ON THE TRANSLATION OF AN ALMOST LINEAR TOPOLOGY

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Abstract: We present some characterizations of $T1, T2$ separation and metrizability for the translation of an almost linear topology.

Key–words: almost linear space, almost linear topological space, translation of a topology.

1 Introduction

Let Γ be a field of scalars and X a linear space over Γ. We denote by $\mathcal{P}(X)$ the family of nonvoid subsets of X. On $\mathcal{P}(X)$ the algebraic operations

$$(A, B) \longmapsto A + B \text{ and } (\lambda, A) \longmapsto \lambda A,$$

with $\lambda \in \Gamma$, verify the most axioms from the definition of the linear space, excepting the existence of the symmetrical element and the distributivity with respect to the sum of scalars. So on $\mathcal{P}(X)$ it is obtained a non-linear structure. This notion was called *almost linear space* (a.l.s.) in Apreutesei [2] and [3], but it is also known as semi-linear space (see, for example, [14]). Godini names another similar notion by almost linear space ([12]). In the sequel, for continuity in terminology we use the term *almost linear space* (a.l.s.). Another studies on classical operations with subsets can be found in [12].

A lot of papers develop the idea to topologize another algebraic structures than the linear spaces. The most used structures are the algebras and the semigroups. This permitted to extend some classical results. For example, the Banach Principle was recently reformulated on JW-algebras ([13]); also the concept of interval-valued intuitionistic fuzzy sets was implemented on K-algebras ([1]).

Now consider the case of almost linear spaces. If X is also a topological space, we must endow $\mathcal{P}(X)$ with a hyperspacial topology and we ask that this hypertopology be compatible with the operations of a.l.s. The answer is affirmative for the linear topology τ_L ([6]), lower and upper Hausdorff topologies τ_H^- and τ_H^+, lower Vietoris topology τ_V^- and proximal topology τ_P (see [2]) and [3]). These examples have suggested us to introduce (in [2]) the notion of *almost linear topological space* (a.l.t.s.).

We notice that for an almost linear topology, a fundamental system of neighbourhoods for a point x_0 isn't, generally, the translation with x_0 of a fundamental system of neighbourhoods for the origin. So we introduce in [3] a new notion, namely *the translation of a topology* on an a.l.t.s.

It is important to precise what are the properties of a linear topological spaces which hold in the case of a.l.t.s. Also we are interested to find the adequate changes which lead to some properties like the continuity of algebraic operations, separation or metrizability on a.l.t.s.

The aim of this paper is to ask this questions for the translation of an almost linear topology.

In Section 2 we recall certain notions, notations and results which we need in this work; we define an almost linear space, almost linear topological space and the translation of an almost linear topology; we give necessary and sufficient conditions which assure that the translation of a topology is almost linear; we describe some important hypertopologies and we apply these theorems in their cases.

Section 3 is dedicated to the results of T1 and T2-separation and metrizability and to some examples.

2 Terminology and notations

Definition 2.1. Let L be a nonvoid set and

$$"+" : L \times L \to L$$

and

$$"\cdot" : \Gamma \times L \to L$$

two operations on L (with Γ a field of scalars) which satisfy the axioms :

S1) $(x+y)+z = x+(y+z), \forall x,y,z \in L$;

S2) there exists an unique element $0 \in L$ such that $x+0 = 0+x = x, \forall x \in L$;

S3) $x+y = y + x, \forall x,y \in L$;

S4) $\lambda(\mu x) = (\lambda\mu)x, \forall \lambda,\mu \in \Gamma, \forall x \in L$;

S5) $1 \cdot x = x, \forall\, x \in L$;

S6) $\lambda(x+y) = \lambda x + \lambda y, \forall \lambda \in \Gamma, \forall\, x,y \in L$.

We say that $(L,+,\cdot)$ is an *almost linear space* (denoted by a.l.s.).

Let present some examples of a.l.s.

We consider X a linear normed space, w the weak topology on X and we denote by $\mathcal{P}(X)$ the family of nonvoid subsets of X. We also denote:

$$\mathcal{C}l(X) = \{A \in \mathcal{P}(X); A \text{ is closed }\},$$

$$\mathcal{P}b(X) = \{A \in \mathcal{P}(X); A \text{ is bounded }\},$$

$$\mathcal{K}(X) = \{A \in \mathcal{P}(X); A \text{ is a compact }\},$$

$$\mathcal{K}^w(X) = \{A \in \mathcal{P}(X); A \text{ is } w\text{-compact }\},$$

$$\mathcal{D}(X) = \{A \in \mathcal{P}(X); A \text{ is open }\}.$$

Except the family $\mathcal{K}^w(X)$, all the above classes can be also defined if X is a metric space.

Example 2.1. $\mathcal{P}(X)$ with usually operations on subsets forms an a.l.s. Obviously we have

$$\mathcal{P}(X) \supset \mathcal{P}b(X) \supset \mathcal{K}(X) \supset \mathcal{K}^w(X),$$

and $\mathcal{P}b(X)$, $\mathcal{K}(X)$, $\mathcal{K}^w(X)$ are also a.l.s.

Definition 2.2. Let $(L,+,\cdot)$ be an a.l.s.. The structure $(L,+,\cdot,\sigma)$ is called *almost linear topological space* (or a.l.t.s.) if the operations

$$"+" : L \times L \to L$$

and

$$"\cdot" : \Gamma \times L \to L$$

are both continuous in the topology σ.

We recall some definitions from linear spaces adjusted to almost linear spaces $(L,+,\cdot)$ with real scalars.

Definition 2.3. A subset $A \subset L$ is called *absorbent* if for every $x \in L$ there exists $\lambda > 0$ such that $\lambda x \in A$.

Definition 2.4. A subset $A \subset L$ is called *balanced* if

$$\lambda A \subset A$$

for any $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

Definition 2.5. By the balanced involving of a set $A \subset L$ (denoted $\mathcal{E}(A)$) we mean the intersection of all balanced subsets of L which contain A.

Remark 2.1. The balanced involving of the set $A \subset L$ can be couched by the formula (valid in linear topological spaces, too)

$$\mathcal{E}(A) = \bigcup_{|\lambda|\leq 1} \lambda A.$$

Definition 2.6. Let $(L,+,\cdot)$ be an a.l.s. and $k > 0$. A subset $M \subset L$ is called k-*balanced* if for every $\lambda \in \Gamma$ with $|\lambda| \leq k$ and every $x \in M$ we have $\lambda x \in M$.

This means that

$$\lambda M \subset M, \forall \lambda \in \Gamma \text{ with } |\lambda| \leq k.$$

(If $k = 1$ one obtains the definition of balanced set.)

These definitions help us in our purpose to give some conditions for a family $\mathcal{V}(0)$ to be a system of neighbourhoods for 0 in a.l.t.s..

We consider the following assertions:

(V0) $0 \in V$ for any $V \in \mathcal{V}(0)$;

(V1) $\forall V_1, V_2 \in \mathcal{V}(0)\ \exists V_3 \in \mathcal{V}(0)$ such that $V_3 \subset V_1 \cap V_2$;

(V2) $\forall V \in \mathcal{V}(0)\ \exists V_1 \in \mathcal{V}(0)$ such that $V_1 + V_1 \subset V$;

(V3) $\forall V \in \mathcal{V}(0)$, V is absorbent set;

(V4) $\forall V \in \mathcal{V}(0)\ \exists V \in \mathcal{V}(0)$ such that $\mathcal{E}(V_1) \subset V$.

The axiom (V2) tell us that the sum is continuous in any point $(x_0, y_0) \in L \times L$; both the axioms (V2) and (V4) assure the continuity of the multiplication with scalars in any point $(0, x_0) \in \Gamma \times L$.

Following the proof from the linear topological spaces we have:

Theorem 2.1 ([3], Theorem 3.1). *If $\mathcal{V}(0)$ is a fundamental system of neighbourhoods of the origin in an a.l.t.s. L, then $\mathcal{V}(0)$ satisfies the axioms (V0)-(V4).*

Now we give some examples of a.l.t.s.

On X we consider

$$S(a, \varepsilon) = \{x \in X; \|a - x\| < \varepsilon\}$$

the ball of center $a \in X$ and radius $\varepsilon > 0$ and

$$B(a, \varepsilon) = \{x \in X; \|a - x\| \leq \varepsilon\}$$

the closed ball of center $a \in X$ and radius $\varepsilon > 0$.

Also $S_\varepsilon(A)$ is the notation for $\varepsilon-$*enlargement* of A :

$$S_\varepsilon(A) = \{x \in X; \exists a \in A \text{ such that } \|a - x\| < \varepsilon\}$$

with $A \subset X, \varepsilon > 0$.

Now we are ready to recall the definitions and the most important informations about some hypertopologies. A lot of hypertopologies on $\mathcal{A} \subset \mathcal{P}(X)$ (Hausdorff, Vietoris, proximal etc.) must be written like a suprema of two topologies, namely a lower topology τ^- and an upper topology τ^+ :

$$\tau = \tau^- \vee \tau^+.$$

Definition 2.7. Let (X, d) be a metric space.

The Hausdorff topology τ_H is defined on $\mathcal{P}(X)$ by

$$\tau_H = \tau_H^- \vee \tau_H^+,$$

where a basic neighbourhoods of a set $A_0 \in \mathcal{P}(X)$ is, respectively:

in τ_H^- (*lower Hausdorff topology*)

$U_-(A_0, \varepsilon) = \{A \in \mathcal{P}(X);\ A_0 \subset S_\varepsilon(A)\}$, with $\varepsilon > 0$,

and in τ_H^+ (*upper Hausdorff topology*)

$U_+(A_0, \varepsilon) = \{A \in \mathcal{P}(X);\ A \subset S_\varepsilon(A_0)\}$, with $\varepsilon > 0$.

This topology is also induced by the extended-valued semi-metric H_d on $\mathcal{P}(X)$, where

$$H_d(A, B) = \sup\{|d(x, A) - d(x, B)|; x \in X\}$$

($H_d : X \to \mathbb{R}_+ \cup \{+\infty\}$ is symmetrically and satisfies the triangle inequality). So τ_H is the topology of uniformly convergence on X of the distance functional

$$A \longmapsto d(x, A),$$

with $A \in \mathcal{P}(X)$.

Equivalently,

$$H_d(A, B) = \max\{e(A, B), e(B, A)\},$$

where

$$e(A, B) = \sup\{d(a, B); a \in A\}$$

is the *Hausdorff excess of A with respect to B.*

If (X, d) is a metric space, the topology τ_{H_d} does not depend by the metric d, but only the uniformity induced by d, namely: if there exist $m, M > 0$ such that the metrics d and ρ on X verify the inequalities $m \cdot d(x, y) \leq \rho(x, y) \leq M \cdot d(x, y)$ for all $x, y \in X$, then τ_{H_d} and τ_{H_ρ} are equivalent topologies. So on the linear normed spaces X we have the same topology for the equivalent norms.

Finally we note that a sequence of subsets $(A_n)_{n\in\mathbb{N}}$ is τ_H^+-convergent to A iff $e(A_n, A) \to 0$ when $n \to \infty$ and $(A_n)_{n\in\mathbb{N}}$ is τ_H^--convergent to A iff $e(A, A_n) \to 0$ when $n \to \infty$.

More generally, if $(A_i)_{i\in I}$ is a net of nonvoid subsets of X and $A \in \mathcal{P}(X)$ is arbitrary, then the following assertions are equivalent:

1. $A \in \tau_H^- - \lim\limits_i A_i$;
2. $\lim\limits_i d(a_i, A_i) = 0$ for all nets $(a_i)_{i\in I}$ of A;
3. $\limsup\limits_i e(B, A_i) \leq e(B, A)$, for all $B \in \mathcal{C}l(X)$.

Also:

1. $A \in \tau_H^+ - \lim_i A_i$;

2. $\lim_i d(a_i, A_i) = 0$ for all nets $(a_i)_{i\in I}$ with $a_i \in A_i$, $i \in I$;

3. $\limsup_i e(A_i, B) \le e(A, B)$, for all $B \in \mathcal{C}l(X)$.

The Hausdorff topology is the most known topology on $\mathcal{P}(X)$. It is used for obtain the continuity of algebraic operations on $\mathcal{C}l(X)$ (see, for instance, [11]), as well as in the study of uniformly autocontinuous non-additive multifunctions ([10]).

Definition 2.8. *The lower Vietoris topology* τ_V^- on $\mathcal{P}(X)$ is given by the following subbase:

$$V^- = \{A \in \mathcal{P}(X); A \cap V \neq \varnothing\},$$

where V is an open subset of X.

τ_V^- is the weakest topology on $\mathcal{P}(X)$ such that all the functionals

$$A \longmapsto d(x, A)$$

are upper semicontinuous, for any $x \in X$.

If (X, d) is a metric space, the topology τ_V^- does not depend by the metric d, but only the topology induced by d.

If $A, (A_i)_{i\in I}$ is a net of closed subsets of X, then the following assertions are equivalent:

1. $A \in \tau_V^- - \lim_i A_i$;

2. $\lim_{n\to\infty} d(a, A_i) = 0$, for all $a \in A$;

3. $\limsup_i d(A_i, B) \le d(A, B)$, for all closed subsets B of X;

4. $e(A, B) \le \liminf_i e(A_i, B)$, for all closed subsets B of X.

Definition 2.9. *The proximal topology* τ_P on $\mathcal{P}(X)$ is

$$\tau_P = \tau_V^- \vee \tau_H^+.$$

A base of neighbourhoods for $A \in \mathcal{P}(X)$ in τ_P is given by

$$S_\varepsilon(A)^{++} \cap S(a_1, \varepsilon)^- \cap ...S(a_n, \varepsilon)^-,$$

with $a_1, ..., a_n \in A$, $n \in \mathbb{N}$, $n > 0$ and $\varepsilon > 0$.

(For E open in X we denote

$$E^{++} = \{A \in \mathcal{P}(X); \exists\varepsilon > 0 \text{ such that } S_\varepsilon(A) \subset E\}).$$

Finally, $\tau_{P(d)} = \tau_{P(\rho)}$ if and only if d and ρ are metrics on X which determine the same uniformity.

Definitions and details on other hypertopologies can be found in [4]-[9], [15]-[17] and [19]-[21].

Examples 2.2. The continuity of algebraic operations with respect to lower Vietoris and Hausdorff topologies are studied, for example, in [20], paragraph 12, and [2], Propositions 3.4 and 3.7, respectively.

So the following spaces are almost linear topological:

$(\mathcal{P}(X), +, \cdot, \tau_V^-)$,
$(\mathcal{P}b(X), +, \cdot, \tau_H^-)$,
$(\mathcal{P}b(X), +, \cdot, \tau_H^+)$,
$(\mathcal{P}b(X), +, \cdot, \tau_H)$,
$(\mathcal{P}b(X), +, \cdot, \tau_P)$.

Because $\mathcal{P}b(X)$, $\mathcal{K}(X)$, $\mathcal{K}^w(X)$ are almost linear subspace of $\mathcal{P}(X)$, then we have also another a.l.t.s.:

$(\mathcal{P}b(X), +, \cdot, \tau_V^-)$,
$(\mathcal{K}(X), +, \cdot, \tau)$,
$(\mathcal{K}^w(X), +, \cdot, \tau)$,
where τ is one of the five above hypertopologies.

Remark 2.2 (see [3], p.8). If $\mathcal{V}(0)$ is a family of subsets of L satisfying the axioms (V0), (V1), (V2), then we can consider the family

(2.1) $\mathcal{U}(x) = \{U \subset L; \exists V \in \mathcal{V}(0)$ such that $x + V \subset U\}$.

If $\mathcal{V}(0)$ is also a fundamental system of neighbourhoods of the origin in L, then we can generate in this way on L a new topology τ in which a fundamental system of neighbourhoods of a point x is given by the above construction.

Definition 2.10. Let σ be a topology on an a.l.t.s. L and $\mathcal{V}(0)$ a fundamental system of neighbourhoods of the origin (which verifies axioms (V1) - (V4)). Then the topology τ on L given by the relation (2.1) is called *the translation of the topology* σ.

Using the above examples we can prove that, generally, an almost linear topology and its translation are different, as it results from:

Example 2.3. Let X be a linear normed space. We consider the a.l.t.s. $\mathcal{P}b(X)$ endowed with upper Hausdorff topology τ_H^+ and its translation τ. For $A \in \mathcal{P}b(X)$, a fundamental system of neighbourhoods in τ_H^+ is formed by the set of closed balls

$\{B_{H^+}(A,\delta);\delta>0\}$; we take

$$A=\{a,a+e_i\},$$

where a is an arbitrary fixed element of X and e_i is a unit vector of the base of the linear space X. For any $\varepsilon>0$ and $\delta>0$ we have

$$B_{H^+}(A,\delta)\nsubseteq A+B_{H^+}(O,\varepsilon):$$

we find the set $A_\delta=\{a+\delta e_i\}$ such that $A_\delta\in B_{H^+}(A,\delta)$ and $A_\delta\notin A+B_{H^+}(O,\varepsilon)$.

Indeed,

$$e(A_\delta,A)=\sup_{b\in A_\delta}\inf_{a\in A}\|a-b\|=$$

$$=\min\{\delta,\|a+(\delta-1)e_i\|\}\le\delta,$$

so $A_\delta\in B_{H^+}(A,\delta)$.

Now, if $B\in\mathcal{P}b(X)$ is an arbitrary nonvoid subset of $B_{H^+}(O,\varepsilon)$ (with $\sup_{b\in B}\|b\|\le\varepsilon$), then $A_\delta\ne A+B$ because A_δ has only one element and $A+B$ has at least two elements (evidently, $a\ne a+\delta e_i$, so A has two elements).

Let observe that the translation topology τ of an almost linear topology σ might not be almost linear:

Example 2.4. Let X be a linear normed space. We endow the family $\mathcal{P}b(X)$ with the translation τ of upper Hausdorff topology τ_H^+ (or another almost linear topology); the multiplication with scalars is not continuos:

We consider the sets

$$A_n=\{\frac{n+1}{n}te_i;t\in[-1,1]\}$$

and

$$A=\{te_i;t\in[-1,1]\},$$

where e_i is a unit vector of a base of X. We take $\mu_n=\left(\frac{n}{n+1}\right)^2$ a scalar sequence. Then $A_n\to^\tau A$ and $\mu_n\to\mu=1$, but $\mu_nA_n\nrightarrow\mu A$.

In order to prove this, we remark that

$$A_n=A+B_n,$$

where

$$B_n=\left\{\frac{1}{n}te_i;\ t\in[-1,1]\right\}$$

verifies the relation

$$e(B_n,0)=\sup_{b\in B_n}|b|=\frac{1}{n}.$$

If k is a real number, we denote by $[k]$ the greatest integer less then k.

For any $\varepsilon>0$ there exists a positive integer $n_\varepsilon=\left[\frac{1}{\varepsilon}\right]+1$ such that for every $n\ge n_\varepsilon$ we have

$$B_n\in B_{H^+}(0,\varepsilon),$$

that implies

$$A_n\in A+B_{H^+}(0,\varepsilon).$$

So $A_n\to^\tau A$.

Now let $\varepsilon_0\in(0,1)$ be fixed. For every $n\in\mathbb{N}$, $n\ne0$, we have

$$\mu_nA_n\nsubseteq\mu A+B_{H^+}(0,\varepsilon_0):$$

otherwise, let be $C_n\in\mathcal{P}b(X)$ with

$$C_n\in B_{H^+}(0,\varepsilon_0)$$

such that

$$\mu_nA_n=\mu A+C_n,$$

that is

$$\left\{\frac{n+1}{n}te_i;t\in[-1,1]\right\}=$$

$$=\{te_i;t\in[-1,1]\}+C_n.$$

If $c\in C_n$ then

$$e_i+c\in\mu A+C_n$$

is an element of μ_nA_n, so

$$e_i+c=\frac{n}{n+1}te_i.$$

Then

$$e(C_n,0)=\sup_{t\in[-1,1]}\left|\frac{n(t-1)-1}{n+1}e_i\right|=$$

$$=\frac{2n+1}{n+1}>1,$$

a contradiction with the hypothesis

$$C_n\in B_{H^+}(0,\varepsilon_0)$$

($\varepsilon_0<1$).

If we are interested by the almost linearity of the translation of an almost linear topology on L we need to introduce some new axioms:

(V3') $\forall V \in \mathcal{V}(0), \forall \lambda_0 \in \mathbf{\Gamma}, \forall x_0 \in L \, \exists \delta > 0$ such that $\forall \lambda \in \mathbf{\Gamma}$ with $|\lambda - \lambda_0| < \delta$ we have $\lambda x_0 \in \lambda_0 x_0 + V$;

(V4') $\forall V \in \mathcal{V}(0), \forall k > 0, \exists V_1 \in \mathcal{V}(0)$ such that $\forall \lambda \in \mathbf{\Gamma}$ with $|\lambda| \leq k$ we have $\lambda V_1 \subset V$.

The axiom **(**V3'**)** represents the continuity in the topology given by the construction (2.1) of the application $\lambda \mapsto \lambda x_0$ in every $\lambda_0 \in \Gamma$, for any $x_0 \in L$.

The axiom (V4') assures that, for every $k > 0$, every neighbourhood of the origin contains a k-balanced neighbourhood.

In fact the system (V1), (V2), (V3') and (V4') gives a necessary and sufficient condition which assures that the translation τ of an almost linear topology on L is almost linear, as we can see in:

Theorem 2.3 ([3], Theorem 3.2). *Let be* $(L, +, \cdot)$ *an almost linear space.*

(i) If the translation of a topology σ *on* L *is almost linear, then any fundamental system of neighbourhoods of the origin in the topology* σ *verifies the axioms (V1), (V2), (V3') and (V4').*

(ii) If a nonvoid family $\mathcal{V}(0) \subset \mathcal{P}(L)$ *satisfies the conditions (V1), (V2), (V3') and (V4'), then the family* $x + \mathcal{V}(0)$ *forms a fundamental system of neighbourhoods of* x *in an almost linear topology.*

Finally we give some sufficient conditions for the almost linearity of the translation topology; we use for this the continuity of the multiplications with scalars in other points than the origin of $\Gamma \times L$. We express these conditions by the following assertions:

(A3) $\forall x_0 \in L, \forall V \in \mathcal{V}(0), \forall \mu_0 \in \Gamma \, \exists \delta > 0$ and $\exists W \in \mathcal{V}(0)$ such that $\forall \mu \in \Gamma$ with $|\mu - \mu_0| < \delta$ we have $\mu x_0 + W \subset \mu_0 x_0 + V$;

(A4) $\forall V \in \mathcal{V}(0)$, $\forall \lambda_0 \in \Gamma \, \exists \lambda > 0$ and $\exists W \in \mathcal{V}(0)$ such that $\forall \lambda \in \Gamma$ with $|\lambda - \lambda_0| < \delta$ we have $\lambda W \subset V$;

(A5) $\forall x_0 \in L, \forall V \in \mathcal{V}(0) \, \exists \delta > 0 \, \exists U \in \mathcal{V}(0)$ such that $\forall \lambda \in \Gamma$ with $|\lambda - 1| < \delta$ we have $\lambda x_0 + U \subset x_0 + V$.

The conditions (A3) and (A5) are effectively related to the translation topology, while hypothesis (A4) refers to the initial topology of L.

In fact the assertion (A3) is a formulation of the idea that any neighbourhoods of $\mu_0 x_0$ in translation topology is also neighbourhood for the points μx_0 "sufficiently close".

(A4) represents the continuity of the multiplication with scalars in the point $(\lambda_0, 0) \in \Gamma \times L$ and the axiom (A5), in the point $(1, x_0) \in \Gamma \times L$.

Proposition 2.4 ([3], Proposition 3.2). *Let* $(L, +, \cdot)$ *be an a.l.s. and* $\mathcal{V}(0) \subset \mathcal{P}(L)$ *a nonvoid family. Then, the conditions (V1), (V2), (V3), (A4) and (A5) are equivalent with (V1), (V2), (A3) and (A4). If* $\mathcal{V}(0)$ *is a fundamental system of neighbourhoods of the origin for a topology in* L*, then both groups of axioms assure the almost linearity for the translation topology.*

For other details on the a.l.s., a.l.t.s. and the translation of an almost topology see [3].

3 Separation and metrizability

In the sequel, our purpose is to characterize the separations T1 and T2 on the a.l.t.s.

Theorem 3.1. *Let* L *be an a.l.s.,* σ *a topology on* L *that satisfies the axioms (V0) – (V2) and its translation* τ*. Then,* (L, τ) *is a T1 separate space if and only if one of the following equivalent properties is fulfilled:*

(3.1) *If* $x, y \in L$ *such as, for any* $V \in \mathcal{V}(0)$ *there exists* $v \in V$ *having the property:* $x = y + v$*, then* $x = y$*;*

(3.2) $\bigcap_{V \in \mathcal{V}(0)} V = \{0\}$*, where* $\mathcal{V}(0)$ *is an arbitrary fundamental system of neighbourhoods of the origin.*

Proof. One can use the fact that a topological space L is a T1 separate space if and only if the singletons are closed sets.

This is similar with the following condition:

$\forall x \in L$ and $\forall y \in L$ such as, for $V \in \mathcal{V}(0)$ one has

$$(y + V) \cap \{x\} = \varnothing \Longrightarrow y = x,$$

from where the (3.1) form derives.

This is also equivalent to:

(3.3) If $x, y \in L$ for which $x \in y + \bigcap_{V \in \mathcal{V}(0)} V$, then $x = y$.

Obviously, (3.2) implies (3.3).

Conversely, let $a \in \bigcap_{V \in \mathcal{V}(0)} V$ (a nonvoid set: contains 0).

Then, $x = y + a$ and $x = y$, i.e., $x = x + a$; from the uniqueness of element 0, postulated by axiom S2, Definition 2.1, it follows that $a = 0$. ■

Theorem 3.2. *If τ is the translation of a topology on an a.l.s. L, then (L,τ) is a T2 separate space if and only if the following condition is fulfilled:*

(3.4) *If $x, y \in L$ such as for any neighbourhood $V \in \mathcal{V}(0)$ there exists $v_1, v_2 \in V$, such as $x + v_1 = y + v_2$, then $x = y$.*

Proof. Let $x, y \in L$. If $x \neq y$, then there is V_1, $V_2 \in \mathcal{V}(0)$ such as $(x+V) \cap (y+V) = \varnothing$. From the axiom (V1), for V_1 and V_2, there exists $V \in \mathcal{V}(0)$ such as $V \subset V_1 \cap V_2$. Then, $(x+V)\cap(y+V) = \varnothing$.

Rephrasing, according to the converse's contrary, it follows that, if $x, y \in L$ such as $(x+V)\cap(y+V) \neq \varnothing$, for any $V \in \mathcal{V}(0)$, then $x = y$, i.e., (3.4). ■

Remark 3.1. The topological condition (3.4) allows us to extract equal elements from an equality relationship, without using the symmetrical elements, thus 'supplying' the existence axiom, for each element of L, of its symmetric.

Thus, one can give a theorem for the metrizability of an a.l.s.

Theorem 3.3. *Let L be an a.l.s., σ a topology on L and $\mathcal{U}(0) = (U_k)_{k\in\mathbb{N}^*}$ a countable, fundamental system of neighbourhoods of the origin, in the topology σ. If $\mathcal{U}(0)$ satisfies the axioms (V0) – (V2) and the condition*

(3.2)' $$\bigcap_{k\in\mathbb{N}^*} U_k = \{0\}$$

then the space L with the translation topology is metrizable.

Proof. One can closely follow the classical proof for the metrizability of linear topological spaces (see, for example, [18]) and adjust it using Theorem 2.3:

I) Let $U_1 \in \mathcal{U}(0)$; from the axiom (V4') there exists a balanced neighbourhood V_1 of the origin with $V_1 \subset U_1$ (see Theorem 2.3). Now, if we consider the neighbourhood $U_2 \cap V_1$ of the origin, from axiom (V2) and Theorem 2.1 we can found the neighbourhood W_2 such that

$$W_2 + W_2 \subset U_2 \cap V_1 \subset V_1.$$

For W_2 there exists V_2 balanced which verifies the relation: $V_2 + V_2 \subset W_2$. Then

$$V_2 + V_2 + V_2 \subset V_2 + V_2 + V_2 + V_2 \subset W_2 + W_2,$$

so

$$V_2 + V_2 + V_2 \subset V_1.$$

We take back this proceeding for the neighbourhood $U_3 \cap V_2$ and we find V_3 (balanced) such that

$$V_3 + V_3 + V_3 \subset V_2.$$

One can recurrently construct a fundamental system of neighbourhoods of the origin $(V_n)_{n\in\mathbb{N}}$ for the topology σ of L, having the following property:

(3.5) $V_{n+1} \cap V_{n+1} \cap V_{n+1} \subset V_n$,

for any $n \in \mathbb{N}$, $n > 0$. All the sets V_n are balanced and $V_n \subset U_n$.

Denote by $\mathcal{V}(0) = (V_n)_{n\in\mathbb{N}}$. Evidently, $\mathcal{U}(0)$ is finer than $\mathcal{V}(0)$ from construction. But V_n are also neighbourhoods in the topology σ, so $\mathcal{V}(0)$ is finer than $\mathcal{U}(0)$. It results that $\mathcal{V}(0)$ and $\mathcal{U}(0)$ are equivalent. For $n = 0$ we put $V_n = L$.

Let τ be the translation topology on L.

II) We consider the function $\phi : L \times L \to \mathbb{R}$,

$\phi(x, y) = \inf\{\frac{1}{2^n}; x \in y + V_n\}$.

From the definition of ϕ we have

(3.6) $\phi(x, y) \leq 1/2^n$ if and only if $x \in y + V_n$.

Let $x, y, u, v \in L$ and $\varepsilon > 0$, such as

$\phi(x, u) \leq \varepsilon$, $\phi(u, v) \leq \varepsilon$, $\phi(v, y) \leq \varepsilon$; if $n \in \mathbb{N}$, $1/2^n \leq \varepsilon$ and $x \in u + V_n$, $u \in v + V_n$, $v \in V_n$, then

$$x \in u + V_n \subset v + V_n + V_n \subset y + V_n + V_n + V_n;$$

from (3.5) we deduce that $x \in y + V_{n-1}$ and $\frac{1}{2^n} \leq \varepsilon$ (in fact, $\frac{1}{2^{n-1}} \leq 2\varepsilon$). It follows that $\phi(x, y) \leq 2\varepsilon$. So

(3.7) $\phi(x, u) \leq \varepsilon$, $\phi(u, v) \leq \varepsilon$, $\phi(v, y) \leq \varepsilon$ $\Longrightarrow \phi(x, y) \leq 2\varepsilon$.

III) We define $d : L \times L \to \mathbb{R}$, $d(x, y) = \inf \sum_{i=0}^{p-1} \phi(u_i, u_{i+1})$,

where infimum is considered on all the finite systems of points $(u_i)_{i=\overline{1,p}}$ for which $u_0 = x$ and $u_p = y$.

Then, the double inequality takes place:

(3.8) $\frac{1}{2}\,\phi(x, y) \leq d(x, y) \leq \phi(x, y), \forall x, y \in L$.

Indeed, the right member of the inequality follows from the definition of d (we take $p = 1$).

For the left member of inequality we prove that

(3.9) $\frac{1}{2}\ \phi(x, y) \ \leq \ \sum_{i=0}^{p-1} \phi(u_i, u_{i+1})$ for all $x, y \in L$ and any $p \in \mathbb{N}$.

This results using the mathematical induction method with respect to p. So we consider that (3.9) is valid for all systems having at the most $p - 1$

points attached of any pair of points. We denote by $s = \sum_{i=0}^{p-1} \phi(u_i, u_{i+1})$. If $s \geq 1/2$ then the relation (3.9) is evidently true because $\phi(x, y) \leq 1$.

Now suppose that $s < 1/2$. We denote by t the biggest integer for which $\sum_{i=0}^{t-1} \phi(u_i, u_{i+1}) < s/2$; so $\sum_{i=0}^{t} \phi(u_i, u_{i+1}) \geq s/2$ and $\sum_{i=t+1}^{p-1} \phi(u_i, u_{i+1}) < s/2$. We observe that $t \leq p-1$ and $p-1-t \leq p-1$. We apply the inductive hypothesis for the pairs of points (u_0, u_t) and (u_{t+1}, u_p) and we found

$$\frac{1}{2}\phi(u_0, u_t) \leq \frac{s}{2} \text{ and } \frac{1}{2}\phi(u_{t+1}, u_p) \leq \frac{s}{2}.$$

Since $\phi(u_t, u_{t+1}) \leq s$ we use (3.7) for u_0, u_t, u_{t+1}, u_p and it follows that $\phi(u_0, u_p) \leq s/2$, i.e. (3.9).

It results that d is a metric on L:

i) from (3.8), $d(x, y) = 0 \Leftrightarrow \phi(x, y) = 0$;

from (3.6), $x \in y + V_n$ for all $n \in \mathbb{N}$, so $x \in y + \bigcap_{n\in\mathbb{N}} V_n$.

From hypothesis $\bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} U_n = \{0\}$, so $x = y$.

ii) $d(x, y) = d(y, x)$ because $\phi(x, y) = \phi(y, x)$.

iii) Let be $x, y, z \in L$ arbitrarily. All the systems $(u_i)_{i=0,1,..,p}$ and $(v_j)_{j=0,1,...q}$ for the pairs of points (x, z) and (z, y) are also systems of points for the pair (x, y), so

$d(x, y) \leq d(x, z) + d(z, y)$.

IV) The topology induced by the metric d is equivalent with τ, because the fundamental system of neighbourhoods $x + \mathcal{V}(0)$ is equivalent with the fundamental system of neighbourhoods $\{B_d(x, 1/2^n);\ n \in \mathbb{N}\}$. This follows from the relation:

$x + V_n \subset B_d\left(x, \frac{1}{2^n}\right) \subset x + V_{n+1}$, for any $x \in L$ and any $n \in \mathbb{N}$:

Consider $u \in x + V_n$; from (3.6) we obtain $\phi(u, x) \leq 1/2^n$ and from (3.8) we have $d(u, x) \leq 1/2^n$, so $u \in B_d\left(x, \frac{1}{2^n}\right)$.

Now, if $u \in B_d\left(x, \frac{1}{2^n}\right)$, then $d(u, x) \leq 1/2^n$ and from (3.8) we deduce that $\frac{1}{2}\ \phi(u, x) \leq 1/2^n$, so (see (3.6)) $u \in x + V_{n+1}$. ■

Remark 3.2.

(i) The metric d constructed in the proof of Theorem 3.3 satisfies the following condition of "semi-invariance" to translations:

$d(x + z, y + z) \leq d(x, y)$, for any $x, y, z \in L$, as the function ϕ defined above fulfils the same inequality: if $x \in y + V_k$, then for any $z \in L$, one has $x + z \in y + z + V_k$, thus

$$\phi(x + z, y + z) \leq \phi(x, y).$$

(ii) The family

$$\left\{x + B_d\left(0, \frac{1}{2^k}\right)\right\}_{k\in\mathbb{N}^*}$$

also constitutes a fundamental system of neighbourhoods for x on the topology τ. One can notice that, if d is any metric on L, the sets $x + B(0, \varepsilon)$ and $B(x, \varepsilon)$ are not necessarily comparable. But if d is "semi-invariant" to translations, from the inequality $d(x + u, x) \leq d(u, 0)$ applied to the elements $u \in B(0, \varepsilon)$ one can find that

$$x + B(0, \varepsilon) \subset B(x, \varepsilon).$$

(iii) If we remove the hypothesis of T1 separation from the Theorem 3.3, then the almost linear topology will be only semi-metrizable.

In the following, we will apply the Theorem 3.3 for the topologies τ_H^-, τ_H^+, τ_V^- and τ_P , in order to find the conditions for semi-metrizability. We will design as $\mathcal{D}(X)$ the family of nonvoid open sets of linear normed space X. Evidently, $(\mathcal{P}b(X), \tau_H)$ is semi-metrizable; we have also a metrizability result for the topology τ_P ([7]):

$(\mathcal{C}l(X), \tau_{P(d)})$ is metrizable if and only if (X, d) is totally bounded.

For the translations of the above hypertopologies we have:

Corollary 3.4.

(i) The translated of topology τ_H^- is semi-metrizable on $\mathcal{P}b(X)$.

(ii) The translated of the topology τ_H^+ is semi-metrizable on $\mathcal{D}(X)$.

(iii) The translated of the topology τ_V^- is semi-metrizable on $\mathcal{P}(X)$.

(iv) The translated of the topology τ_P is semi-metrizable on $\mathcal{D}(X)$.

Proof. (i) A fundamental system $\mathcal{V}$ of neighbourhoods of the origin in τ_H^- on $\mathcal{P}b(X)$ will be formed by the sets having the form

$$V_H^-(0; B, \varepsilon) = \{A \in \mathcal{P}b(X);\ B \subset S_\varepsilon(A)\}$$

with $B \in \mathcal{P}b(X)$ containing the origin and $\varepsilon > 0$.

Let be the family $\mathcal{V}'$ of the neighbourhoods for the origin of τ_H^- having the type

$$V_H^-(0; B(0, p), \frac{1}{n})$$

with $p, n \in \mathbb{N}^*$. The system $\mathcal{V} \subset \mathcal{V}'$ is countable and defines the same topology as $\mathcal{V}$, as: B is bounded, then there exists $p \in \mathbb{N}^*$ such that $B \subset B(0,p)$; we take $n = [6/\varepsilon] + 1$ (where $[\alpha]$ is the greatest integer less than the real number α). Then for any $A \in \mathcal{P}b(X)$ having the property: $B(0,p) \subset S_{1/n}(A)$, the following inclusion is also valid:

$$B \subset S_\varepsilon(A) \ \left(\text{so } V_H^-(0; B(0,p), \frac{1}{n}) \subset V_H^-(0; B, \varepsilon) \right).$$

(ii) For the topology τ_H^+ on $\mathcal{D}(X)$, a fundamental system $\mathcal{V}$ of neighbourhoods of the origin is formed by sets of the type:

$$V_H^+(0; B, \varepsilon) = \{A \in \mathcal{D}(X);\ A \subset S_\varepsilon(B)\},$$

where $B \in \mathcal{D}(X)$ with $0 \in B$ and $\varepsilon > 0$.

For such a set B, there exists $p \in \mathbb{N}^*$ such that $S(0, 1/p) \subset B$, and if $n = [6/\varepsilon] + 1$, then for any $A \in \mathcal{D}(X)$ such that $A \subset S_{1/n}(S(0, 1/p))$, it follows that $A \in S_\varepsilon(B)$. From here, one can deduce that the family $\mathcal{V}' \subset \mathcal{V}$ formed by the sets of type

$\{A \in \mathcal{D}(X); A \subset S_{1/n}(S(0, 1/p))\}$ with $n, p \in \mathbb{N}^*$ also constitutes a fundamental system of neighbourhoods equivalent to $\mathcal{V}$.

(iii) Let be $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$, with $n \in \mathbb{N}^*$ and $U_{V^-} = S(0, \varepsilon_1)^- \cap S(0, \varepsilon_2)^- \cap ... \cap S(0, \varepsilon_n)^- = S(0, \varepsilon)^-$, where $\varepsilon = \min\{\varepsilon_j;\ j = \overline{1,n}\}$ is a fundamental neighbourhood of the origin in τ_V^-.

By choosing for every $\varepsilon > 0$ a $n \in \mathbb{N}^*$ i.e., $n = [6/\varepsilon]$, one can obtain

$$S(0, 1/n)^- \subset S(0, \varepsilon)^-,$$

that is, the family

$$\mathcal{V}' = \{S(0, 1/n)^-; n \in \mathbb{N}^*\}$$

is contained in $\mathcal{V} = \{S(0, \varepsilon)^-; \varepsilon > 0\}$; also $\mathcal{V}'$ represents a fundamental system of neighbourhoods for the origin in τ_V^-.

(iv) This follows from (ii) and (iii), as $\tau_P = \tau_V^- \vee \tau_H^+$.■

Now we offer a sufficient condition for a metric d, in order to induce a topology which is almost linear.

Theorem 3.5. *Let $(L, +, .)$ be an a.l.s. and d a semi-metric on L, satisfying the properties:*

$I)$ $d(a + c, b + c) \le d(a, b)$, for any $a, b, c \in L$,

$II)$ $d(\lambda a, \lambda b) \le |\lambda|\, d(a, b)$, for any $a, b, \in L$, $\lambda \in \Gamma$

$III)$ $d(\lambda a, \mu a) \le |\lambda - \mu| \cdot d(a, 0)$, for any $\lambda, \mu \in \Gamma$ and any $a \in L$.

Then the topology induced on L by the semi-metric d is almost linear.

Proof. Let $x_0, y_0 \in L$ and $(x_n)_{n \in \mathbb{N}^*}, (y_n)_{n \in \mathbb{N}^*} \subset L$, with $d(x_n, x_0) \to 0$, $d(y_n, y_0) \to 0$.

We have the inequalities:

$d(x_n + y_n, x_0 + y_0) \le d(x_n + y_n, x_0 + y_n) + d(x_0 + y_n, x_0 + y_0) \le d(x_n, x_0) + d(y_n, y_0)$, so $d(x_n + y_n, x_0 + y_0) \to 0$.

Now, let $\lambda_n, \lambda \in \Gamma$, $x_n, x_0 \in L$ with $d(x_n, x_0) \to 0$ and $\lambda_n \to \lambda$ in Γ.

In this case,

$d(\lambda_n x_n, \lambda_0 x_0) \le d(\lambda_n x_n, \lambda_n x_0) + d(\lambda_n x_0, \lambda_0 x_0) \le |\lambda_n| \cdot d(x_n, x_0) + |\lambda_n - \lambda_0| \cdot d(x_0, 0)$, hence $d(\lambda_n x_n, \lambda_0 x_0) \to 0$. ■

For discuss the Theorem 3.5 we need first to give some properties of H_d. The below proposition describes the behavior of the Hausdorff "distance" with respect to latticeal and algebraic operations :

Proposition 3.6. *Let X be a linear normed space over the scalar field $\Gamma$and H the Hausdorff extended semi-metric on $\mathcal{P}(X)$. Then the following assertions hold:*

(i) $H(A \cup C, B \cup C) =$
$= \max\{\sup_{a \in A} \min\{d(a, B), d(a, C)\},$
$\sup_{b \in B} \min\{d(b, A), d(b, C)\}\ \}$
and
$H(A \cup C, B \cup C) \le H(A, B)$,
for all $A, B, C \in \mathcal{P}(X)$.

(ii) $H(A + C\ ,\ B + C) \le H(A\ ,\ B)$,
for all $A, B, C \in \mathcal{P}(X)$;

(iii) $H(A, B) \le H(A - B, 0)$,
for all $A, B \in \mathcal{P}(X)$;

i*(iv)* $|H(A, 0) - H(B, 0)| \le H(A, B)$,
for all $A, B \in \mathcal{P}(X)$;

(v) $H(\lambda A, \mu A) \le |\lambda - \mu| \cdot H(A, 0)$,
for all $\lambda, \mu \in \Gamma$ and $A \in \mathcal{P}(X)$;

(vi) $H(\lambda A, \lambda B) = |\lambda| \cdot H(A, B)$,
for all $\lambda \in \Gamma$ and $A, B \in \mathcal{P}(X)$.

Proof. The conditions (i), (ii) and (vi) are prove in [2], Proposition 4.5.

(iii) We have

$e(A, B) \le \sup_{a \in A} \sup_{b \in B} \|a - b\| = e(A - B, 0)$

and symmetrically we found $e(B, A) \le e(A - B, 0)$.

But $e(0, A - B) = \inf\{\|a - b\|\, , a \in A, b \in B\}$.

It follows that

$$H(A - B, 0) = e(A - B, 0) \ge$$

$$\geq \max\{e(A,B), e(B,A)\} = H(A,B).$$

(iv) If $a \in A$ and $b \in B$, first we take the infimum for all $a \in A$ in the formula $\|a\| - \|b\| \leq \|a - b\|$; second we take the supremum for all $b \in B$ and we obtain$H(A,0) - H(B,0) \leq e(A,B)$. Changing A for B, it implies that$H(B,0) - H(A,0) \leq e(B,A)$ and it results the inequality from our assertion.

(v) In order to calculate $H(\lambda A, \mu A)$ we estimate $e(\lambda A, \mu A)$. So

$$e(\lambda A, \mu A) \leq \sup_{a \in A} \|\lambda a - \mu a\| =$$

$$= |\lambda - \mu| \cdot \sup_{a \in A} \|a\| = |\lambda - \mu| \cdot H(A, 0).$$

Evidently, $e(\mu A, \lambda A) \leq |\lambda - \mu| \cdot H(A,0)$, so we obtain the desired inequality.■

Remark 3.3. The conditions of Theorem 3.5 are consistent. As an example of distance which induces a topology almost linear on an a.l.t.s., the conditions I, II, III from Theorem 3.5 are fulfilled by the semimetric H on the family $\mathcal{P}b(X)$: see Proposition 3.6.

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