Asymptotics of Solution and Finite Difference Scheme to a Nonlinear Integro-Differential Equations Associated with the Penetration of a Magnetic Field into a Substance

TEMUR JANGVELADZE
Ilia Chavchavadze State University
Department of Physics and Mathematics
32 Chavchavadze Ave., 0179 Tbilisi
Ivane Javakhishvili Tbilisi State University
Department of Mathematics
2 University St., 0186 Tbilisi
GEORGIA
tjangv@yahoo.com

ZURAB KIGURADZE
Ilia Chavchavadze State University
Department of Physics and Mathematics
32 Chavchavadze Ave., 0179 Tbilisi
Ivane Javakhishvili Tbilisi State University
Department of Computer Science
2 University St., 0186 Tbilisi
GEORGIA
zkigur@yahoo.com

Abstract: Asymptotics of solution and finite difference approximation of the nonlinear integro-differential equations associated with the penetration of a magnetic field into a substance is studied. Asymptotic properties of solutions for the initial-boundary value problem with homogeneous as well as nonhomogeneous Dirichlet boundary conditions are considered. The corresponding finite difference scheme is studied. The convergence of this scheme is proven. Numerical experiments are carried out.

Key–Words: Nonlinear integro-differential equation, asymptotic behavior, finite difference scheme

1 Introduction

Many practical problems are described by integro-differential models (see, for example, [1], [4], [5], [8], [10], [19], [22], [29]). One of such model arise in the study of electromagnetic field penetration into a substance. As it is known this process is modeled by Maxwell's system of partial differential equations (see, for example, [15]). If the coefficients of thermal heat capacity and electro-conductivity of the substance depend on temperature, then the Maxwell's system can be reduced to the integro-differential model, one-dimensional scalar analogue of which has the following form [9]:

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial W}{\partial x}\right)^2 d\tau\right)\frac{\partial W}{\partial x}\right], \quad (1)$$

where $a = a(S)$ is a given function defined for $S \in [0, \infty)$.

Principal characteristic peculiarity of the equation (1) is connected with the appearance in the coefficient with derivative of higher order nonlinear term depended on the integral in time.

Note that the integro-differential equation of type (1) is complex and only special cases were investigated (see, for example, [6], [7], [9], [12]-[14], [16], [17]).

In some restrictions by modeling the same process in [16] integro-differential model is received, one-dimensional scalar analogue of which has the following form

$$\frac{\partial W}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial W}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 W}{\partial x^2}. \quad (2)$$

The existence and uniqueness of the solutions of the initial-boundary value problems for the equations of type (1) and (2) are studied in [6], [7], [9], [16], [17] and in a number of other works as well. The existence theorems, proved in [6], [7], [9], are based on Galerkin method and compactness arguments as in [18], [27] for nonlinear problems.

Asymptotic behavior of solution as $t \to \infty$ and numerical solution of initial-boundary value problem for equation (2) in the case $a(S) = 1 + S$ is given in [14]. Note That in this work asymptotic behavior of solution of initial-boundary value problem with non-homogeneous boundary condition on part of lateral boundary has a power-like form.

Many authors study the finite difference approximation for a integro-differential models (see, for example, [2], [3], [11], [20], [21], [25], [26], [28], [30]).

In the present work we strengthening result given in [14] for the solution of first initial-boundary value problem for equation (2). We also discuss finite difference scheme in the case $a(S) = 1+S$ for the equation (1).

The rest of the paper is organized as follows. In the second section we will state problem and consider large time behavior of solutions of first initial-boundary value problems for equations (1) and (2). In the third section finite difference scheme for equation (1) is discussed. In the fourth section we conclude with some remarks on numerical implementations. In the fifth part of this note some conclusions are given.

2 Asymptotic behavior of solutions as $t \to \infty$

In the area $Q = (0,1) \times (0,\infty)$, let us consider following initial-boundary value problem:

$$\frac{\partial W}{\partial t} - \frac{\partial}{\partial x}\left[a(S)\frac{\partial W}{\partial x}\right] = 0, \tag{3}$$

$$W(0,t) = W(1,t) = 0, \tag{4}$$

$$W(x,0) = W_0(x), \tag{5}$$

where

$$S(x,t) = \int_0^t \left(\frac{\partial W}{\partial x}\right)^2 d\tau, \tag{6}$$

or

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial W}{\partial x}\right)^2 dx d\tau, \tag{7}$$

$W_0 = W_0(x)$ is a given function.

Asymptotic behavior of solution as $t \to \infty$ of initial-boundary value problem (3)-(6) is investigated in [12]. One of main result of investigations made in this work can be stated as follows.

Theorem 1 *If $a(S) = (1+S)^p$, $0 < p \le 1$; $W_0 \in H^2(0,1) \cap H_0^1(0,1)$, then the solution of the problem (3)-(6) satisfies the following estimate*

$$\left|\frac{\partial W(x,t)}{\partial x}\right| + \left|\frac{\partial W(x,t)}{\partial t}\right| \le C\exp\left(-\frac{t}{2}\right).$$

Here and below, H^k and H_0^k denote usual Sobolev spaces, while C denotes positive constant independent of t.

If instead boundary conditions (4) following non-homogeneous boundary condition on part of lateral boundary is considered

$$W(0,t) = 0, \quad W(1,t) = \mu, \tag{8}$$

then we derive again one of main result of [12], which can be formulated as a following statement.

Theorem 2 *If $a(S) = (1+S)^p$, $0 < p \le 1$; $W_0 \in H^2(0,1)$, $W_0(0) = 0$, $W_0(1) = \mu$, $\mu = Const > 0$, then the solution of the problem (3),(5),(6),(8) satisfies the following estimates:*

$$\left|\frac{\partial W(x,t)}{\partial x} - \mu\right| \le Ct^{-p-1}, \quad \left|\frac{\partial W(x,t)}{\partial t}\right| \le Ct^{-1}.$$

Now consider problem (3),(5),(7),(8). Let us introduce the notation

$$U(x,t) = W(x,t) - \mu x. \tag{9}$$

So, instead (3),(5),(7),(8) we have following problem:

$$\frac{\partial U}{\partial t} = a(S)\frac{\partial^2 U}{\partial x^2}, \quad (x,t) \in Q, \tag{10}$$

$$U(0,t) = U(1,t) = 0, \quad t \ge 0, \tag{11}$$

$$U(x,0) = W_0(x) - \mu x, \quad x \in [0,1], \tag{12}$$

where

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x} + \mu\right)^2 dx d\tau.$$

Theorem 3 *If $a(S) = (1+S)^p$, $0 < p \le 1$; $W_0 \in H_0^1(0,1)$, then the solution of the problem (3),(5),(7),(8) satisfies the following estimate*

$$\|W - \mu x\| + \left\|\frac{\partial W}{\partial x} - \mu\right\| \le C\exp\left(-\frac{t}{2}\right).$$

Proof. Let us multiply (10) by U and integrate over $(0,1)$. After integrating by parts and using the boundary conditions (11) we get

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \int_0^1 (1+S)^p \left(\frac{\partial U}{\partial x}\right)^2 dx = 0.$$

Since $(1+S)^p \ge 1$ we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 \le 0. \tag{13}$$

Using Poincare-Friedrichs inequality from (13) we obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \|U\|^2 \le 0. \tag{14}$$

Now multiply (10) by $\dfrac{\partial^2 U}{\partial x^2}$ and integrate over $(0,1)$. Using again integration by parts and the boundary conditions (11) we get

$$\left.\frac{\partial U}{\partial t}\frac{\partial U}{\partial x}\right|_0^1 - \int_0^1 \frac{\partial^2 U}{\partial x \partial t}\frac{\partial U}{\partial x}dx = \int_0^1 (1+S)^p \left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx,$$

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2 + (1+S)^p\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 = 0, \tag{15}$$

or

$$\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2 \le 0. \tag{16}$$

From (13),(14) and (16) we find

$$\frac{d}{dt}\left[\exp(t)\left(\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2\right)\right] \le 0.$$

This inequality using initial condition (12) immediately proves Theorem 3.

Note that Theorem 3 gives exponential stabilization of the solution of the problem (3),(5),(7),(8) in the norm of the space $H^1(0,1)$. Let us show that the stabilization is also achieved in the norm of the space $C^1(0,1)$. In particular, let us show that the following statement takes place.

Theorem 4 *If $a(S) = (1+S)^p$, $0 < p \le 1$; $W_0 \in H^4(0,1) \cap \overset{\circ}{H}{}^1_0(0,1)$, then the solution of the problem (3),(5),(7),(8) satisfies the following estimates:*

$$\left|\frac{\partial W(x,t)}{\partial x} - \mu\right| \le C\exp\left(-\frac{\alpha t}{2}\right),$$

$$\left|\frac{\partial W(x,t)}{\partial t}\right| \le C\exp\left(-\frac{\beta t}{2}\right),$$

where $\alpha = Const$, $\beta = Const$, $0 < \beta < \alpha < 1$.

To this end we need following auxiliary result.

Theorem 5 *If $a(S) = (1+S)^p$, $0 < p \le 1$; $W_0 \in H^3(0,1) \cap \overset{\circ}{H}{}^1_0(0,1)$, then for the solution of the problem (3),(5),(7),(8) the following estimate holds*

$$\left\|\frac{\partial U}{\partial t}\right\| \le C\exp\left(-\frac{\alpha t}{2}\right).$$

Proof. Let us differentiate (10) with respect to t,

$$\begin{gathered}\frac{\partial^2 U}{\partial t^2} = (1+S)^p\frac{\partial^3 U}{\partial x^2 \partial t} + \\ p(1+S)^{p-1}\left[\int_0^1\left(\frac{\partial U}{\partial x} + \mu\right)^2 dx\right]\frac{\partial^2 U}{\partial x^2}.\end{gathered} \tag{17}$$

Multiply (17) by $\dfrac{\partial U}{\partial t}$ and integrate over $(0,1)$. Using the boundary conditions (11) we deduce

$$\begin{gathered}\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx + \\ 2(1+S)\int_0^1\left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx = \\ -2p(1+S)^{p-1}\left[\int_0^1\left(\frac{\partial U}{\partial x} + \mu\right)^2 dx\right]\times \\ \times\int_0^1\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t}dx.\end{gathered} \tag{18}$$

Let us estimate the right hand side of the equality (18).

$$\begin{gathered}-2p(1+S)^{p-1}\left[\int_0^1\left(\frac{\partial U}{\partial x} + \mu\right)^2 dx\right]\times \\ \int_0^1\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t}dx = -2p\int_0^1\left\{(1+S)^{p/2-1}\times\right. \\ \left.\left[\int_0^1\left(\frac{\partial U}{\partial x} + \mu\right)^2 dx\right]\frac{\partial U}{\partial x}\right\}\times \\ \left\{(1+S)^{p/2}\frac{\partial^2 U}{\partial x \partial t}\right\}dx.\end{gathered} \tag{19}$$

From this, using the Schwarz's inequality we get

$$\begin{gathered}-2p(1+S)^{p-1}\left[\int_0^1\left(\frac{\partial U}{\partial x} + \mu\right)^2 dx\right]\times \\ \int_0^1\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t}dx \le (2-\alpha)(1+S)^p\times \\ \int_0^1\left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx + \frac{p^2}{2-\alpha}(1+S)^{p-2}\times \\ \left[\int_0^1\left(\frac{\partial U}{\partial x} + \mu\right)^2 dx\right]^2\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx \le\end{gathered}$$

$$\begin{gathered}\le (2-\alpha)(1+S)^p\int_0^1\left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx + \\ +\frac{8p^2}{2-\alpha}(1+S)^{p-2}\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]^3 + \\ +\frac{8p^2\mu^4}{2-\alpha}(1+S)^{p-2}\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx.\end{gathered} \tag{20}$$

Combining (18)-(20) we have

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \alpha(1+S)^p \int_0^1 \left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx \le \frac{8p^2}{2-\alpha}(1+S)^{p-2}\left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx\right]^3 + \frac{8p^2\mu^4}{2-\alpha}(1+S)^{p-2}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx.$$

Using Poincare-Friedrichs inequality, notation (9), Theorem 3, restrictions on p and nonnegativity of $S(t)$ we arrive at

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \alpha \int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \le C\exp(-t).$$

After multiplying by $\exp(\alpha t)$, the last inequality gives

$$\frac{d}{dt}\left(\exp(\alpha t)\left\|\frac{\partial U}{\partial t}\right\|^2\right) \le C\exp(-(1-\alpha)t).$$

Therefore,

$$\exp(\alpha t)\left\|\frac{\partial U}{\partial t}\right\|^2 \le C\int_0^t \exp(-(1-\alpha)\tau)d\tau \le \frac{C}{1-\alpha},$$

or

$$\left\|\frac{\partial U}{\partial t}\right\| \le C\exp\left(-\frac{\alpha t}{2}\right).$$

So, Theorem 5 is proven.

Proof of Theorem 4. Let us estimate $\frac{\partial^2 U}{\partial x^2}$ in the norm of the space $L_1(0,1)$. From (10) we have

$$\frac{\partial^2 U}{\partial x^2} = (1+S)^{-p}\frac{\partial U}{\partial t}. \tag{21}$$

Integrating (21) on $(0,1)$ and using Schwarz's inequality we get

$$\int_0^1 \left|\frac{\partial^2 U}{\partial x^2}\right| dx = \int_0^1 \left|(1+S)^{-p}\frac{\partial U}{\partial t}\right| dx \le \left[\int_0^1 (1+S)^{-2p}dx\right]^{1/2}\left[\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx\right]^{1/2}.$$

Applying Theorem 5 and taking into account the nonnegativity of $S(t)$ we derive

$$\int_0^1 \left|\frac{\partial^2 U}{\partial x^2}\right| dx \le C\exp\left(-\frac{\alpha t}{2}\right).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y}dy + \int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi dy$$

and the boundary conditions (11) it follows that

$$\left|\frac{\partial U(x,t)}{\partial x}\right| = \left|\int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi dy\right| \le \int_0^1 \left|\frac{\partial^2 U(y,t)}{\partial y^2}\right| dy \le C\exp\left(-\frac{\alpha t}{2}\right).$$

So, for the solution of the initial-boundary value problem (3),(5),(7),(8) we have

$$\left|\frac{\partial W(x,t)}{\partial x} - \mu\right| \le C\exp\left(-\frac{\alpha t}{2}\right).$$

Now let us estimate $\frac{\partial U}{\partial t}$ in the norm of the space $C^1(0,1)$. Let us multiply (10) by $\frac{\partial^3 U}{\partial x^2 \partial t}$ and integrate over $(0,1)$. Using integration by parts we get

$$\left.\frac{\partial U}{\partial t}\frac{\partial^2 U}{\partial x \partial t}\right|_0^1 - \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 = (1+S)^p \int_0^1 \frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2 \partial t}dx. \tag{22}$$

Taking into account boundary conditions (11) we arrive at

$$\frac{(1+S)^p}{2}\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 = 0,$$

or

$$\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 \le 0. \tag{23}$$

Note that from (22) we have

$$\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 \le \frac{(1+S)^p}{2}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{(1+S)^p}{2}\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2. \tag{24}$$

Now multiply (17) by $\frac{\partial^3 U}{\partial x^2 \partial t}$ scalarly and integrate the left hand side by parts

$$\left.\frac{\partial^2 U}{\partial t^2}\frac{\partial^2 U}{\partial x \partial t}\right|_0^1 - \int_0^1 \frac{\partial^3 U}{\partial x \partial t^2}\frac{\partial^2 U}{\partial x \partial t}dx = (1+S)^p\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 +$$

$$p(1+S)^{p-1}\left[\int_0^1\left(\frac{\partial U}{\partial x}+\mu\right)^2dx\right]\int_0^1\frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2\partial t}dx.$$

Taking into account the boundary conditions (11) we have

$$\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+2(1+S)\left\|\frac{\partial^3 U}{\partial x^2\partial t}\right\|^2=$$

$$-2p(1+S)^{p-1}\left[\int_0^1\left(\frac{\partial U}{\partial x}+\mu\right)^2dx\right]\int_0^1\frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2\partial t}dx.$$

We estimate the right hand side in a similar fashion to (19),(20). It is easy to see that

$$\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+(1+S)^p\left\|\frac{\partial^3 U}{\partial x^2\partial t}\right\|^2\le$$
$$p^2(1+S)^{p-2}\left[\int_0^1\left(\frac{\partial U}{\partial x}+\mu\right)^2dx\right]^2\times$$
$$\int_0^1\left(\frac{\partial^2 U}{\partial x^2}\right)^2dx\le 8p^2(1+S)^{p-2}\times$$
$$\left\{\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2dx\right]^2+\mu^4\right\}\int_0^1\left(\frac{\partial^2 U}{\partial x^2}\right)^2dx.$$

Using equation (21) and Theorems 3 and 5 we have

$$\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+ (1+S)^p\left\|\frac{\partial^3 U}{\partial x^2\partial t}\right\|^2\le C\exp(-\alpha t). \tag{25}$$

Combining (13)-(15), (23), (24) and (25) we get

$$\|U\|^2+\frac{d}{dt}\|U\|^2+\left\|\frac{\partial U}{\partial x}\right\|^2+\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2+$$
$$2(1+S)^p\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+$$
$$\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+$$
$$(1+S)^p\left\|\frac{\partial^3 U}{\partial x^2\partial t}\right\|^2\le\frac{(1+S)^p}{2}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+$$
$$\frac{(1+S)^p}{2}\left\|\frac{\partial^3 U}{\partial x^2\partial t}\right\|^2+C\exp(-\alpha t).$$

From this, keeping in mind the nonnegativity of $S(t)$ and inequalities $0<\beta<\alpha<1$, we deduce

$$\beta\|U\|^2+\frac{d}{dt}\|U\|^2+\beta\left\|\frac{\partial U}{\partial x}\right\|^2+\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2+$$
$$\beta\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+$$
$$\beta\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\le C\exp(-\alpha t).$$

After multiplying by the function $\exp(\beta t)$ we get

$$\frac{d}{dt}\left[\exp(\beta t)\left(\|U\|^2+\left\|\frac{\partial U}{\partial x}\right\|^2+\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\right.\right.$$
$$\left.\left.\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\right)\right]\le C\exp(-(\alpha-\beta)t).$$

Since $\beta<\alpha$ we get

$$\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\le C\exp(-\beta t).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial t}=\int_0^1\frac{\partial U(y,t)}{\partial t}dy+\int_0^1\int_y^x\frac{\partial^2 U(\xi,t)}{\partial t\partial\xi}d\xi dy$$

and Theorem 5, we obtain

$$\left|\frac{\partial W(x,t)}{\partial t}\right|=\left|\frac{\partial U(x,t)}{\partial t}\right|\le C\exp\left(-\frac{\alpha t}{2}\right)+$$
$$C\exp\left(-\frac{\beta t}{2}\right)\le C\exp\left(-\frac{\beta t}{2}\right).$$

Thus Theorem 4 has been proven.

3 Finite difference scheme

In order to describe the finite difference method for problem (3)-(6) (case $a(S)=1+S$), on $Q_T=(0,1)\times(0,T)$, where T is a positive constant, we introduce a net whose mesh point are denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,...,M$; $j=0,1,...,N$ with $h=\frac{1}{M}$, $\tau=\frac{T}{N}$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is designed by w_i^j and the exact solution to the problem (3)-(6) by W_i^j. We will use the following known notations [24]:

$$w_{x,i}^{j+1}=\frac{w_{i+1}^{j+1}-w_i^{j+1}}{h},\quad w_{\bar{x},i}^{j+1}=\frac{w_i^{j+1}-w_{i-1}^{j+1}}{h}.$$

$$w_{t,i}^{j}=\frac{w_i^{j+1}-w_i^{j}}{\tau},\quad w_{\bar{t},i}^{j}=w_{t,i}^{j-1}=\frac{w_i^{j}-w_i^{j-1}}{\tau}.$$

Let us correspond to the problem (3)-(6) with nonzero right part f in (3) the following difference scheme:

$$\frac{w_i^{j+1}-w_i^j}{\tau}-\left\{\left[1+\tau\sum_{k=1}^{j+1}(w_{\bar{x},i}^k)^2\right]w_{\bar{x},i}^{j+1}\right\}_x=f_i^j, \tag{26}$$

$$i=1,2,...,M-1;\;\; j=0,1,...,N-1,$$

$$w_0^j=w_M^j=0,\;\; j=0,1,...,N, \tag{27}$$

$$w_i^0=W_{0,i},\;\; i=0,1,...,M. \tag{28}$$

Introduce inner products and norms:

$$(u,v)_h=\sum_{i=1}^{M-1}u_iv_ih,\quad (u,v]_h=\sum_{i=1}^{M}u_iv_ih,$$

$$\|u\|_h=(u,u)_h^{1/2},\quad \|u]|_h=(u,u]_h^{1/2}.$$

Multiplying (26) scalarly by $w^{j+1}=(w_1^{j+1}, w_2^{j+1},\ldots,w_{M-1}^{j+1})$, using the discrete analogue of the integration by parts, after simple transformations it is not difficult to get

$$\|w^n\|_h^2+\tau\sum_{j=1}^{n}\|w_{\bar{x}}^j]|_h^2\le C,\quad n=1,2,...,N. \tag{29}$$

Here and below C is a positive constant independent from τ and h.

The a-priori estimate (29) guarantees the stability of the scheme (26)-(28).

The main result of this section is the following statement.

Theorem 6 *If the problem (3)-(6) has a sufficiently smooth solution $W=W(x,t)$, then the solution $w^j=(w_1^j,w_2^j,\ldots,w_{M-1}^j)$, of the finite difference scheme (26)-(28) tends to the $W^j=(W_1^j,W_2^j,\ldots,W_{M-1}^j)$, as $\tau\to 0,\;\; h\to 0$ and the following estimate is true*

$$\|w^j-W^j\|_h\le C(\tau+h),\quad j=1,2,\ldots,N. \tag{30}$$

Proof. For the exact solution $W=W(x,t)$ of the problem (3)-(6) we have

$$\frac{W_i^{j+1}-W_i^j}{\tau}-\left\{\left[1+\tau\sum_{k=1}^{j+1}(W_{\bar{x},i}^k)^2\right]W_{\bar{x},i}^{j+1}\right\}_x=f_i^j-\psi_i^j, \tag{31}$$

$$W_0^j=W_M^j=0, \tag{32}$$

$$W_i^0=W_{0,i}, \tag{33}$$

where

$$\psi_i^j=O(\tau+h).$$

Solving (26)-(28) instead of the problem (3)-(6) we have the error $z_i^j=w_i^j-W_i^j$. From (26)-(28) and (31)-(33) we get

$$\frac{z_i^{j+1}-z_i^j}{\tau}-\left\{\left[1+\tau\sum_{k=1}^{j+1}(w_{\bar{x},i}^k)^2\right]w_{\bar{x},i}^{j+1}-\left[1+\tau\sum_{k=1}^{j+1}(W_{\bar{x},i}^{j+1})^2\right]W_{\bar{x},i}^{j+1}\right\}_x=\psi_i^j, \tag{34}$$

$$z_0^j=z_M^j=0, \tag{35}$$

$$z_i^0=0. \tag{36}$$

Multiplying (34) scalarly by $z^{j+1}=(z_1^{j+1}, z_2^{j+1}, \ldots, z_{M-1}^{j+1})$, using (35), and the discrete analogue of integration by parts we get

$$\begin{gathered}\|z^{j+1}\|^2-(z^{j+1},z^j)+\\ \tau h\sum_{i=1}^{M}\left\{\left[1+\tau\sum_{k=1}^{j+1}(w_{\bar{x},i}^k)^2\right]w_{\bar{x},i}^{j+1}-\left[1+\tau\sum_{k=1}^{j+1}(W_{\bar{x},i}^k)^2\right]W_{\bar{x},i}^{j+1}\right\}z_{\bar{x},i}^{j+1}=\\ \tau(\psi^j,z^{j+1}).\end{gathered} \tag{37}$$

Taking into account the relations:

$$(z^{j+1},z^j)=\frac{1}{2}\|z^{j+1}\|^2+\frac{1}{2}\|z^j\|^2-\frac{1}{2}\|z^{j+1}-z^j\|^2,$$

$$\left[(w_{\bar{x},i}^k)^2w_{\bar{x},i}^{j+1}-(W_{\bar{x},i}^k)^2W_{\bar{x},i}^{j+1}\right](w_{\bar{x},i}^{j+1}-W_{\bar{x},i}^{j+1})\ge \frac{1}{2}\left[(w_{\bar{x},i}^k)^2-(W_{\bar{x},i}^k)^2\right]\left[(w_{\bar{x},i}^{j+1})^2-(W_{\bar{x},i}^{j+1})^2\right],$$

from (37) we have

$$\begin{gathered}\|z^{j+1}\|^2+\frac{1}{2}\|z^{j+1}-z^j\|^2-\frac{1}{2}\|z^{j+1}\|^2-\\ \frac{1}{2}\|z^j\|^2+\tau\|z_{\bar{x}}^{j+1}\|^2+\\ \frac{\tau^2h}{2}\sum_{i=1}^{M-1}\sum_{k=1}^{j+1}\left[(w_{\bar{x},i}^k)^2-(W_{\bar{x},i}^k)^2\right]\times\\ \left[(w_{\bar{x},i}^{j+1})^2-(W_{\bar{x},i}^{j+1})^2\right]\le\\ \frac{\tau}{2\varepsilon}\|\psi^j\|^2+2\varepsilon\tau\|z^{j+1}\|^2,\\ j=0,1,...,N-1.\end{gathered} \tag{38}$$

Here ε is an arbitrary positive constant.

Introduce the notations

$$\xi_i^j = \tau \sum_{k=1}^{j} \left[(w_{\bar{x},i}^k)^2 - (W_{\bar{x},i}^k)^2 \right],$$

then

$$\xi_{i,t}^j = \left[(w_{\bar{x},i}^{j+1})^2 - (W_{\bar{x},i}^{j+1})^2 \right].$$

So, from (38) we get

$$\begin{gathered} \|z^{j+1}\|^2 - \|z^j\|^2 + \tau^2 \|z_{\bar{t}}^{j+1}\|^2 + \\ \tau \|z_{\bar{x}}^{j+1}\|^2 + \tau^2 \|\xi_t^j\|^2 + \tau(\xi^j, \xi_t^j) \leq \\ \frac{\tau}{\varepsilon} \|\psi^j\|^2 + 4\varepsilon\tau \|z^{j+1}\|^2. \end{gathered} \tag{39}$$

Using (36) the discrete analogue of Poincare's inequality [24]

$$\|z^{j+1}\|^2 \leq \frac{1}{8} \|z_{\bar{x}}^{j+1}]|^2$$

and the relation

$$\tau(\xi^j, \xi_t^j) = \frac{1}{2} \|\xi^{j+1}\|^2 - \frac{1}{2} \|\xi^j\|^2 - \frac{\tau^2}{2} \|\xi_t^j\|^2,$$

we have from (39)

$$\begin{gathered} \|z^n\|^2 + \tau^2 \sum_{j=0}^{n-1} \|z_{\bar{t}}^{j+1}\|^2 + \frac{\tau}{2} \sum_{j=0}^{n-1} \|z_{\bar{x}}^{j+1}]|^2 + \\ \frac{\tau^2}{2} \sum_{j=0}^{n-1} \|\xi_t^j\|^2 + \frac{1}{2} \|\xi^n\|^2 \leq C \sum_{j=0}^{n-1} \|\psi^j\|^2 \tau, \\ n = 1, 2, ..., N. \end{gathered} \tag{40}$$

From (40) we get (30) and thus Theorem 6 has been proven.

Note that analogical theorem is hold for problem (3),(5),(7),(8) (see, [14]). Note also, that according to the scheme of proving convergence theorem, the uniqueness of the solution of the scheme (26)-(28) can be proven. In particular, assuming existence of two solutions w and $\bar{w}$ of the scheme (26)-(28), for the difference $\bar{z} = w - \bar{w}$ we get $\|\bar{z}^n\|_h \leq 0, \quad n = 1, 2, ..., N$. So, $\bar{z} \equiv 0$.

4 Numerical implementation remarks

We now comment on the numerical implementation of the discrete problem (26)-(28). Note that (26) can be rewritten as:

$$\begin{gathered} \frac{w_i^{j+1} - w_i^j}{\tau} - \\ \frac{1}{h} \left\{ \left[1 + \tau \sum_{k=1}^{j+1} \left(\frac{w_{i+1}^k - w_i^k}{h} \right)^2 \right] \frac{w_{i+1}^{j+1} - w_i^{j+1}}{h} - \right. \\ \left. \left[1 + \tau \sum_{k=1}^{j+1} \left(\frac{w_i^k - w_{i-1}^k}{h} \right)^2 \right] \frac{w_i^{j+1} - w_{i-1}^{j+1}}{h} \right\} = f_i^j, \\ i = 1, \dots, M-1. \end{gathered}$$

Let

$$A_i^\ell = 1 + \tau \sum_{k=1}^{\ell} \left[\left(\frac{w_{i+1}^k - w_i^k}{h} \right)^2 \right],$$

$$i = 0, 1, \dots, M-1,$$

then (26) becomes

$$\begin{gathered} \frac{w_i^{j+1} - w_i^j}{\tau} - \frac{1}{h} \left\{ A_i^{j+1} \frac{w_{i+1}^{j+1} - w_i^{j+1}}{h} - \right. \\ \left. A_{i-1}^{j+1} \frac{w_i^{j+1} - w_{i-1}^{j+1}}{h} \right\} = f_{1,i}^j, \\ i = 1, 2, \dots, M-1. \end{gathered} \tag{41}$$

The system (41) can be written in matrix form

$$\mathbf{P}\left(\mathbf{w}^{j+1}\right) \equiv \mathbf{G}\left(\mathbf{w}^{j+1}\right) - \frac{1}{\tau}\mathbf{w}^j - \mathbf{f}^j = 0.$$

The vector $\mathbf{w}$ containing all the unknowns $w_1, \dots, w_{M-1}$ at the level indicated. The vector $\mathbf{G}$ is given by

$$\mathbf{G}\left(\mathbf{w}^{j+1}\right) = \mathbf{T}^{j+1}\mathbf{w}^{j+1},$$

where the $(M-1) \times (M-1)$ matrix $\mathbf{T}$ is symmetric and tridiagonal with elements:

$$\mathbf{T}_{rs}^\ell = \begin{cases} -\dfrac{1}{h^2} A_{r-1}^\ell, & s = r-1, \\ \dfrac{1}{h^2}\left(A_r^\ell + A_{r-1}^\ell\right), & s = r, \\ -\dfrac{1}{h^2} A_r^\ell, & s = r+1, \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

Newton's method for the system is given by

$$\nabla \mathbf{P}\left(\mathbf{w}^{j+1}\right)\Big|^{(n)} \left(\mathbf{w}^{j+1}\Big|^{(n+1)} - \mathbf{w}^{j+1}\Big|^{(n)}\right) = -\mathbf{P}\left(\mathbf{w}^{j+1}\right)\Big|^{(n)}.$$

The elements of the matrix $\nabla \mathbf{P}\left(\mathbf{w}^{j+1}\right)$ require the derivative of A. The elements are:

$$\frac{\partial A_{r-1}^{j+1}}{\partial w_s^{j+1}} = -\frac{\tau}{h^2}\frac{\partial}{\partial w_s^{j+1}}\left\{\left(\frac{w_r^{j+1}-w_{r-1}^{j+1}}{h}\right)^2\right\} = \begin{cases} \frac{2\tau}{h^3}\, w_{\bar{x},r}^{j+1}, & s=r-1,\\ -\frac{2\tau}{h^3}\, w_{\bar{x},r}^{j+1}, & s=r,\\ 0, & \text{otherwise,}\end{cases} \tag{43}$$

and

$$\frac{\partial A_{r}^{j+1}}{\partial w_s^{j+1}} = -\frac{\tau}{h^2}\frac{\partial}{\partial w_s^{j+1}}\left[\left(\frac{w_{r+1}^{j+1}-w_{r}^{j+1}}{h}\right)^2\right] = \begin{cases} \frac{2\tau}{h^3}\, w_{x,r}^{j+1}, & s=r,\\ -\frac{2\tau}{h^3}\, w_{x,r}^{j+1}, & s=r+1,\\ 0, & \text{otherwise.}\end{cases} \tag{44}$$

Combining (42)-(44) we have

$$\nabla \mathbf{P}\left(\mathbf{w}^{j+1}\right)\Big|_{r\,s} =$$

$$\begin{cases} -\frac{1}{h^2}A_{r-1}^{j+1} - \frac{2\tau}{h^2}\left(w_{\bar{x},r}^{j+1}\right)^2, & s=r-1,\\ \frac{1}{\tau}+\frac{1}{h^2}\left(A_r^{j+1}+A_{r-1}^{j+1}\right)+ \\ \frac{2\tau}{h^2}\left(w_{x,r}^{j+1}\right)^2+\frac{2\tau}{h^2}\left(w_{\bar{x},r}^{j+1}\right)^2, & s=r,\\ -\frac{1}{h^2}A_{r}^{j+1} - \frac{2\tau}{h^2}\left(w_{x,r}^{j+1}\right)^2, & s=r+1,\\ 0, & \text{otherwise.}\end{cases}$$

Let us state well known theorem (see, for example, [23]).

Theorem 7 *Given the nonlinear system of equations*

$$g_i\left(x_1,\ldots,x_{M-1}\right) = 0, \;\; i=1,2,\ldots,M-1.$$

If g_i are three times continuously differentiable in a region containing the solution $\xi_1,\ldots,\xi_{M-1}$ and the Jacobian does not vanish in that region, then Newton's method converges at least quadratically.

The Jacobian is the matrix ∇P computed above. The term $\frac{1}{\tau}$ on diagonal ensures that the Jacobian doesn't vanish. The differentiability is guaranteed, since ∇P is quadratic. Newton's method is costly, because the matrix changes at every step of the iteration. One can use modified Newton (keep the same matrix for several iterations) but the rate of convergence will be slower.

In the first numerical experiment we have chosen the right hand side of equation (3) so that the exact solution is given by

$$W(x,t) = x(1-x)\cos t,$$

which satisfy homogeneous boundary conditions (4).

The parameters used are $M = 100$ which dictates $h = 0.01$. Since the method is implicit we can use $\tau = h$ and we took 100 time steps. In the Fig. 1 and Fig. 2 we plotted the numerical solution and the exact solutions at $t = 0.5$ (Fig. 1) and $t = 1.0$ (Fig. 2). As it is visible from these pictures, the numerical and exact solutions are almost identical.

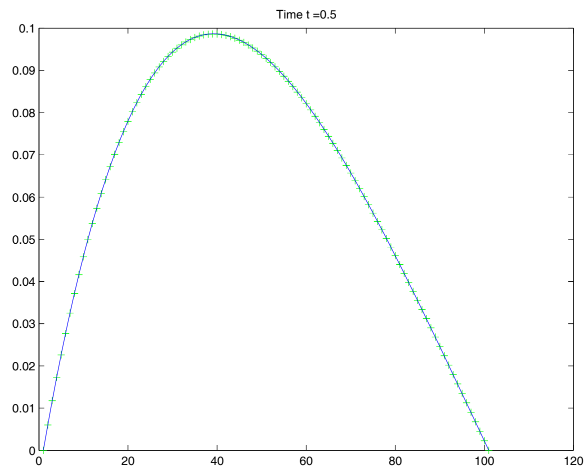

Figure 1: The solution at $t = 0.5$. The exact solution is solid line and the numerical solution is marked by $+$.

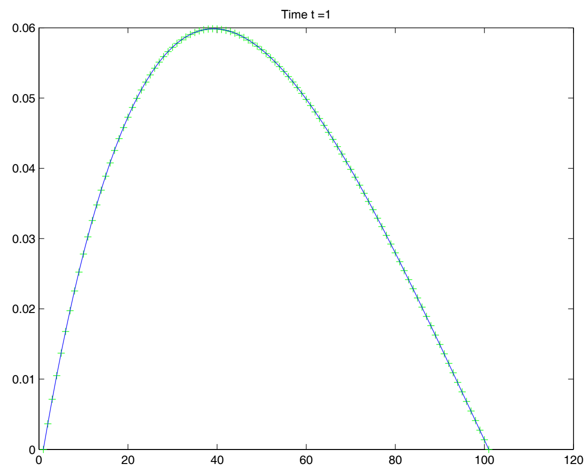

Figure 2: The solution at $t = 1.0$. The exact solution is solid line and the numerical solution is marked by $+$.

In the second experiment we have taken zero right hand side and initial data given by

$$W(x,0) = x(1-x)\cos(4\pi x).$$

In this case, we know that the solution will decay in time. The parameters M, h, τ are as before. In Fig. 3 we plotted the initial data and in Fig. 4 we have the numerical solution at four different times. It is clear that the numerical solution is approaching zero for all x.

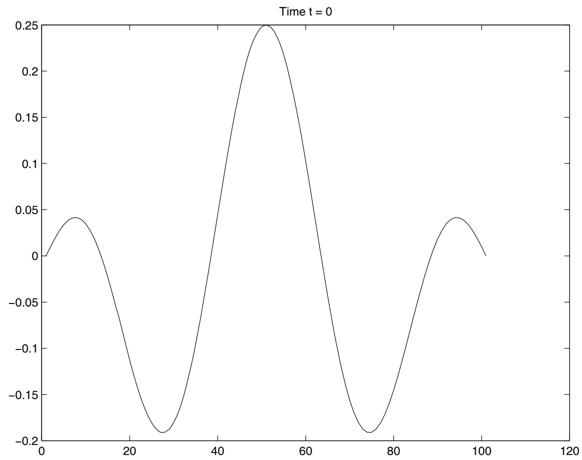

Figure 3: The initial data for homogeneous boundary conditions.

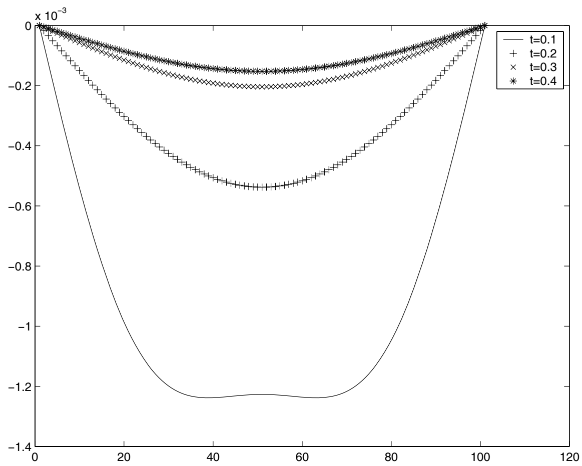

Figure 4: The numerical solution at $t = 0.1, 0.2, 0.3, 0.4$ for homogeneous boundary conditions.

The numerical experiments for problem (3),(5),(6),(8) was carried out as well. For our next experiment we have taken zero right hand side and initial data given by

$$W(x,0) = x(1-x)\cos(4\pi x) + 0.001x.$$

In this case, we know that the solution will approach to the steady-state solution, which in this case is $W(x) = 0.001x$. The parameters M, h, τ are as before. In Fig. 5 we plotted the initial data and in Fig. 6 we have the numerical solution at four different times. It is clear that the numerical solution is approaching steady-state solution for all x.

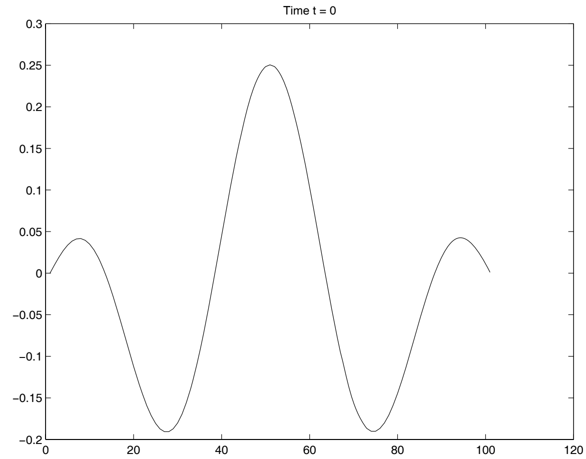

Figure 5: The initial data for nonhomogeneous boundary condition on part of lateral boundary.

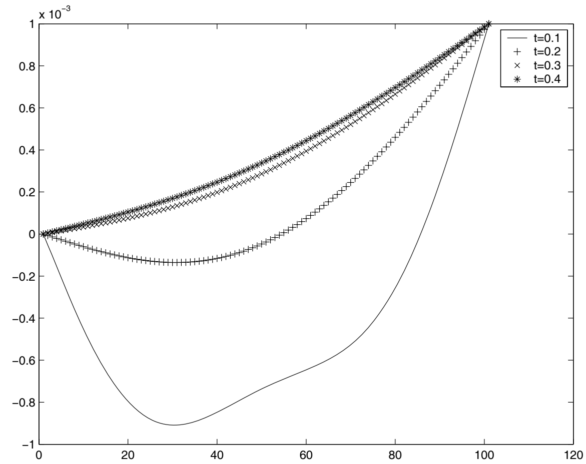

Figure 6: The numerical solution at $t = 0.1, 0.2, 0.3, 0.4$ for nonhomogeneous boundary condition on part of lateral boundary.

5 Conclusion

We have experimented with several other initial data for both inial-boundary value problems (3)-(6) and (3),(5),(6),(8). In all cases we noticed that numerical solutions are approaching steady-state solution as it is shown in theoretical researches.

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