On Existence and Uniqueness of the Cauchy Problem for Parabolic Equations with Unbounded Coefficients

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Abstract: A new kind of entropy solution to the Cauchy problem for strong degenerate parabolic equations with unbounded coefficients,

$$\frac{\partial u}{\partial t} = \Delta A(u) + \mathtt{div}(uE),\ A'(u) \geq 0,$$

is quoted. Suppose that $u_0 \in L^\infty(R^N)$, $E = \{E_i\} \in (L^2(Q_T))^N$ and div$E \in L^2(Q_T)$, by a modified regularization method and choosing a suitable test function, the BV estimates are got, the existence of the entropy solution is obtained. At last, by Kruzkov bi-variables method, the stability of the solutions is obtained too.

Key–Words: Cauchy problem, Degenerate parabolic equation, Existence, Unbounded coefficient.

1 Introduction

This paper is to study the existence and uniqueness of BV-solution of the Cauchy problem for nonlinear degenerate parabolic equation of the form

$$\frac{\partial u}{\partial t} = \Delta A(u) + \mathtt{div}(uE),\ in\ Q_T = R^N \times (0,T), \tag{1}$$

$$u(x,0) = u_0(x), \tag{2}$$

where $E = \{E_i\} \in (L^2(Q_T))^N$ and

$$A(u) = \int_0^u a(s)ds, \quad a(s) \geq 0. \tag{3}$$

When $a(s) \geq \alpha > 0$, some applicative models related to the equation (1) were studied in [16], the existence of weak solutions of the first initial boundary problem of (1) was got in [15] when $u_0 \in L^1(\Omega)$, $E = \{E_i\} \in (L^2(Q_T))^N$. Especially, when $|E| \in L^r(0,T;L^q(\Omega))$ with $\frac{2}{r} + \frac{N}{q} < 1$ and $u_0 \in L^\infty(\Omega)$, then the weak solution $u \in L^\infty(Q) \cap L^2(0,T;H_0^1(\Omega))$, where Ω is a bounded open set in R^N, $Q = (0,T) \times \Omega$. When $a(s) \geq 0$, equation (1) arises in many applications, e.g., heat flow in materials with temperature dependent conductivity, fluid mechanics, flow in a porous medium, and the boundary layer theory (see [1], [10] et al.). If $E(x,t)$ is a bounded and suitable smooth function, the paper [2] by A. I. Vol'pert and S.I.Hudjaev was the first to be devoted to the solvability of (1.1). After that, many mathematicians (e.g. Bénilan, Brezis, DiBenedetto, Carrillo, Gagneux, Madaune-Tort, Wittbold, and Wu-Zhao et al.) continued to study its solvability, and got many excellent results, one can refer to chapter 3 of the book [10], the papers [2]-[6], [12], [13], [14], [17], [18], [20], [21] et al. and the references therein for details.

The difficulties of problem (1.1)-(1.2) come from three obstacles. The first obstacle is the strong degeneration of a, so the solutions generally are discontinuous even if $u_0(x)$ is smooth. The second obstacle is the unboundedness of E, even if $a(s) \geq \alpha > 0$, one only can proved that there exists a weak solution of (1.1)-(1.2) and it seems difficult to prove the existence of the classical solution, so the maximal principle can not be used directly, this adds the difficulty to get the estimates we need. The third obstacle is also in the unboundedness of E, which makes the estimating method used in [12]-[18] et al. not effective. To overcome these difficulties, solved as in [12], we put forward a new definition of BV-entropy solution for (1.1)-(1.2). By modifying the classical initial value regularizing method, we get the BV estimated formulas, this method is completely different from that used in [2]-[4], [12]-[14] et al. To this aim, some restrictions in E are added.

As for the uniqueness problem, it can be similarly solved as [12], for the completeness of the paper, we give the outline of the proof.

2 Definitions and Main Results

Following reference [8], $f \in BV(Q_T)$ if and only if that the generalized derivatives of every function $f(x,t)$ in $BV(Q_T)$ are regular measures on Q_T, i.e.

$$\int\int_{Q_T} |\frac{\partial f}{\partial t}| < \infty, \int\int_{Q_T} |\frac{\partial f}{\partial x_i}| < \infty, i = 1, 2, \cdots, N.$$

A basic property of BV function is that [25]: let $f \in BV(O_T)$. Then there exists a sequence $\{f_n\} \subset C^\infty(Q_T)$ such that

$$\lim_{n\to\infty} | f_n - f | dx = 0,$$

$$\lim_{n\to\infty} \int\int_{Q_T} | Df_n | dx = \int\int_{Q_T} | Df | .$$

So, it can be defined the trace of the functions in BV space as in Sobolev space. Moreover, the BV functions are the most weakly functions which can be defined the traces.

Let

$$S_\eta(s) = \int_0^s h_\eta(\tau)d\tau$$

for small $\eta > 0$, where $h_\eta(s) = \frac{2}{\eta}(1 - \frac{|s|}{\eta})_+$. Obviously $h_\eta(s) \in C(R)$, and

$$h_\eta(s) \geq 0, \ \ | sh_\eta(s) | \leq 1, \ | S_\eta(s) | \leq 1;$$

$$\lim_{\eta\to 0} S_\eta(s) = \mathtt{sgn}s, \ \lim_{\eta\to 0} sS'_\eta(s) = 0.$$

Definition 1 *A function u is said to be a weak solution of the Cauchy problem (1)-(2), if*

1.

$$u \in BV(Q_T) \cap L^\infty(Q_T),$$

$$\frac{\partial}{\partial x_i}\int_0^u \sqrt{a(s)ds} \in L^2(B_K \times (0,T)), \ \forall K > 0. \tag{4}$$

2. For any $\varphi \in C_0^2(Q_T), \ \varphi \geq 0, \ k \in R, \ \eta > 0, u$ satisfies

$$\begin{aligned}&\int\int_{Q_T}\{I_\eta(u-k)\varphi_t - E_i I_\eta(u-k)\varphi_{x_i}\\ &+A_\eta(u,k)\Delta\varphi - S'_\eta(u-k) \mid \nabla\int_0^u \sqrt{a(s)}ds \mid^2 \varphi\\ &+\int_k^u sS'_\eta(s-k)dsE_{ix_i}\varphi\}dxdt \geq 0,\end{aligned} \tag{5}$$

3.

$$\lim_{t\to 0}\int_{B_R} | u(x,t) - u_0(x) | dx = 0, \ \forall R > 0, \tag{6}$$

where the pairs of equal indices imply a summation from 1 up to N, and

$$A_\eta(s) = \int_k^u a(s)S_\eta(s-k)ds,$$

$$I_\eta(u-k) = \int_0^{u-k} S_\eta(s)ds. \tag{7}$$

Clearly if u is a weak solution in Definition 1, then u is a entropy solution in [2].

We will prove the following Theorems:

Theorem 2 *Suppose that $A(s)$ has at least two order derivatives, $a(s) = A'(s) \geq 0$; $u_0(x) \in L^\infty(R^N)$, $u_0(x) \in C^1(R^N)$; $E = \{E_i\} \in (L^2(Q_T))^N$, $divE \in L^2(Q_T)$, then problem (1)-(2) has a generalized solution in the sense of Definition 1.*

Theorem 3 *Let u, v be solutions of (1)-(2) with initial values $u_0(x), v_0(x) \in L^\infty(R^N) \cap L^2(R^N)$ respectively. Suppose that $A(s)$ has at least two order derivatives, $a(s) = A'(s) \geq 0$, $u_0(x), v_0(x) \in C^1(R^N)$. Then*

$$\int_{R^N} | u(x,t) - v(x,t) | \omega_\lambda(x)dx$$

$$\leq c\int_{R^N} | u_0 - v_0 | \omega_\lambda(x)dx$$

where c, λ are positive constants and

$$\omega_\lambda(x) = \exp\{-\lambda\sqrt{1+ | x |^2}\}. \tag{8}$$

Corollary 4 *The solution of (1)-(2) is unique.*

3 The Regularized Problem

Suppose that $A(s), u_0(x)$ are appropriately smooth and $u_0(x) \in L^\infty(R^N) \cap L^2(R^N)$, $E = \{E_i\} \in (L^2(Q_T))^N$, div$E \in L^2(Q_T)$, $Q_T = R^N \times (0,T)$. For any given large positive numbers K, let us introduce the following modified regularized equation.

$$\frac{\partial u}{\partial t} = \Delta A(u) + \frac{1}{K}\Delta u + \mathtt{div}(u\delta_\varepsilon * T_K E), \tag{9}$$

$$u(x,0) = u_{0K}(x), \tag{10}$$

where δ_ε is the mollifier as usual, i.e. let $x = (x_1, \cdots, x_N, t)$, and

$$\delta(x) = \begin{cases} \frac{1}{A}e^{\frac{1}{|x|^2-1}}, & \text{if } \ |x| < 1, \\ 0, & \text{if } \ |x| \geq 1, \end{cases}$$

where

$$A = \int_{B_1(0)} e^{\frac{1}{|x|^2-1}}dx.$$

For any given $\varepsilon > 0$, let

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon^{N+1}}\delta(\frac{x}{\varepsilon}).$$

Here, we choose $\varepsilon = \frac{1}{K}$ especially, and

$$\delta_\varepsilon * T_K(E) = (\delta_\varepsilon(E_1) * T_K(E_1), \delta_\varepsilon(E_2) * T_K(E_2),$$

$$\cdots, \delta_\varepsilon(E_N) * T_K(E_N)),$$

$$T_K(s) = \min\{K, \max\{-K, s\}\}.$$

Moreover, we suppose that $\mathtt{supp} u_{0K} \subset B_K = \{x : |x| < K\}$, and it satisfies

$$\lim_{n\to\infty} \|u_{0K} - u_0\|_{L^2(R^N)} = 0,$$

$$\|u_{0K}\|_{L^\infty} \leq \|u_0\|_{L^\infty(R^N)}. \tag{11}$$

It is well-known that there is a classical solutions $u_K \in C^{2,1}(Q_T)$ of (9)-(10). By this fact and using the maximum principle in problem of (9)-(10), we have

$$\|u_{K\varepsilon}\|_{L^\infty} \leq \|u_0\|_{L^\infty}. \tag{12}$$

Let $grad u_K = (u_{x_1}, u_{x_2}, \cdots, u_{x_N}, u_{x_{N+1}})$ and $u_{x_{N+1}} = u_t$. For simplicity, we denote u_K as u in the following calculation. Let us derivation on $x_s, s = 1, 2, \cdots, N, N+1$ in (2.1). Then multiplying with $u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}\varphi$ on the two sides, $0 \leq \varphi \in C_0^\infty(Q_T)$, and integrating over B_K, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{R_N} I_\eta(|gradu|)\varphi dx\\
&-\frac{1}{K}\int_{R_N} (\triangle u_{x_s}) u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}\varphi dx\\
&-\int_{R_N} \triangle A(u_{x_s}) u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}\varphi dx\\
&-\int_{R_N} \nabla u_{x_s}\cdot \delta_\varepsilon * T_K(E) u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}\varphi dx\\
&-\int_{R_N} \mathtt{div}(u\frac{\partial \delta_\varepsilon * T_K(E)}{\partial x_s}) u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}\varphi dx\\
&= 0.
\end{aligned} \tag{13}$$

Integrating by part, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{R_N} I_\eta(|gradu|)\varphi dx\\
&+\frac{1}{K}\int_{R_N} u_{x_sx_i}u_{x_px_s}\frac{\partial^2 I_\eta(|gradu|)}{\partial\xi_s\xi_p}\varphi dx\\
&+\frac{1}{K}\int_{R_N} \frac{S_\eta(|gradu|)}{|gradu|}u_{x_sx_i}u_{x_s}\varphi_{x_i}dx\\
&+\int_{R_N} a(u)u_{x_sx_i}u_{x_px_s}\frac{\partial^2 I_\eta(|gradu|)}{\partial\xi_s\xi_p}\varphi dx\\
&+\int_{R_N} a(u)\frac{S_\eta(|gradu|)}{|gradu|}u_{x_sx_i}u_{x_s}\varphi_{x_i}dx\\
&+\int_{R^N} a'(u)u_{x_i}I_\eta(|gradu|)\varphi_{x_i}dx\\
&-\int_{R_N} (\frac{\partial}{\partial x_i}a'(u))u_{x_i}(|gradu|S_\eta(|gradu|)\\
&-I_\eta(|gradu|))\varphi dx\\
&-\int_{R_N} a'(u)\triangle u(|gradu|S_\eta(|gradu|)\\
&-I_\eta(|gradu|))\varphi dx\\
&-\sum_{i=1}^N\int_{R_N} \delta_\varepsilon * T_K(E_i)(|gradu|S_\eta(|gradu|)\\
&-I_\eta(|gradu|))\varphi dx\\
&-\int_{R_N} \mathtt{div}(\frac{\partial[u\delta_\varepsilon * T_K(E)]}{\partial x_s})u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}\varphi dx\\
&= 0.
\end{aligned} \tag{14}$$

For the last term of the left side in (2.6),

$$\int_{R_N} \mathtt{div}[u\frac{\partial\delta_\varepsilon * T_K(E)}{\partial x_s}]u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}\varphi dx$$

$$= \sum_{i=1}^N\int_{R_N}[u_{x_i}\frac{\partial(\delta_\varepsilon * T_K(E))}{\partial x_s} + u\frac{\partial^2(\delta_\varepsilon * T_K(E))}{\partial x_s\partial x_i}]$$

$$\cdot u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}\varphi dx. \tag{15}$$

If we notice that $\varepsilon = \frac{1}{K}$, then

$$\frac{\partial(\delta_\varepsilon * T_K(E_i))}{\partial x_s}$$

$$= -\int_{\{y:|K(x-y)|<1\}}\frac{2K^2(x_s-y_s)}{(|K(x-y)|^2-1)^2}\frac{K^{N+1}}{A}$$

$$\cdot e^{\frac{1}{|K(x-y)|^2-1}}T_K(E_i(y,s))dyds,$$

where $x = (x_1, \cdots, x_N, t)$ as before.

Moreover, it is well known that

$$\frac{1}{(|K(x-y)|^2-1)^2}e^{\frac{1}{|K(x-y)|^2-1}} \leq \frac{e^2}{4},$$

so, by the facts of that

$$|K(x-y)| < 1,\ |T_K(E_i))| \leq K,$$

$$|\frac{\partial(\delta_\varepsilon * T_K(E_i))}{\partial x_s}|$$

$$\leq c\int_{\{y:|K(x-y)|<1\}}\frac{1}{(|K(x-y)|^2-1)^2}$$

$$\frac{K^{N+3}}{A}e^{\frac{1}{|K(x-y)|^2-1}}dyds$$

$$\leq cK^{N+3}.$$

Thus, if we choose that

$$\varphi(x) = \frac{1}{K^{N+4}}\varphi_1(x), \varphi_1 \in C_0^\infty(R^N),$$

$$I_{1s} = \int_{R_N} [u_{x_i} \frac{\partial(\delta_\varepsilon * T_K(E))}{\partial x_s} u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|}\varphi dx$$

$$\leq \frac{c}{K}\int_{R_N} |gradu|\varphi_1 dx. \quad (16)$$

Similarly, we are able to show that

$$|\frac{\partial^2(\delta_\varepsilon * T_K(E))}{\partial x_s \partial x_i}| \leq cK^{N+4},$$

then

$$I_{2s} = \int_{R_N} u\frac{\partial^2(\delta_\varepsilon * T_K(E))}{\partial x_s \partial x_i} u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|}\varphi dx$$

$$\leq c\int_{R_N} \varphi_1 dx. \quad (17)$$

By the following facts

$$\frac{1}{K}\int_{R_N} \frac{S_\eta(|gradu|)}{|gradu|} u_{x_s x_i} u_{x_s}\varphi_{x_i} dx$$

$$= -\frac{1}{K^{N+5}}\int_{R_N} I_\eta(|gradu|)\triangle\varphi_1 dx,$$

$$\int_{R_N} a(u)u_{x_s x_i}u_{x_p x_s}\frac{\partial^2 I_\eta(|gradu|)}{\partial\xi_s\xi_p}\varphi dx \geq 0,$$

$$\int_{R_N} a(u)\frac{S_\eta(|gradu|)}{|gradu|}u_{x_s x_i}u_{x_s}\varphi_{x_i} dx$$

$$+\int_{B_K} a'(u)u_{x_i}I_\eta(|gradu|)\varphi_{x_i} dx$$

$$= -\frac{1}{K^{N+4}}\int_{R_N} a(u)I_\eta(|gradu|)\triangle\varphi_1 dx,$$

and as $\eta \to 0$,

$$|gradu|S_\eta(|gradu|) - I_\eta(|gradu|)$$

$$= \int_0^{|gradu|} \tau h_\eta(\tau)d\tau \to 0.$$

By a process of limit, one can assume that

$$\varphi_1 = \omega_\lambda(x) = \exp(-\lambda\sqrt{1+|x|^2}),$$

where λ is a positive constant. Then

$$\omega_{\lambda x_i} = \omega_\lambda\frac{-\lambda x_i}{\sqrt{1+|x|^2}},$$

$$|\nabla\omega_\lambda| \leq c_\lambda\omega_\lambda,\ |\triangle\omega_\lambda| \leq c_\lambda\omega_\lambda.$$

Let $\eta \to 0$ in (14). We have

$$\frac{d}{dt}\int_{R_N} |gradu|\omega_\lambda dx \leq c_1 + c_2\int_{R_N} |gradu|\omega_\lambda dx,$$

equivalently,

$$\int_{R_N} |gradu|\omega_\lambda dx \leq c_1 + c_2\int_0^t ds\int_{R_N} |gradu|\omega_\lambda dx,$$

by Gronwall Lemma, we have

$$\int_{R^N} |gradu|\omega_\lambda dx \leq c(T, \lambda, \|u_0\|_{L^\infty}). \quad (18)$$

By (18), from (9), it is easy to show that

$$\int\int_{Q_T} (a(u_K) + \frac{1}{K}) \mid \nabla u_K \mid^2 \leq c(T, \lambda, \|u_0\|_{L^\infty}). \quad (19)$$

By (12), (19) and Kolomogroff's Theorem, there exists a subsequence $\{u_{K_n}\}$ of the family $\{u_K\}$ of solutions of regularized problems (9)-(10), which converges strongly in $L^1(Q_T)$. Thus the limit function $u \in BV(Q_T) \cap L^\infty(Q_T)$ and $u_{K_n} \to u$ a.e. on Q_T.

4 The Proof of Theorem 2

We need the following Lemma, which can be found in [22].

Lemma 5 *Assume that $U \subset R^N$ is an open bounded set and let $f_k, f \in L^q(U)$ as $k \to \infty$,*

$$f_k \rightharpoonup f \text{ weakly in } L^q(U), 1 \leq q < \infty.$$

Then

$$\lim\inf_{k\to\infty} \| f_k \|^q_{L^q(U)} \geq \| f \|^q_{L^q(U)}. \quad (20)$$

We now prove that u is a generalized solution of (1)-(2). From (19), we have

$$\frac{\partial}{\partial x_i}\int_0^{u_K}\sqrt{a(s)}ds \rightharpoonup \frac{\partial}{\partial x_i}\int_0^u \sqrt{a(s)}ds,$$

weakly in $L^2_{loc}(R^N \times (0,T)), i = 1, 2, \cdots, N$. This implies

$$\frac{\partial}{\partial x_i}\int_0^u \sqrt{a(s)}ds \in L^2_{loc}(R^N \times (0,T)), \forall R > 0,$$

$$i = 1, 2, \cdots, N.$$

Thus u satisfies (1) in Definition 1.

Let $\varphi \in C_0^2(Q_T),\ \varphi \geq 0,\ k \in R,\ \eta > 0$. Multiplying (9) by $\varphi S_\eta(u_K - k)$ and integrating over Q_T, we obtain

$$-\int\int_{Q_T} I_\eta(u_K-k)\varphi_t dxdt$$
$$+\frac{1}{K}\int\int_{Q_T} S_\eta(u_K-k)\frac{\partial u_K}{\partial x_i}\varphi_{x_i}dxdt$$
$$+\frac{1}{K}\int\int_{Q_T} S'_\eta(u_K-k)\frac{\partial u_K}{\partial x_i}\frac{\partial u_K}{\partial x_i}\varphi dxdt$$
$$-\int\int_{Q_T} S_\eta(u_K-k)(A(u_K)-A(k))\Delta\varphi dxdt$$
$$-\int\int_{Q_T} S'_\eta(u_K-k)(A(u_K)-A(k))\frac{\partial u_K}{\partial x_i}\varphi_{x_i}dxdt$$
$$+\int\int_{Q_T} S'_\eta(u_K-k)a(u_K)\frac{\partial u_K}{\partial x_i}\frac{\partial u_K}{\partial x_i}\varphi dxdt$$
$$+\int\int_{Q_T} S_\eta(u_K-k)\ T_K(E_i)u_K\varphi_{x_i}dxdt$$
$$+\int\int_{Q_T} S'_\eta(u_K-k)T_K(E_i)u_K\frac{\partial u_K}{\partial x_i}\varphi dxdt = 0. \quad (21)$$

Notice that the second term trends to zero as $K \to \infty$, the third term is nonnegative, and by Lemma 5,

$$\liminf_{K\to\infty}\int\int_{Q_T} S'_\eta(u_K-k)a(u_K)\frac{\partial u_K}{\partial x_i}\frac{\partial u_K}{\partial x_i}\varphi dxdt$$
$$\geq \int\int_{Q_T} S'_\eta(u_K-k) \mid \nabla\int_0^{u_K}\sqrt{a(s)}ds \mid^2 \varphi dxdt. \quad (22)$$

At the same time, we have

$$\int\int_{Q_T} S'_\eta(u_K-k)(A(u_K)-A(k))\frac{\partial u_K}{\partial x_i}\varphi_{x_i}dxdt$$
$$+\int\int_{Q_T} S_\eta(u_K-k)(A(u_K)-A(k))\Delta\varphi dxdt$$
$$= -\int\int_{Q_T}\int_k^{u_K} S'_\eta(s-k)(A(s)-A(k))ds\Delta\varphi dxdt$$
$$+\int\int_{Q_T} S_\eta(u_K-k)(A(u_K)-A(k))\Delta\varphi dxdt$$
$$= \int\int_{Q_T}\int_k^{u_K} S_\eta(s-k)a(s)ds\Delta\varphi dxdt, \quad (23)$$

$$\int\int_{Q_T} S_\eta(u_K-k)T_K(E_i)u_K\varphi_{x_i}dxdt$$
$$+\int\int_{Q_T} S'_\eta(u_K-k)T_K(E_i)u_K\frac{\partial u_K}{\partial x_i}\varphi dxdt$$
$$= \int\int_{Q_T}[\int_k^{u_K} d(sS_\eta(s-k))\varphi_{x_i}$$
$$+\frac{\partial}{\partial x_i}\int_k^{u_K} sS'_\eta(s-k)ds\varphi]T_K(E_i)dxdt$$
$$= \int\int_{Q_T}[\int_k^{u_K} S_\eta(s-k)ds$$
$$+\int_k^{u_K} sS'_\eta(s-k)ds]\varphi_{x_i}T_K(E_i)dxdt$$
$$-\int\int_{Q_T}\int_k^{u_K} sS'_\eta(s-k)ds$$
$$\cdot(\varphi_{x_i}T_K(E_i)+\varphi T_{Ki}(E_i))dxdt$$
$$= \int\int_{Q_T}[T_K(E_i)I_\eta(u_K-k)\varphi_{x_i}$$
$$+\int_k^{u_K} sS'_\eta(s-k)dsT_{Ki}(E_i)\varphi]dxdt. \quad (24)$$

where $T_{Ki}(E_i) = \frac{\partial E_i(x,t)}{\partial x_i}$.

Noticing that $E = \{E_i\} \in (L^2(Q_T))^N$ and $\text{div}E \in L^2(Q_T)$, let $K \to \infty$ in (21). By (22)-(24), we get (5).

Now, we will prove that the above u satisfies the initial condition (2). This is the direct corollary of the following Theorem 6.

Let us choose the K in (9) to be $m, l \in N$, the initial condition (10) be

$$u_m(x,0) = u_0(x), u_l(x,0) = u_0(x),$$

respectively. Then we have

Theorem 6 *For any given $R > 0$ and when m, l are large enough,*

$$\int_{B_R} |u_m(x,t) - u_l(x,t)|dx$$
$$\leq \int_{B_{2R}} |u_{0m}(x) - u_{0l}(x)|dx + C_R(t). \quad (25)$$

where $C_R(t)$ is independent of m, l, and moreover

$$\lim_{t\to 0} C_R(t) = 0.$$

Proof. Denoting that $v = u_m - u_l$, by (9), for any given t, choosing $\varphi(x,t) \in C_0^1(R^N)$,

$$\int_0^t\int_{B_{2R}} \varphi v_t dxdt$$
$$+\int_0^t\int_{B_{2R}}[a(u_m)\frac{\partial u_m}{\partial x_i} - a(u_l)\frac{\partial u_l}{\partial x_i}]\frac{\partial\varphi}{\partial x_i}dxd\tau$$
$$+\int_0^t\int_{B_{2R}}[\frac{1}{m}\frac{\partial u_m}{\partial x_i} - \frac{1}{l}\frac{\partial u_l}{\partial x_i}]\frac{\partial\varphi}{\partial x_i}dxd\tau$$
$$+\int_0^t\int_{B_{2R}} E_i(u_m - u_l)\frac{\partial\varphi}{\partial x_i}dxd\tau$$
$$= 0.$$

Let
$$\varphi(x,t)=\zeta(x)S_\eta(v),$$
where
$$\zeta(x)\in C_0^1(B_{2R}),0\le\zeta\le 1,\zeta|_{B_R}=1.$$
Then
$$\begin{aligned}&\int_0^t\int_{B_{2R}}\zeta(x)(\int_0^v S_\eta(s)ds)_\tau dxd\tau\\&-\int_0^t\int_{B_{2R}}[a(u_m)\frac{\partial u_m}{\partial x_i}-a(u_l)\frac{\partial u_l}{\partial x_i}]\\&\cdot[\zeta_{x_i}(x)S_\eta(v)+\zeta(x)S'_\eta(v)\frac{\partial v}{\partial x_i}]dxd\tau\\&-\int_0^t\int_{B_{2R}}[\frac{1}{m}\frac{\partial u_m}{\partial x_i}-\frac{1}{l}\frac{\partial u_l}{\partial x_i}]\\&\cdot[\zeta_{x_i}(x)S_\eta(v)+\zeta(x)S'_\eta(v)\frac{\partial v}{\partial x_i}]dxd\tau\\&+\int_0^t\int_{B_{2R}}E_i(u_m-u_l)\frac{\partial\varphi}{\partial x_i}dxd\tau=0.\end{aligned}\tag{26}$$

Noticing that
$$|\nabla\zeta|\le\frac{1}{R},\lim_{\eta\to 0}S'_\eta(s)s=0,S'_\eta(s)\ge 0.$$
and
$$\begin{aligned}&(\frac{1}{m}\frac{\partial u_m}{\partial x_i}-\frac{1}{l}\frac{\partial u_l}{\partial x_i})\frac{\partial v}{\partial x_i}\\&=(\frac{1}{m}\frac{\partial u_m}{\partial x_i}-\frac{1}{l}\frac{\partial u_l}{\partial x_i})(\frac{\partial u_m}{\partial x_i}-\frac{\partial u_l}{\partial x_i})\\&=\frac{1}{m}(\frac{\partial u_m}{\partial x_i})^2-(\frac{1}{m}+\frac{1}{l})\frac{\partial u_m}{\partial x_i}\frac{\partial u_l}{\partial x_i}+\frac{1}{l}(\frac{\partial u_l}{\partial x_i})^2\\&\ge\frac{1}{m}(\frac{\partial u_m}{\partial x_i})^2-(\frac{1}{m}+\frac{1}{l})|\frac{\partial u_m}{\partial x_i}\frac{\partial u_l}{\partial x_i}|+\frac{1}{l}(\frac{\partial u_l}{\partial x_i})^2\\&\ge\frac{1}{m}(\frac{\partial u_m}{\partial x_i})^2-\frac{2}{\sqrt{ml}}|\frac{\partial u_m}{\partial x_i}\frac{\partial u_l}{\partial x_i}|+\frac{1}{l}(\frac{\partial u_l}{\partial x_i})^2\\&\ge 0.\end{aligned}$$

Then, let $\eta\to 0$ in (26). One gets the conclusion.

By Theorem 6, for any given $R>0$ and m,l large enough, if let
$$u_{0m}(x)=u_{0l}(x)=u_0(x),$$
we have
$$\begin{aligned}&\int_{B_R}|u(x,t)-u_0(x)|dx\\&\le\int_{B_R}|u(x,t)-u_m(x,t)|dx+\int_{B_R}|u_{0m}(x)-u_{0l}(x)|dx\\&+C_R(t)+\int_{B_R}|u_{0l}(x)-u_0(x)|dx\\&+\int_{B_R}|u_l(x,t)-u_0(x)|dx.\end{aligned}$$
Let $t\to 0$ and notice that $u_m(x,t),u_l(x,t)$ is classical solutions of (9). We know (2) is true in the sense of (6).

5 The Proof of Theorem 3

Let Γ_u be the set of all jump points of $u\in BV(Q_T)$, v the normal of Γ_u at $X=(x,t)$, $u^+(X)$ and $u^-(X)$ the approximate limits of u at $X\in\Gamma_u$ with respect to $(v,Y-X)>0$ and $(v,Y-X)<0$ respectively. For continuous function $p(u,x,t)$ and $u\in BV(Q_T)$, define
$$\widehat{p}(u,x,t)=\int_0^1 p(\tau u^++(1-\tau)u^-,x,t)d\tau,$$
$$\overline{u}=\frac{1}{2}(u^++u^-),$$
which is called the composite mean value of p and u. For a given t, we denote $\Gamma_u^t,\ H^t,(v_1^t,\ \cdots,v_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, Hausdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$ respectively. By [8], if $f(s)\in C^1(R),\ u\in BV(Q_T)$, then $f(u)\in BV(Q_T)$ and
$$\frac{\partial f(u)}{\partial x_i}=\widehat{f}'(u)\frac{\partial u}{\partial x_i},\ i=1,2,\cdots,N.$$

Lemma 7 *Let u be a solution of (1)-(2). Then*
$$a(s)=0,\ s\in I(u^+(x,t),u^-(x,t)),\tag{27}$$
a.e. on Γ^u, where $I(\alpha,\beta)$ denote the closed interval with endpoints α and β.

Proof. Denote
$$\Gamma_1=\{(x,t)\in\Gamma_u,v_1(x,t)=\cdots=v_N(x,t)=0\}$$
$$\Gamma_2=\{(x,t)\in\Gamma_u,v_1^2(x,t)=\cdots=v_N^2(x,t)>0\}.$$

First prove $a(s)=0,\ s\in I(u^+(x,t),u^-(x,t))$, a.e. on Γ_1. Since any measurable subset of Γ_1 can be expressed as the union of a Borel set and a set of measure zero, it suffices to prove
$$a(s)=0,\ s\in I(u^+(x,t),u^-(x,t)),$$
a.e. on $U\subset\Gamma_1$,where U is a Borel subset of Γ_1. We may suppose $\overline{U}$ is compact. By Lemma 3.7.8 in [10],

for any bounded function $f(x,t)$, which is measurable with respect to measure $\frac{\partial u}{\partial x_i}$, we have

$$\int\int_U f(x,t)\frac{\partial u}{\partial x_i} = \int_0^T dt\int_{U^t} f(x,t)\frac{\partial u}{\partial x_i}, \quad (28)$$

where $U^t = \{x : (x,t) \in U\}$. By [11], for any Borel subset $S \subset U$,

$$\frac{\partial u}{\partial x_i}(s) = \int_S (u^+(x,t) - u^-(x,t))v_i dH,$$

$$\frac{\partial u(\cdot,t)}{\partial x_i}(S^t) = \int_{S^t}(u^t_+ - u^t_-)v_i dH^t.$$

(28) is equivalent to

$$\int\int_U f(x,t)(u^+(x,t) - u^-(x,t))v_i dH$$

$$= \int_0^T dt\int_{U^t} f(x,t)(u^t_+(x,t) - u^t_-(x,t))v^t_i dH^t.$$

The definition of Γ_1 implies that the left hand side vanishes, so we have

$$\int_0^T dt\int_{U^t} f(x,t)(u^t_+(x,t) - u^t_-(x,t))v^t_i dH^t = 0.$$

Choose $f(x,t) = \chi_u(x,t)\mathrm{sgn}(u^t_+(x,t) - u^t_-(x,t))$ $\mathrm{sgn}v^t_i$, where $\chi_u(x,t)$ denote the characteristic function of U and sum up for i from 1 up to N. Then we obtain

$$\int_G dt\int_{U^t}(u^t_+(x,t)-u^t_-(x,t))(| v^t_1 | + \cdots + | v^t_N |)dH^t$$

$$= 0, \quad (29)$$

where G is the projection of U on the t-axis. (29) implies for almost all $t \in G$,

$$\int_{U^t}(u^t_+(x,t) - u^t_-(x,t))(| v^t_1 | + \cdots + | v^t_N |)dH^t$$

$$= 0$$

and hence for almost all $t \in G$,

$$v^t_1 = \cdots = v^t_N = 0,$$

H^t-almost everywhere on U^t, which is impossible unless $\mathrm{mes}G = 0$.

For any α, β with $0 < \alpha < \beta < T$, we choose $\psi_j(t) \in C^\infty_0(0,T)$ such that

$$0 \le \psi_j(t) \le 1,$$

$$\lim_{j\to\infty}\psi_j(t) = \chi_{[\alpha,\beta]}(t), \ \forall t \in [0,T],$$

By [4], we can choose $\varphi_n \in C^\infty_0(Q_T)$ such that

$$| \varphi_n(x,t) | \le 1,$$

$$\lim_{n\to\infty}\varphi_n = \chi_U$$

in $L^1(Q_T, | \frac{\partial u}{\partial t} |)$.

Now from the definition of BV-function, we have

$$\int\int_{Q_T}\varphi_n(x,t)\psi_j(t)\frac{\partial u}{\partial t}$$

$$= \int\int_{Q_T} A(u)\Delta\varphi_n(x,t)\psi_j(t)dxdt$$

$$-\int\int_{Q_T} E_i u\frac{\partial}{\partial x_i}\varphi_n(x,t)\psi_j(t)dxdt.$$

Letting $j \to \infty$ leads to

$$\int\int_{Q_T}\varphi_n(x,t)\chi_{[\alpha,\beta]}(t)\frac{\partial u}{\partial t}$$

$$= \int\int_{Q_T} A(u)\Delta\varphi_n(x,t)\chi_{[\alpha,\beta]}(t)dxdt$$

$$-\int\int_{Q_T} E_i u\frac{\partial}{\partial x_i}\varphi_n(x,t)\chi_{[\alpha,\beta]}(t)dxdt.$$

Clearly, this equality also holds if $[\alpha,\beta]$ is replaced by (α,β) and hence it holds even if $[\alpha,\beta]$ is replace by any open set I with $\overline{I} \subset (0,T)$. Since G is a Borel set, by approximation we may conclude that

$$\int\int_{Q_T}\varphi_n(x,t)\chi_G(t)\frac{\partial u}{\partial t}$$

$$= \int\int_{Q_T} A(u)\Delta\varphi_n(x,t)\chi_G(t)dxdt$$

$$-\int\int_{Q_T} E_i u\frac{\partial}{\partial x_i}\varphi_n(x,t)\chi_G(t)dxdt.$$

Since $\mathrm{mes}G = 0$, the two terms on the right hand vanish and

$$\int\int_{Q_T}\varphi_n(x,t)\chi_G(t)\frac{\partial u}{\partial t} = 0.$$

Letting $n \to \infty$ gives

$$\int\int_U \frac{\partial u}{\partial t}$$

$$= \int\int_{Q_T}\chi_U(x,t)\chi_G\frac{\partial u}{\partial t} = 0.$$

Hence

$$\int_U (u^+(x,t) - u^-(x,t))v_t dH = 0,$$

which implies $H(U)=0$ and $H(\Gamma_1)=0$ by the arbitrariness of U.

Next we prove $H(\Gamma_2)=0$. Let U be any Borel subset of Γ_2 which is compact in Q_T, Since U is a set of $N+1$-dimensional measure zero and $\nabla A(u)\in L^2_{loc}(Q_T)$, we have

$$\int\int_U \frac{\partial}{\partial x_i}A(u)dxdt=0,$$

$$i=1,\cdots,N,$$

and hence

$$\int_U (A(u^+(x,t))-A(u^-(x,t)))\nu_t dH=0,\ i=1,\cdots,N.$$

Form this it follows by the definition of Γ_2 that

$$\int_{u^-(x,t)}^{u^+(x,t)} a(s)ds=0,$$

a.e. on Γ_2.Thus the lemma is proved.

Proof of Theorem 3. Let u,v be two generalized solutions of (1) with initial values

$$u(x,0)=u_0(x),\ v(x,0)=v_0(x).$$

By Definition 1, we have for any $\varphi\in C_0^2(Q_T)$, $\varphi\geq 0,\ k,l\in R$,

$$\int\int_{Q_T}\{I_\eta(u-k)\varphi_t-E_i(x,t)I_\eta(u-k)\varphi_{x_i}$$
$$+A_\eta(u,k)\Delta\varphi-S'_\eta(u-k)\mid\nabla\int_0^u\sqrt{a(s)}ds\mid^2\varphi$$
$$-\int_k^u sS'_\eta(s-k)dsE_{ix_i}\varphi\}dxdt\geq 0, \qquad (30)$$

$$\int\int_{Q_T}\{I_\eta(v-l)\varphi_t-E_i(y,\tau)I_\eta(v-l)\varphi_{y_i}$$
$$+A_\eta(v,l)\Delta\varphi-S'_\eta(v-l)\mid\nabla\int_0^v\sqrt{a(s)}ds\mid^2\varphi$$
$$-\int_l^v sS'_\eta(s-k)dsE_{iy_i}\varphi\}dxdt\geq 0. \qquad (31)$$

Let $\psi(x,t,y,\tau)\geq 0$, $\psi\in C^2(Q_T\times Q_T)$, supp $\psi(\cdot,\cdot,\tau,y)\subset Q_T$ if $(\tau,y)\in Q_T$, supp$\psi(x,t,\cdot,\cdot)\subset Q_T$. We choose $k=v(y,\tau)$, $l=u(x,t)$, $\varphi=\psi(x,t,y,\tau)$ in (30) (31) and integrate over Q_T, to get

$$\int\int_{Q_T}\int\int_{Q_T}\{I_\eta(u-v)(\psi_t+\psi_\tau)-(E_i(x,t)\psi_{x_i}$$
$$+E_i(y,\tau)\psi_{y_i})I_\eta(u-v)$$
$$+A_\eta(u,v)\Delta_x\psi+A_\eta(v,u)\Delta_y\psi)$$
$$-S'_\eta(u-v)(\mid\nabla\int_0^u\sqrt{a(s)}ds\mid^2$$
$$+\mid\nabla\int_0^v\sqrt{a(s)}ds\mid^2)\psi$$
$$-(E_{ix_i}-E_{iy_i})\int_v^u sS'_\eta(s-k)ds\psi\}dxdtdyd\tau$$
$$\geq 0. \qquad (32)$$

Let $\psi(x,t,y,\tau)=\phi(x,t)j_h(x-y,t-\tau)$. Where $\phi(x,t)\geq 0$, $\phi(x,t)\in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau)=\omega_h(t-\tau)\Pi_{i=1}^N\omega_h(x_i-y_i),$$

$$\omega_h(s)=\frac{1}{h}\omega(\frac{s}{h})$$

$$\omega(s)\in C_0^\infty(R),\ \omega(s)\geq 0,\ \omega(s)=0,$$

if $\mid s\mid>1$,

$$\int_{-\infty}^\infty\omega(s)ds=1.$$

it is clear of that

$$\frac{\partial j_h}{\partial t}+\frac{\partial j_h}{\partial\tau}=0,$$

$$\frac{\partial j_h}{\partial x_i}+\frac{\partial j_h}{\partial y_i}=0,$$

$$\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial\tau}=\frac{\partial\phi}{\partial t}j_h,$$

$$\frac{\partial\psi}{\partial x_i}+\frac{\partial\psi}{\partial y_i}=\frac{\partial\phi}{\partial x_i}j_h.$$

If we notice that

$$E_i(x,t)I_\eta(u-v)=\int_v^u E_i(x,t)S_\eta(s-v)ds$$

$$=\int_v^u(E_i(x,t)s-vE_i(y,\tau))'S_\eta(s-v)ds,$$

and

$$\lim_{\eta\to 0}\int_v^u(E_i(x,t)s-vE_i(y,\tau))'S_\eta(s-v)ds$$

$$=\mathtt{sgn}(u-v)(E_i(x,t)u-E_i(y,\tau)v),$$

because $E_i\in L^2(Q_T)$ and $\psi\in C_0^\infty(Q_T\times Q_T)$, by the control convergent theorem, we have

$$\lim_{\eta\to 0}\int\int_{Q_T}\int\int_{Q_T}(E_i(x,t)\psi_{x_i}+E_i(y,\tau)$$
$$\psi_{y_i})I_\eta(u-v)dxdtdyd\tau$$
$$=\int\int_{Q_T}\int\int_{Q_T}\mathtt{sgn}(u-v)(E_i(x,t)u-E_i(y,\tau)v)$$
$$\phi_{x_i}j_h dxdtdyd\tau.$$

Let $h \to 0$ in the above equality. We have

$$\lim_{h\to 0}\int\int_{Q_T}\int\int_{Q_T}\mathrm{sgn}(u-v)(E_i(x,t)u-E_i(y,\tau)v)\phi_{x_i}j_h dxdtdyd\tau = \int\int_{Q_T}E_i(x,t)|u-v|\phi_{x_i}dxdt. \quad (33)$$

At the same time, it is clear of that

$$-\lim_{h\to 0}\lim_{\eta\to 0}\int\int_{Q_T}\int\int_{Q_T}(E_{ix_i}-E_{iy_i})\int_v^u sS'_\eta(s-k)ds\psi dxdtdyd\tau = 0. \quad (34)$$

For the third terms in (32), by Lemma 7, we can deal with it as [12, 13], and get

$$\lim_{\eta\to 0}(A_\eta(u,v)\phi_{x_i}j_{hx_i}+A_\eta(u,v)\phi_{y_i}j_{hy_i}) = 0. \quad (35)$$

Combing (32)-(35), and letting $\eta \to 0, h \to 0$ in (32), we get

$$\int\int_{Q_T}\{u(x,t)-v(x,t)\phi_t-\mathrm{sgn}(u-v)(E_iu-E_iv)\phi_{x_i} +\mathrm{sgn}(u-v)(A(u)-A(v))\Delta\phi\} \geq 0. \quad (36)$$

Let

$$\eta(t) = \int_{\tau-t}^{s-t}\alpha_\varepsilon(\sigma)d\sigma, \;\; \varepsilon < \min\{\tau, T-s\},$$

where $\alpha_\varepsilon(t)$ is the kernel of mollifier with $\alpha_\varepsilon(t) = 0$ for $t \notin (-\varepsilon, \varepsilon)$.

By approximation, we can replace ϕ in (36) by $\phi(x,t) = \omega_\lambda(x)\eta(t)$, where $\omega_\lambda(x)$ is the function of (8), $\eta(t) \in C_0^1(0,T)$. Using the estimates

$$|\nabla\omega_\lambda| \leq C_\lambda\omega_\lambda(x),$$

$$|\Delta\omega_\lambda(x)| \leq C_\lambda\omega_\lambda(x),$$

we obtain from (36)

$$\int_{R^N}|u(x,t)-v(x,t)|\omega_\lambda(x)dx \leq \int_{R^N}|u(x,\tau)-v(x,\tau)|\omega_\lambda(x)dx +c\int_\tau^s\int_{R^N}|u(x,t)-v(x,t)|\omega_\lambda(x)dxdt$$

By Gronwall Lemma

$$\int_{R^N}|u(x,s)-v(x,s)|\omega_\lambda(x)dx \leq c\int_{R^N}|u(x,\tau)-v(x,\tau)|\omega_\lambda(x)dx.$$

Letting $\tau \to 0$, the proof of Theorem 2 is complete.

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