Diagonally Stable Tridiagonal Switched Linear Systems

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Abstract: A stability analysis is carried out for certain classes of switched linear systems with tridiagonal structure, under arbitrary switching signal. This analysis is made using diagonal common quadratic Lyapunov functions. Namely, necessary and sufficient conditions for the existence of such Lyapunov functions are proposed for second order switched systems and for third order switched systems with Toeplitz tridiagonal structure.

Key–Words: switched linear systems, quadratic Lyapunov stability, diagonal stability

1 Introduction

The stability theory of switched systems has been widely studied specially during the last two decades, see, for instance, [1, 13, 12, 14, 9, 5]. This fact is significantly explained by the development of multi-control switching schemes. In these schemes instead of using a unique controller for a given system, a bank of controllers is taken and the control technique consists in switching among the controllers. Moreover, switched systems may also appear as a direct result of a modeling process. In this paper we consider a switched linear system as a family of time invariant linear systems (the bank of the switched system) together with some switching laws. A switching law determines which of the linear system within the family is active at each time instant. That is, a switching signal defines how the time invariant systems commute among themselves. In some applications this switching process occurs with very fast switching rates, [13]. In this cases, the switched system is usually considered under arbitrary switching. Clearly, in this situation, the question of finding conditions that guarantee that the obtained switched system is stable for every switching control law is a crucial one.

Several attempts have been made in order to solve the stability question of a switched system under arbitrary switching, see, for instance, [13, 1, 8, 18, 19, 12, 15]. Most of those attempts are based in the well known fact that the existence of a common quadratic Lyapunov function (CQLF) for the invariant linear systems in the switched system bank is a sufficient condition for asymptotic stability, under arbitrary switching. Indeed, most of the until now known sufficient conditions for stability are sufficient conditions for the existence of a CQLF. It was early established that simultaneous triangularization of the corresponding system matrices is a sufficient condition for the existence of a CQLF [16, 12]. Since similarity preserves the CQLF existence property, the essential argument was showing that a family of upper triangular systems has always a CQLF. Moreover, this was made using diagonal quadratic Lyapunov functions, proving that triangular (upper or lower) systems always have a diagonal CQLF.

Linear invariant systems with diagonal quadratic Lyapunov functions are called, with some language abuse, diagonally stable. Similarly, a switched linear systems with diagonal CQLF is called diagonally stable.

As pointed out by Kaszkurewicz and Bhaya, [10], diagonal stability appears with significant importance in different kinds of applications. Namely, in fish population studies, robust stability of a mechanical system, Lotka-Volterra ecosystem model, convergence of asynchronous computations and in global stability of neural networks. Also, in [2, 3] it can be found the use of CQLF with diagonal structure in the analysis the robust stability of 2-D systems.

The diagonal stability problem for a switched linear system with general structure is not very easy to tackle. It is trivial to note that, diagonal stability of each time invariant systems in the bank is a necessary condition (not usually sufficient) for the switched system diagonal stability. But, even the diagonal stability of time invariant systems is not completely understood. In particular, only for very low dimensions (1, 2 or 3) there are easy verifiable conditions for diagonal stability [17, 6], or for special matrix structures.

In this paper we consider switched linear systems

with tridiagonal structure since for tridiagonal matrices the class of diagonal stable matrices is well identified. So, your goal is to pinpoint, within this class of matrices, sets that are simultaneous diagonally stable, and, consequently, identify banks of invariant linear systems that originate diagonally stable switched linear systems. We approach this question in a constructive manner. In fact, by identifying all possible diagonal quadratic Lyapunov functions for each system in the bank, we are able to determine if there exists a diagonal CQLF and calculate it. First, we propose a very easy verifiable necessary and sufficient condition for the existence of diagonal CQLF for second order switched systems. Using that condition, we obtain necessary and sufficient conditions for the diagonal stability of switched systems with 3-order Toeplitz tridiagonal matrix structure.

2 Preliminaries

Throughout the paper, $\mathbb{R}^{n\times n}$ is used to denote the set of square real matrices of order n. Given $A \in \mathbb{R}^{n\times n}$, $A > 0$ and $A < 0$ means that A is positive definite or negative definite, respectively.

Let $\mathcal{P} = 1, 2, \ldots, N$ be a finite index set, $\Sigma = \{\Sigma_p, p \in \mathcal{P}\}$ a family of time invariant linear systems such that

$$\Sigma_p \colon \dot{x}(t) = A_p x(t)\,,$$

where $A_p \in \mathbb{R}^{n\times n}$, $p \in \mathcal{P}$ and $\sigma \colon [0, +\infty[\to \mathcal{P}$ a piecewise constant function. The family Σ is called the bank of the switched system and each σ is called a switching signal. The corresponding switched system has the following representation

$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{1}$$

where the state x is considered to be a continuous function of time, for each switching signal σ.

We denote by $\mathcal{A}$ the set of all system matrices of Σ, that is, $\mathcal{A} = \{A_p : p \in \mathcal{P}\}$. The switched system as defined above is denoted by $\Sigma_{\mathcal{A}}$. Moreover, $\mathcal{A}$ is said to be the matrix bank of $\Sigma_{\mathcal{A}}$.

Definition 1 ([13]) *The system $\Sigma_{\mathcal{A}}$ is globally uniformly exponentially stable if there exist two positive real numbers c, λ such that, for every initial condition $x(t_0) = x_0$ and every switching signal σ, the solution $x(t)$ of (1), satisfies*

$$||x(t)|| \leq c\, e^{-\lambda(t-t_0)} ||x_0||\,,$$

for $t \geq t_0$.

For simplicity, when $\Sigma_{\mathcal{A}}$ satisfies the previous definition of stability, we refer to it as being stable. It is trivial to notice that a necessary condition for the stability of $\Sigma_{\mathcal{A}}$ is the stability of each individual Σ_p, but is not sufficient. However, if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A_p^T P + P A_p < 0\,, \text{ for every } p \in \mathcal{P}\,,$$

that is, if the systems Σ_p share a common quadratic Lyapunov function $V(x) = x^T\, P\, x$, then the switched system is stable, [13]. V is said a common quadratic Lyapunov function (CQLF) for the systems Σ_p, and also for the switched system $\Sigma_{\mathcal{A}}$. $\Sigma_{\mathcal{A}}$ is said to be diagonally stable, if there exists a diagonal matrix D such that $V(x) = x^T\, D\, x$ is a CQLF for the switched system $\Sigma_{\mathcal{A}}$. Similarly, the matrices A_p are said to be simultaneous diagonally stable and the diagonal matrix D is called a common Lyapunov solution for $\mathcal{A} = \{A_p : p \in \mathcal{P}\}$. Moreover, for invariant systems and for a single matrix we have the correspondent definitions. That is, a matrix $A \in \mathbb{R}^{n\times n}$ (an invariant system $\dot{x} = Ax(t)$) is said to be diagonally stable if there exists a diagonal matrix $D > 0$ such that

$$A^T D + DA < 0\,. \tag{2}$$

In the sequel, we shall use the following classical necessary conditions of matrix diagonal stability, [11, 6, 10].

Proposition 2 *If A is diagonally stable matrix, then*

(i) all principal minors of even order of A are positive and all principal minors of odd order of A are negative, that is, $-A$ is a P-matrix.

(ii) all principal submatrices of A are diagonally stable.

In almost every cases, condition (ii) is not sufficient, unless we consider A to be a submatrix of A, but, then the statement becomes obvious. On the other hand, condition (i) is also sufficient in cases like second order matrices, tridiagonal matrices and Metzler matrices, that is, matrices with non-negative off-diagonal entries, see [7].

Notice that, matrix diagonal stability implies other type stability concepts. As is easy to see from the content of this section, a diagonal stable matrix is always a stable matrix. Furthermore, a diagonal stable matrix is always D-stable and additive D-stable, see [10], for instance, for further insight.

Remark 1: It is a well known fact that matrix stability is preserved by similarity, and that it a real square matrix is stable if and only if the real part of its eigenvalues are negative. However, for diagonal stability

no such criterium can be established. In fact, diagonal stability is not preserved by similarity. For instance,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$

are similar. A is diagonally stable, but B is not diagonally stable. □

3 Second order case

In this section we consider second order systems and give a necessary and sufficient condition for the existence of a diagonal CQLF. The next proposition is the reformulation of condition (i) of Proposition 2, as a necessary and sufficient condition, for second order matrices, [6].

Proposition 3 *Let A be a real matrix of order 2,*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

A is diagonally stable if and only if $\det(A) > 0$, $a_{11} < 0$ *and* $a_{22} < 0$.

First, we show how to construct, in case A is diagonally stable matrix of order two, all of the diagonal solution to the Lyapunov inequality (2).

Proposition 4 *Let $A = [a_{ij}]$ be a diagonally stable matrix of order 2 and D a positive definite diagonal matrix. D is a Lyapunov solution for A if and only if $D = \alpha \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$, for $\alpha \in \mathbb{R}^+$ and $d \in \left]\left(\frac{a_{12}}{2\sqrt{\det A}}\right)^2, +\infty\right[$, if $a_{21} = 0$, or $d \in \left]\left(\frac{\sqrt{\det A}-\sqrt{a_{11}a_{22}}}{a_{21}}\right)^2, \left(\frac{\sqrt{\det A}+\sqrt{a_{11}a_{22}}}{a_{21}}\right)^2\right[$, if $a_{21} \neq 0$.*

Proof: Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is diagonally stable, that is, according with Proposition 3, $a_{11} < 0$, $a_{22} < 0$ and $\det A > 0$.

Without loss of generality, let us consider $D = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$, where $d > 0$. The following is divided into two cases: $a_{21} = 0$ and $a_{21} \neq 0$.

CASE 1: $a_{21} = 0$

$$A^T D + DA = \begin{bmatrix} 2a_{11} & a_{12} \\ a_{12} & 2a_{22}d \end{bmatrix}$$

So, since $a_{11} < 0$, $A^T D + DA < 0$ if and only if $d > \left(\frac{a_{12}}{2\sqrt{\det A}}\right)^2$.

CASE 2: $a_{21} \neq 0$

$$A^T D + DA = \begin{bmatrix} 2a_{11} & a_{12} + a_{21}d \\ a_{12} + a_{21}d & 2a_{22}d \end{bmatrix}$$

So, since $a_{11} < 0$, $A^T D + DA < 0$ if and only if

$$4a_{11}a_{22}d - (a_{12} + a_{21}d)^2 > 0$$

which is equivalent to

$$a_{21}^2 d^2 - 2(\det A + a_{11}a_{22})d + a_{12}^2 < 0 \qquad (3)$$

Therefore, since $\det A + a_{11}a_{22} > 0$, there exists $d > 0$ satisfying the previous inequality if and only if

$$4(\det A + a_{11}a_{22})^2 - 4a_{21}^2 a_{12}^2 > 0$$

which is equivalent to

$$a_{11}a_{22} \det A > 0\,.$$

It follows that such a d exists, since by hypothesis $\det(A) > 0$, $a_{11} < 0$ and $a_{22} < 0$. Moreover, d can and must be chosen inside the interval defined by the two positive real roots of the following quadratic equation

$$a_{21}^2 d^2 - 2(\det A + a_{11}a_{22})d + a_{12}^2 = 0 \qquad (4)$$

By straightforward computation, we get the following roots of (4)

$$l_1 = \left(\frac{\sqrt{\det A} - \sqrt{a_{11}a_{22}}}{a_{21}}\right)^2$$

$$l_2 = \left(\frac{\sqrt{\det A} + \sqrt{a_{11}a_{22}}}{a_{21}}\right)^2.$$

Thus, $A^T D + DA < 0$ if and only if $d \in]l_1, l_2[$. □

Remark 2: Notice that, using the previous proposition, if A is diagonally stable we may identify diagonal solution of the Lyapunov inequality in the following simple manner. If $a_{21} = a_{12} = 0$, take any $D > 0$. If $a_{21} = 0$ and $a_{12} \neq 0$, consider, for instance,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{a_{12}^2}{2\det A} \end{bmatrix}.$$

If $a_{21} \neq 0$, take, for instance,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\det A + a_{11}a_{22}}{a_{21}^2} \end{bmatrix}.$$

□

From Proposition 4 is possible to formulate the next necessary and sufficient condition for the existence of diagonal CQLF.

Proposition 5 *Let*

$$\mathcal{A} = \left\{ A_p \colon A_p = \begin{bmatrix} a_{11}^{(p)} & a_{12}^{(p)} \\ a_{21}^{(p)} & a_{22}^{(p)} \end{bmatrix}, p \in \mathcal{P} \right\}$$

be a set of diagonally stable matrices and define

$$l_1^{(p)} = \begin{cases} \left(\frac{a_{12}^{(p)}}{2\sqrt{\det A_p}} \right)^2 & \text{if} \quad a_{21}^{(p)} = 0 \\ \left(\frac{\sqrt{\det A_p} - \sqrt{a_{11}^{(p)} a_{22}^{(p)}}}{a_{21}^{(p)}} \right)^2 & \text{if} \quad a_{21}^{(p)} \neq 0 \end{cases}$$

and

$$l_2^{(p)} = \begin{cases} +\infty & \text{if} \quad a_{21}^{(p)} = 0 \\ \left(\frac{\sqrt{\det A_p} + \sqrt{a_{11}^{(p)} a_{22}^{(p)}}}{a_{21}^{(p)}} \right)^2 & \text{if} \quad a_{21}^{(p)} \neq 0 \,. \end{cases}$$

The following statements are equivalent.

(i) $\Sigma_{\mathcal{A}}$ *is diagonally stable;*

(ii) $\bigcap_{p \in \mathcal{P}}]l_1^{(p)}, l_2^{(p)}[\neq \emptyset$

(iii) $\max\{l_1^{(p)} : p \in \mathcal{P}\} < \min\{l_2^{(p)} : p \in \mathcal{P} \wedge a_{21}^{(p)} \neq 0\}$

Proof: Since for every A_p the interval $]l_1^{(p)}, l_2^{(p)}[$ defines all admissible values for d_p such that

$$D = \begin{bmatrix} 1 & 0 \\ 0 & d_p \end{bmatrix}$$

is Lyapunov solution for A_p, there exists a common Lyapunov solution for all A_p, with diagonal structure, if and only if

$$\bigcap_{p \in \mathcal{P}}]l_1^{(p)}, l_2^{(p)}[\neq \emptyset \,.$$

Furthermore, each value $d \in \bigcap_{p \in \mathcal{P}}]l_1^{(p)}, l_2^{(p)}[$ defines a common Lyapunov solution for $\mathcal{A}$.

Finally, condition *(iii)* is simply an equivalent form of condition *(ii)*. □

Example 1: The following diagonal stable matrices

$$A_1 = \begin{bmatrix} -2 & 1 \\ 1.9 & -1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} -2 & 2.9 \\ 2 & -3 \end{bmatrix}$$

do not have a diagonal common Lyapunov solution. In fact, applying Proposition 5 to $\mathcal{A} = \{A_1, A_2\}$,

$$\begin{aligned} l_1^{(1)} &\approx 0.334 \qquad l_2^{(1)} \approx 0.829 \\ l_1^{(2)} &\approx 1.002 \qquad l_2^{(2)} \approx 2.097. \end{aligned}$$

Since $l_1^{(2)} > l_2^{(1)}$, $\mathcal{A}$ do not has a diagonal common Lyapunov solution. It is worthwhile to notice that $\mathcal{A}$ has a common Lyapunov solution, but not with diagonal form. The conclusion that $\mathcal{A}$ has a common Lyapunov solution may be obtained noting that the eigenvalues of the matrices $A_1 A_2$ and $A_1 A_2^{-1}$ are all positive, using a necessary and sufficient condition proposed in [18] for the order two case. □

Example 2: The following diagonal stable matrices

$$A_1 = \begin{bmatrix} -2 & -1 \\ 1.9 & -1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} -2 & 2.9 \\ 2 & -3 \end{bmatrix}$$

have a diagonal common Lyapunov solution. In fact, applying Proposition 5 to $\mathcal{A} = \{A_1, A_2\}$,

$$\begin{aligned} l_1^{(1)} &\approx 0.087 \qquad l_2^{(1)} \approx 3.181 \\ l_1^{(2)} &\approx 1.002 \qquad l_2^{(2)} \approx 2.097. \end{aligned}$$

Since $l_1^{(2)} < l_2^{(1)}$, $\mathcal{A}$ has a diagonal common Lyapunov solution. Furthermore, in this case, the identity matrix is a diagonal common Lyapunov solution. □

4 Tridiagonal systems

In this section we address the problem of diagonal stability of switched systems where the matrix bank is formed by diagonally stable tridiagonal matrices. That is, we consider switched systems, $\Sigma_{\mathcal{A}}$, such that $\mathcal{A} = \{A_p \colon p \in \mathcal{P}\}$ where

$$A_p = \begin{bmatrix} a_1^{(p)} & b_1^{(p)} & & 0 \\ c_1^{(p)} & a_2^{(p)} & b_2^{(p)} & \\ \ddots & \ddots & \ddots & \\ & c_{n-2}^{(p)} & a_{n-1}^{(p)} & b_{n-1}^{(p)} \\ 0 & & c_{n-1}^{(p)} & a_n^{(p)} \end{bmatrix}$$

and A_p is diagonally stable.

Our interest in this particular type of matrix this do to the hardness of the general problem and motivated by the fact that the case of tridiagonal matrices is one of the few cases were there is a very simple characterization of matrix diagonal stability. A tridiagonal matrix A is diagonally stable if and only if $-A$ is a P-matrix, [4].

Next proposition identifies a special class of tridiagonal switched systems that are diagonally stable.

Proposition 6 *Let $\Sigma_{\mathcal{A}}$ be a switched system such that $\mathcal{A} = \{A_p : p \in \mathcal{P}\}$ where*

$$A_p = \begin{bmatrix} a_1^{(p)} & b_1^{(p)} & & 0 \\ c_1^{(p)} & a_2^{(p)} & b_2^{(p)} & \\ \ddots & \ddots & \ddots & \\ & c_{n-2}^{(p)} & a_{n-1}^{(p)} & b_{n-1}^{(p)} \\ 0 & & c_{n-1}^{(p)} & a_n^{(p)} \end{bmatrix}$$

is an irreducible tridiagonal and diagonally stable matrix. If $\left|\frac{b_i^{(p)}}{c_i^{(p)}}\right| = d_i$, for some d_i, $i = 1, 2, \ldots, n-1$, for every $p \in \mathcal{P}$, then $\Sigma_{\mathcal{P}}$ is diagonally stable.

Proof: Following the construction proposed in [4], the matrix

$$D_p = \begin{bmatrix} 1 & & & & \\ & \left|\frac{b_1^{(p)}}{c_1^{(p)}}\right| & & & 0 \\ & & \left|\frac{b_1^{(p)} b_2^{(p)}}{c_1^{(p)} c_2^{(p)}}\right| & & \\ & & & \ddots & \\ & 0 & & & \left|\frac{b_1^{(p)} \cdots b_{n-1}^{(p)}}{c_1^{(p)} \cdots c_{n-1}^{(p)}}\right| \end{bmatrix}$$

is a Lyapunov solution of A_p. Since, by hypothesis, $\left|\frac{b_i^{(p)}}{c_i^{(p)}}\right| = d_i$, for some d_i, $i = 1, 2, \ldots, n-1$, for every $p \in \mathcal{P}$,

$$D = \begin{bmatrix} 1 & & & & \\ & d_1 & & & 0 \\ & & d_1 d_2 & & \\ & & & \ddots & \\ & 0 & & & \prod_{i=1}^{n-1} d_i \end{bmatrix}$$

is a diagonal CQLF for $\mathcal{A}$. Therefore, $\Sigma_{\mathcal{P}}$ is diagonally stable. □

Notice that not every set of tridiagonal diagonally stable matrices has a diagonal common Lyapunov solution. In fact, a necessary condition is that every set of principal submatrices formed from the original ones by the same choice of lines and columns has a common diagonal Lyapunov solution. For example, the following diagonally stable matrices

$$\begin{bmatrix} -2 & 1 & 0 \\ 1.9 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 & 2.9 & 0 \\ 2 & -3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

do not have a diagonal common Lyapunov solution, since the submatrices $\begin{bmatrix} -2 & 1 \\ 1.9 & -1 \end{bmatrix}$ and $\begin{bmatrix} -2 & 2.9 \\ 2 & -3 \end{bmatrix}$ do not have have a diagonal common Lyapunov solution, see Example 1. Furthermore, that necessary condition is not sufficient, as we shall see. In order to clearly conclude this and with the goal of pointing out some other classes of diagonally stable tridiagonal switched systems, we next study the diagonal stability of switched systems with 3-order Toeplitz tridiagonal matrix structure.

4.1 3-Order Toeplitz Tridiagonal Systems

In this subsection we consider switched systems with matrix bank of the form

$$A_p = \begin{bmatrix} a_p & b_p & 0 \\ c_p & a_p & b_p \\ 0 & c_p & a_p \end{bmatrix}, p \in \mathcal{P}$$

and diagonally stable. We take these matrices partitioned as follows

$$A_p = \left[\begin{array}{cc|c} & & 0 \\ & M_p & b_p \\ \hline 0 & c_p & a_p \end{array}\right], \text{ where } M_p = \begin{bmatrix} a_p & b_p \\ c_p & a_p \end{bmatrix}.$$

First of all, note that the second order principal submatrices of A_p are M_p or $\text{diag}(a_p, a_p)$. Since $\text{diag}(a_p, a_p)$ admits every positive diagonal matrix as Lyapunov solution, it is natural to ask if the existence of a diagonal common Lyapunov solution for $\{M_p : p \in \mathcal{P}\}$ implies the existence of a diagonal common Lyapunov solution for $\{A_p : p \in \mathcal{P}\}$. As we shall see the answer to this question is negative.

Let us begin by clearly identify the diagonally stable 3-order Toeplitz tridiagonal matrices. Indeed we may state the following proposition.

Proposition 7 *Let A be a matrix of the form*

$$A = \begin{bmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{bmatrix}.$$

The following propositions are equivalent:

(i) A *is stable*

(ii) A *is diagonally stable*

(iii) $a < 0 \wedge a^2 - 2cb > 0$

Proof: First let us prove that *(ii)* and *(iii)* are equivalent. Since, A is tridiagonal, A is diagonally stable if and only if $-A$ is P-matrix, that is, if and only if all principal minors of even order of A are positive and all principal minors of odd order of A are negative. So, A is diagonally stable if and only if

$$a < 0 \wedge a^2 - bc > 0 \wedge -cba + a(a^2 - bc) < 0\,,$$

that is, if and only if

$$a < 0 \wedge a^2 - bc > 0 \wedge a^2 - 2bc > 0\,.$$

Since $a^2 - bc > 0$ whenever $a^2 - 2bc > 0$, we may conclude that A is diagonally stable if and only if

$$a < 0 \wedge a^2 - 2bc > 0\,.$$

Considering that *(ii)* implies *(i)*, it remains to prove that *(i)* implies *(iii)*. If A is stable, then the real parts of its eigenvalues are negative. Let us determine the eigenvalues of A.
$\det(I\lambda - A) = 0$ if and only if

$$(\lambda - a)\det\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - a \end{bmatrix} - cb(\lambda - a) = 0\,.$$

So, a is one of the eigenvalues and the other two are the roots of the following polynomial

$$(\lambda - a)^2 - 2cb\ .$$

If $cb > 0$, then the eigenvalues of A are a, $a - \sqrt{2cb}$ and $a + \sqrt{2cb}$. Since they are negative, we conclude that $a < 0$ and $a^2 > 2cb$. On the other hand, if $cb < 0$, then $a^2 - 2cb > 0$ and $a < 0$ (a is one of the eigenvalues of A). □

In the sequel, consider a diagonally stable matrix A such that

$$A = \left[\begin{array}{cc|c} & M & \begin{matrix}0\\ b\end{matrix} \\ \hline 0 & c & a \end{array}\right], \qquad (5)$$

where $M = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$. Next we identify all possible diagonal solutions for a matrix A of the form given by (5).

Clearly, A is diagonally stable if and only if $a < 0$ and $\det M - cb > 0$, by Proposition 7. Let us consider a positive definite diagonal matrix

$$D(x,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{bmatrix}.$$

$D(x,z)$ is a Lyapunov solution for A if and only if

$$Q(x,z) = \begin{bmatrix} 2a & cx+b & 0 \\ cx+b & 2ax & cz+bx \\ 0 & cz+bx & 2az \end{bmatrix} < 0$$

i.e., if and only if

$$2a < 0\,; \qquad (6)$$

$$q_2(x) = \det\begin{bmatrix} 2a & cx+b \\ cx+b & 2ax \end{bmatrix} > 0\,; \qquad (7)$$

$$\det Q(x,z) < 0\,. \qquad (8)$$

Conditions (6) and (7) are equivalent to say that $\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$ is a diagonal Lyapunov solution for M. On the other hand, condition (8) is equivalent to

$$-2a(cz+bx)^2 + 2azq_2(x) < 0$$

i.e., equivalent to

$$c^2z^2 - (q_2(x) - 2bcx)z + b^2x^2 < 0 \qquad (9)$$

In order to analyse the solutions of (9), let us consider the following two cases: $c = 0$ and $c \neq 0$.

CASE 1: $c = 0$

From (9) we have

$$z > \frac{b^2x^2}{4a^2x - b^2}\,.$$

Therefore, $D(x,z)$ is a Lypunov solution for A if and only if

$$x \in]l_1, +\infty[, \quad \text{where} \quad l_1 = \frac{b^2}{4a^2},$$

$$z \in]z_1(x), +\infty[, \quad \text{where} \quad z_1(x) = \frac{b^2x^2}{4a^2x - b^2}\,.$$

CASE 2: $c \neq 0$

Applying Proposition 4 to M, (6) and (7) if and only if $x \in]l_1, l_2[$, where

$$l_1 = \frac{\det M + a^2 - 2\sqrt{a^2 \det M}}{c^2} \qquad (10)$$

$$l_2 = \frac{\det M + a^2 + 2\sqrt{a^2 \det M}}{c^2}\,. \qquad (11)$$

Since $z \in \mathbb{R}^+$, (9) is equivalent to

$$q_2(x)(q_2(x) - 4bcx) > 0 \wedge z \in]z_1(x), z_2(x)[, \quad (12)$$

where

$$z_1(x) = \frac{q_2(x) - 2bcx - \sqrt{q_2(x)(q_2(x) - 4bcx)}}{2c^2} \quad (13)$$

$$z_2(x) = \frac{q_2(x) - 2bcx + \sqrt{q_2(x)(q_2(x) - 4bcx)}}{2c^2}. \quad (14)$$

If $bc \leq 0$, then

$$q_2(x)(q_2(x) - 4bcx) > 0$$

is satisfied for all $x \in \mathbb{R}^+$. So, in this case, $D(x, z)$ is a Lypunov solution for A if and only if

$$\begin{array}{l} x \in]l_1, l_2[\,, \text{ where } l_1, l_2 \text{ are as in (10)-(11)} \\ z \in]z_1(x), z_2(x)[\,, \text{ where } z_1(x), z_2(x) \text{ are} \\ \qquad\qquad\qquad\qquad \text{as in (13)-(14).} \end{array}$$

If $bc > 0$, then $q_2(x)(q_2(x) - 4bcx) > 0$ if and only if

$$c^2x^2 - 2(2\det M - bc)x + b^2 < 0\,. \quad (15)$$

Notice that the discriminant of the corresponding quadratic equation, Δ, is as follows

$$\begin{aligned} \Delta &= 4(2\det M - bc)^2 - 4c^2b^2 \\ &= 4(2\det M - 2bc)(2\det M) \\ &= 16\det M(\det M - bc)\,. \end{aligned}$$

Since $\det M > 0$ and $\det M - cb > 0$, (15) holds if and only if $x \in]s_1, s_2[$, where

$$s_1 = \frac{2\det M - bc - 2\sqrt{\det M(\det M - bc)}}{c^2} \quad (16)$$

$$s_2 = \frac{2\det M - bc + 2\sqrt{\det M(\det M - bc)}}{c^2}\,. \quad (17)$$

Then, in case $bc > 0$, $D(x, z)$ is a Lypunov solution for A if and only if

$$\begin{array}{l} x \in]s_1, s_2[\,, \text{ where } s_1, s_2 \text{ are as in (16)-(17)} \\ z \in]z_1(x), z_2(x)[\,, \text{ where } z_1(x), z_2(x) \text{ are} \\ \qquad\qquad\qquad\qquad \text{as in (13)-(14).} \end{array}$$

Note that, as expectable, it is possible to show that $]s_1, s_2[\subset]l_1, l_2[$.

The previous analysis enable us to give a necessary and sufficient condition for diagonal stability of switched systems with matrix bank $\mathcal{A} = \{A_p : p \in \mathcal{P}\}$, where

$$A_p = \left[\begin{array}{cc|c} & & 0 \\ & M_p & b_p \\ \hline 0 & c_p & a_p \end{array}\right], \quad (18)$$

where $M_p = \begin{bmatrix} a_p & b_p \\ c_p & a_p \end{bmatrix}, p \in \mathcal{P}$, is diagonally stable. Consider associated to each A_p the following constants:

$$s_1^{(p)} = \begin{cases} \left(\frac{b_p}{2a_p}\right)^2\,, \text{ if } c_p = 0 \\ \left(\frac{\sqrt{\det M_p} - \sqrt{a_p^2}}{c_p}\right)^2\,, \text{ if } c_p \neq 0 \wedge b_pc_p \leq 0 \\ \left(\frac{\sqrt{\det M_p} - \sqrt{\det M_p - b_pc_p}}{c_p}\right)^2\,, \text{ if } b_pc_p > 0 \end{cases}$$

$$s_2^{(p)} = \begin{cases} +\infty\,, \text{ if } c_p = 0 \\ \left(\frac{\sqrt{\det M_p} + \sqrt{a_p^2}}{c_p}\right)^2\,, \text{ if } c_p \neq 0 \wedge b_pc_p \leq 0 \\ \left(\frac{\sqrt{\det M_p} + \sqrt{\det M_p - b_pc_p}}{c_p}\right)^2\,, \text{ if } b_pc_p > 0 \end{cases}$$

Proposition 8 *Let $\Sigma_{\mathcal{A}}$ a switched system such that $\mathcal{A}$ is defined as in (18) and, for each A_p, $s_1^{(p)}, s_2^{(p)}$ defined as above. $\Sigma_{\mathcal{A}}$ is diagonally stable if and only if there exists*

$$x \in \bigcap_{p \in \mathcal{P}}]s_1^{(p)}, s_2^{(p)}[\quad (19)$$

such that

$$\bigcap_{p \in \mathcal{P}}]z_1^{(p)}(x), z_2^{(p)}(x)[\neq \emptyset \quad (20)$$

where, if $c_p \neq 0$,

$$z_1^{(p)}(x) = \frac{q_2^{(p)}(x) - 2b_pc_px - \sqrt{q_2^{(p)}(x)(q_2^{(p)}(x) - 4b_pc_px)}}{2c_p^2}$$

$$z_2^{(p)}(x) = \frac{q_2^{(p)}(x) - 2b_pc_px + \sqrt{q_2^{(p)}(x)(q_2^{(p)}(x) - 4b_pc_px)}}{2c_p^2}$$

or else

$z_1^{(p)}(x) = \frac{b_p^2x^2}{4a_p^2x - b_p^2}$ *and* $z_2^{(p)}(x) = +\infty$.

Proposition 8 gives a necessary and sufficient condition for the diagonal stability of the switched system $\Sigma_{\mathcal{A}}$, but, in general, it is not very easy to use. However, from it we may derive a necessary condition simple to check.

Corollary 9 *Let $\Sigma_{\mathcal{A}}$ a switched system such that $\mathcal{A}$ is defined as in (18). If $\Sigma_{\mathcal{A}}$ is diagonally stable, then*

$$\bigcap_{p\in\mathcal{P}}]s_1^{(p)}, s_2^{(p)}[\neq \emptyset \tag{21}$$

Using this last corollary, we present an example where the diagonal stability of $\{M_p\colon p \in \mathcal{P}\}$ is not followed by the diagonal stability of $\mathcal{A} = \{A_p\colon p \in \mathcal{P}\}$.

Example 3: Although the switched system with matrix bank $\mathcal{M} = \{M_1, M_2\}$, where

$$M_1 = \begin{bmatrix} -4 & 1 \\ 6 & -4 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} -4 & 5 \\ 1 & -4 \end{bmatrix},$$

is diagonally stable, the switched system $\Sigma_{\mathcal{P}}$ with matrix bank $\mathcal{A} = \{A_1, A_2\}$, where

$$A_1 = \left[\begin{array}{cc|c} -4 & 1 & 0 \\ 6 & -4 & 1 \\ \hline 0 & 6 & -4 \end{array}\right] \text{ and } A_2 = \left[\begin{array}{cc|c} -4 & 5 & 0 \\ 1 & -4 & 5 \\ \hline 0 & 1 & -4 \end{array}\right],$$

is not diagonally stable. In fact, applying Proposition 5 to $\mathcal{M}$,

$$l_1^{(1)} \approx 0.0195 \qquad l_2^{(1)} \approx 1.4249$$
$$l_1^{(2)} \approx 0.467 \qquad l_2^{(2)} \approx 53.5329.$$

Therefore, $\mathcal{M}$ has a diagonal common Lyapunov solution. On the other hand, applying Corollary 9 to $\mathcal{A}$,

$$s_1^{(1)} \approx 0.0375 \qquad s_2^{(1)} \approx 0.7402$$
$$s_1^{(2)} \approx 0.7519 \qquad s_2^{(2)} \approx 33.2480.$$

So, $\mathcal{A}$ does not have a diagonal common Lyapunov solution. □

Example 4: Notice that, the necessary condition in Corollary 9 is not sufficient. For example, $\mathcal{A} = \{A_1, A_2\}$, where

$$A_1 = \begin{bmatrix} -3 & 2.2 & 0 \\ 2 & -3 & 2.2 \\ 0 & 2 & -3 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} -4 & 1 & 0 \\ 6 & -4 & 1 \\ 0 & 6 & -4 \end{bmatrix},$$

satisfies (21). In fact,

$$s_1^{(1)} \approx 0.721 \qquad s_2^{(1)} \approx 0.03$$
$$s_1^{(2)} \approx 1.67 \qquad s_2^{(2)} \approx 0.74.$$

However, $\mathcal{A}$ does not have a diagonal common Lyapunov solution. This can be concluded, using Proposition 8, showing that

$$\bigcap_{p=1,2}]z_1^{(p)}(x), z_2^{(p)}(x)[= \emptyset\,,$$

for every $x \in \bigcap_{p=1,2}]s_1^{(p)}, s_2^{(p)}[$. □

Remark 3: A 3-order Toeplitz tridiagonal matrix,

$$A = \begin{bmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{bmatrix},$$

always has a diagonal Lyapunov solutions of the form

$$D(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^2 \end{bmatrix}.$$

Indeed, if A is irreducible, then

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & |\frac{a}{c}| & 0 \\ 0 & 0 & (\frac{a}{c})^2 \end{bmatrix}$$

is a Lyapunov solution for A. If $c = 0$, it is sufficient to choose x sufficiently large, otherwise, if $b = 0$, it is sufficient to choose x sufficiently small. □

Proposition 8 has been established by totally characterizing the diagonal Lyapunov solutions of a diagonally stable 3-order Toeplitz tridiagonal matrix. An alternative sufficient condition can be obtained by characterizing the diagonal Lyapunov solution of the type mentioned in Remark 3.

Actually, by going back to (9) and taking $z = x^2$, we conclude that $D(x)$ is a Lyapunov solution for A, as defined in (5), if and only if the following two conditions hold

$$q_2(x) > 0 \tag{22}$$
$$c^2x^2 - 2\det M x + b^2 < 0\,. \tag{23}$$

Clearly, (22) is equivalent to

$$x \in]l_1, l_2[$$

where l_1 and l_2 are as follows

$$l_1 = \begin{cases} \frac{b^2}{4a^2} & \text{if} \quad c = 0 \\ \frac{\det M + a^2 - 2\sqrt{a^2 \det M}}{c^2} & \text{if} \quad c \neq 0 \end{cases} \tag{24}$$

$$l_2 = \begin{cases} +\infty & \text{if} \quad c = 0 \\ \frac{\det M + a^2 - 2\sqrt{a^2 \det M}}{c^2} & \text{if} \quad c \neq 0\,. \end{cases} \tag{25}$$

In order to analyse the solutions of (23), let us consider the following two cases: $c = 0$ and $c \neq 0$.

CASE 1: $c = 0$

From (23) we have $x > \frac{b^2}{2 \det M}$., that is,

$$x > \frac{b^2}{2a^2}\,.$$

Since $\frac{b^2}{2a^2} > \frac{b^2}{4a^2}$, considering (24)-(25), we conclude that $D(x)$ is a Lypunov solution for A if and only if

$$x \in]r_1, +\infty[, \quad \text{where} \quad r_1 = \frac{b^2}{2a^2},$$

CASE 2: $c \neq 0$

Notice that the discriminant of the corresponding quadratic equation, Δ, is as follows

$$\begin{aligned} \Delta &= 4 \det M^2 - 4c^2b^2 \\ &= 4(\det M - bc)(\det M + cb) \\ &= 4a^2(\det M - bc)\,. \end{aligned}$$

Since $\det M - cb > 0$, (23) holds if and only if $x \in]r_1, r_2[$, where

$$r_1 = \frac{\det M - \sqrt{a^2(\det M - bc)}}{c^2} \tag{26}$$

$$r_2 = \frac{\det M + \sqrt{a^2(\det M - bc)}}{c^2}\,. \tag{27}$$

Clearly, $]r_1, r_2[\subset]l_1, l_2[$. Then, in this case , $D(x)$ is a Lypunov solution for A if and only if

$$x \in]r_1, r_2[\,, \text{ where } r_1, r_2 \text{ are as in (26)-(27)}\,.$$

Next we give an easily verifiable sufficient condition, based on these previous conclusions.

Proposition 10 *Let $\Sigma_{\mathcal{A}}$ a switched system such that $\mathcal{A}$ is defined as in (18) and define*

$$r_1^{(p)} = \begin{cases} \frac{b_p^2}{2a_p^2}\,, \text{ if } c_p = 0 \\ \frac{\det M_p - \sqrt{a_p \det A_p}}{c_p^2}\,, \text{ if } c_p \neq 0 \end{cases}$$

$$r_2^{(p)} = \begin{cases} +\infty\,, \text{ if } c_p = 0 \\ \frac{\det M_p + \sqrt{a_p \det A_p}}{c_p^2}\,, \text{ if } c_p \neq 0\,. \end{cases}$$

If

$$\bigcap_{p \in \mathcal{P}}]r_1^{(p)}, r_2^{(p)}[\neq \emptyset\,,$$

then $\Sigma_{\mathcal{A}}$ is diagonally stable.

Proof: Based on the previously made study, since $\bigcap_{p \in \mathcal{P}}]r_1^{(p)}, r_2^{(p)}[\neq \emptyset$, we may conclude that every diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^2 \end{bmatrix}, \text{ with } x \in \bigcap_{p \in \mathcal{P}}]r_1^{(p)}, r_2^{(p)}[$$

defines a CQLF for $\Sigma_{\mathcal{A}}$. □

Example 5: Let $\Sigma_{\mathcal{A}}$ be a switched system where $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, for

$$A_1 = \begin{bmatrix} -1 & 2.5 & 0 \\ 2 & -1 & 2.5 \\ 0 & 2 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -3 & -1 & 0 \\ 2 & -3 & -1 \\ 0 & 2 & -3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -5 & 2 & 0 \\ 4 & -5 & 2 \\ 0 & 4 & -5 \end{bmatrix},\ A_4 = \begin{bmatrix} -4 & -1 & 0 \\ 6 & -4 & -1 \\ 0 & 6 & -4 \end{bmatrix}.$$

It is possible to conclude the diagonal stability of $\Sigma_{\mathcal{A}}$ using Proposition 10. In fact,

$$\begin{aligned} r_1^{(1)} &\approx 0.671 \qquad r_2^{(1)} \approx 2.329 \\ r_1^{(2)} &\approx 0.046 \qquad r_2^{(2)} \approx 5.454 \\ r_1^{(3)} &\approx 0.125 \qquad r_2^{(3)} \approx 2 \\ r_1^{(4)} &\approx 0.023 \qquad r_2^{(4)} \approx 1.199\,. \end{aligned}$$

Note that, for instance, $1 \in \bigcap_{p \in \{1,2,3,4\}}]r_1^{(p)}, r_2^{(p)}[$. So, the identity matrix defines a CQLF for $\Sigma_{\mathcal{A}}$. □

Example 6: Let $\Sigma_{\mathcal{A}}$ be a switched system where $\mathcal{A} = \{A_1, A_2\}$, for

$$A_1 = \begin{bmatrix} -1 & 2.5 & 0 \\ 2 & -1 & 2.5 \\ 0 & 2 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 0.2 & 0 \\ 2 & -1 & 0.2 \\ 0 & 2 & -1 \end{bmatrix}.$$

In this case if we try to use Proposition 10, we obtain

$$\begin{aligned} r_1^{(1)} &\approx 0.671 \qquad r_2^{(1)} \approx 2.329 \\ r_1^{(2)} &\approx 0.038 \qquad r_2^{(2)} \approx 0.261\,, \end{aligned}$$

which is inconclusive. However, using Corollary 9 we may conclude that in fact $\Sigma_{\mathcal{A}}$ is not diagonally stable. □

5 Conclusion

In this paper we have study the diagonal stability of switched linear systems. Since the this study for a general order and general matrix structure is very

hard, we have consider the second order case and the tridiagonal case. In particular, we have given full characterizations of diagonally stable switched systems of order 2 and of order 3 with Toeplitz tridiagonal matrix structure.

The proposed characterization of diagonally stable switched systems of order 3 with Toeplitz tridiagonal matrix structure is not very simple to use. In order to remedy this disadvantage, we have proposed an alternative sufficient conditions. Moreover, during the numerous computations made in our research, this sufficient condition behaved like a necessary condition. However, we were not able to prove that is indeed the case.

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References:

[1] A. A. Agrachev and D. Liberzon. Lie-algebraic stability criteria for switched systems. *SIAM Journal Control Optimization*, 40, 2001, pp. 253–269.

[2] M. Alfidi and A. Hmamed. Control for stability and positivity of 2-D linear discrete-time systems. *WSEAS Transactions on Systems and Control*, 2(12), 2007, pp.547–556.

[3] M. Alfidi and A. Hmamed. Robust stability analysis for 2-d continuous-time systems via parameter-dependent lyapunov functions. *WSEAS Transactions on Systems and Control*, 2(11), 2007, pp. 497–503.

[4] D. Carlson, B. N. Datta, and C. R. Johnson. A semi-definite Lyapunov theorem and the characterization of tridiagonal D-stable matrices. *SIAM J. Alg. Disc. Meth.*, 3(3), 1982, pp. 223–304.

[5] L. Ciprian, U. Andreea, P. Dumitru, and F. Cristian. Stabile algorithms switching for multiple models control systems. *WSEAS Transactions on Systems*, 8(4), 2009, pp. 451–460.

[6] G. W. Cross. Theree types of matrix stability. *Linear Algebra and its Applications*, 20, 1978, pp. 253–263.

[7] D. Hershkowitz. Recent directions in matrix stability. *Linear Algebra and its Applications*, 171, 1992, pp. 161–186.

[8] J. P. Hespanha and A. S. Morse. Switching between stabilizing controllers. *Automatica*, 38, 2002, pp. 1905–1917.

[9] J. P. Hespanha, P. Santesso, and G. Stewart. Optimal controller initialization for switching between stabilizing controllers. In *Proceedings of the 46th IEEE Conference on Decision and Control New Orleans, LA, USA*, December 2007, pp. 5634–5639.

[10] E. Kaszkurewicz and A. Bhaya. *Matrix Diagonal Stability in Systems and Computation*. Birkhuser, Boston, 2000.

[11] J. F. B. M. Kraaijevanger. A characterization of Lyapunov diagonal stability using Hadamard products. *Linear Algebra and its Applications*, 151, 1991, pp. 245–254.

[12] D. Liberzon, J. P. Hespanha, and A. S. Morse. Stability of switched systems: a lie-algebraic condition. *Systems Control Letters*, 37(3), 1999, pp. 117–122.

[13] D. Liberzon and S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems*, 19, 1999, pp. 59–70.

[14] H. Lin and P. J. Antsakis. Stability and stabilizability of switched linear systemss a short survey of recent results. In *Proceedings of the 2005 IEEE International Symposium on Intelligent Control, Limassol, Cyprus*, June 2005, pp. 24–29.

[15] O. Mason and R. Shorten. On the simultaneous diagonal stability of a pair of positive linear systems. *Linear Algebra and its Applications*, 413, 2006, pp. 13–23.

[16] Y. Mori, T. Mori, and Y. Kuroe. A solution to the common Lyapunov function problem for continuous-time systems. In *Proceedings of the 36th IEEE Conference on Decision and Control, San Diego, California*, December 1997, pp. 3530–3531.

[17] R. Shorten, O. Mason, and C. King. An alternative proof of the Baker, Berman, Plemmons (BBP) result on diagonal stability and extensions. *Linear Algebra and its Applications*, 430, 2009, pp. 34–40.

[18] R. N. Shorten and K. S. Narendra. Necessary and sufficient condictions for the existence of a commom quadratic Lyapunov function for a finite number of stable second order linear time-invariant systems. *Int. J. Adapt. Control Signal Process*, 16, 2002, pp. 709–728.

[19] R. N. Shorten and K. S. Narendra. On commom quadratic Lyapunov functions for pairs of stable lti systems whose sytem matrices are in companion form. *IEEE Transactions on Automatic Control*, 48(4), 2003, pp. 618–621.