every $i = \overline{1, l_n}$,

$$\int_{A_i^n} |f_n - f|^p d\nu \geq \int_{A_i^n} \delta^p d\nu = \int_{A_i^n} \delta^p d\overline{\nu} = \delta^p \overline{\nu}(A_i^n),$$

whence

$$\delta^p \sum_{i=1}^{l_n} \overline{\nu}(A_i^n) \leq \sum_{i=1}^{l_n} \int_{A_i^n} |f_n - f|^p d\nu$$
$$= \int_{\bigcup_{i=1}^{l_n} A_i^n} |f_n - f|^p d\nu \leq$$
$$\leq \int_T |f_n - f|^p d\nu = ||f_n - f||_p^p,$$

so $\sum\limits_{i=1}^{l_n} \nu(A_i^n) \leq \sum\limits_{i=1}^{l_n} \overline{\nu}(A_i^n) \leq \frac{1}{\delta^p} ||f_n - f||_p^p$.

Consequently, taking the supremum on the left side overall sequences (A_i^n), we obtain:

$$\overline{\nu}(B_n(\delta)) = \sup\{\sum_{i=1}^{l_n} \nu(A_i^n)\} \leq \frac{1}{\delta^p} ||f_n - f||_p^p.$$

Since $\lim\limits_{n\to\infty} ||f_n - f||_p = 0$, then for every $\delta > 0$, $\lim\limits_{n\to\infty} \overline{\nu}(B_n(\delta)) = 0$, as claimed.

(iii) It immediately follows from the inequality:

$$|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p, \ \ \forall n \in \mathbb{N}.$$

□

Theorem 3.10. [13] **(Fatou Lemma)**
Suppose $\mathcal{A}$ is a σ-algebra, $\nu : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation so that $\widetilde{\nu}$ is o-continuous on $\mathcal{P}(T)$.

Let $(f_n)_n$ be a sequence of uniformly bounded, totally-measurable functions $f_n : T \to \mathbb{R}$. Then

$$\int_T \liminf_n f_n d\nu \leq \liminf_n \int_T f_n d\nu.$$

For establishing the next theorem, we shall need the following lemma.

Lemma 3.11.
Let $x, a_n, b_n \in \mathbb{R}$, for every $n \in \mathbb{N}$ so that $x \geq 0$, $b_n \geq 0$, for all $n \in \mathbb{N}$, $a_n \to x$ and

$$x \leq \liminf_{n\to\infty} (a_n - b_n).$$

Then $b_n \to 0$.

Theorem 3.12.
Suppose $\mathcal{A}$ is a σ-algebra and $\nu : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation so that $\widetilde{\nu}$ is o-continuous on $\mathcal{P}(T)$.

Suppose $(f_n) \subset \mathcal{L}^p$ is uniformly bounded and pointwise converges to $f \in \mathcal{L}^p$. If $f_n \xrightarrow{sn} f$, then $f_n \xrightarrow{\mathcal{L}^p} f$.

Proof.
As in Florescu [6], we use the inequality

$$(16) \qquad |a + b|^p \leq 2^{p-1}(|a|^p + |b|^p),$$

for every $a, b \in \mathbb{R}$.

Consider the sequence (g_n) defined for every $n \in \mathbb{N}$ by:

$$g_n = 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p.$$

By the inequality (16) it results that $g_n \geq 0$, for every $n \in \mathbb{N}$.

One can easily see that (g_n) is uniformly bounded and g_n is totally-measurable, for every $n \in \mathbb{N}$.

Now, we apply Fatou Lemma (Theorem 3.10) for (g_n) and we have:

$$(17) \qquad \int_T \liminf_n g_n d\nu \leq \liminf_n \int_T g_n d\nu.$$

Since $f_n \xrightarrow[T]{p} f$, it results $\liminf_n g_n = 2^p \cdot |f|^p$.

And $\int_T g_n d\nu = 2^{p-1}(\|f\|_p^p + \|f_n\|_p^p) - \|f_n - f\|_p^p$.

So, from (17) it follows

$$2^p \|f\|_p^p \leq \liminf_{n\to\infty} [2^{p-1}(\|f\|_p^p + \|f_n\|_p^p) - \|f_n - f\|_p^p].$$

Since $f_n \xrightarrow{sn} f$, it results $\lim_{n\to\infty} 2^{p-1}(\|f\|_p^p + \|f_n\|_p^p) = 2^p \cdot \|f\|_p^p$.

According to Lemma 3.11, we have:

$$\lim_{n\to\infty} \|f_n - f\|_p^p = 0,$$

which proves that $f_n \xrightarrow{\mathcal{L}^p} f$. □

In what follows, we present several counterexamples:

Example 3.13.

Let μ be the real Lebesgue measure and consider the submeasure ν defined by $\nu(A) = \sqrt{\mu(A)}$, for any $A \in \mathcal{A}$.

I) For every $n \in \mathbb{N}^*$ and $x \in (0, 1]$, let

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x \leq \frac{1}{n} \\ 0, & \frac{1}{n} < x. \end{cases}$$

Then $f_n \xrightarrow[(0,1]]{p} 0$ and $f_n \xrightarrow[(0,1]]{au} 0$, but $f_n \not\xrightarrow[(0,1]]{u} 0$.

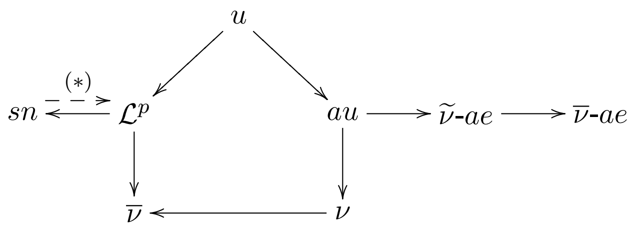
II) For every $n \in \mathbb{N}$ and $x \in [0, +\infty)$, let

$$f_n(x) = \begin{cases} 1, & n \leq x < n+1 \\ 0, & otherwise. \end{cases}$$

Then $f_n \xrightarrow[[0,+\infty)]{p} 0$, but $f_n \not\xrightarrow[[0,+\infty)]{\nu} 0$.

Concluding remarks. In this paper, we establish some relationship among different types of convergences for sequences of totally-measurable functions, such as convergence in submeasure, convergence in variation, almost uniformly convergence, uniform convergence and convergence in $\mathcal{L}^P$ spaces.

These relationships are synthetized in the following scheme:



where $(*)$ means the hypothesis of Theorem 3.12.

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