On One Generalization of Boundary Value Problem for Ordinary Differential Equations on Graphs in the Three-dimensional Space

D.G.GORDEZIANI
Department of Mathematics, Faculty of Exact and Natural Sciences,
Tbilisi State University,
2, University str.,0143, Tbilisi
GEORGIA
dgord37@hotmail.com

H.V. MELADZE
Department of Computer Sciences,
St Andrea First Georgian University
53-a, Chavchavadze aven., Tbilisi
GEORGIA
h_meladze@hotmail.com

T.D. DAVITASHVILI
Department of Computer Sciences, Faculty of Exact and Natural Sciences,
Tbilisi State University,
2, University str.,0143, Tbilisi
GEORGIA
t_davitashvili@hotmail.com

Abstract: - The present work is the generalization of boundary value problem for ordinary differential equations on graphs. This problem is investigated and correctness of the stated problem is proved in [1]. The special attention is given to construction and research of difference analogues. Estimation of precision is given. The formulas of double-sweep method type are suggested for finding the solution of obtained difference scheme.

In this work the boundary-value problems for Poisson's equations in the three-dimensional space on some two-dimensional structures with one-dimensional common part is given and investigated. This technique of investigation can be easily applied to the more complex initial data and equations. The difference scheme for numerical solution of this problem is constructed and estimation of precision is given.

Such problems have practical sense and they can be used for mathematical modeling of specific problems of physics, engineering, ecology and so on.

Key-Words: - Differential equations on graphs, Difference scheme.

1 Introduction

In the work [1] the boundary value problem for ordinary differential equations on graphs is investigated; correctness of the stated problem is proved. The special attention is given to construction and research of difference analogues, which is a little concern in papers of other authors. Estimation of precision is given; double-sweep method type formulas are suggested for finding the solution of difference scheme ([2]-[3]).

It's possible to note some works, devoted to the theoretical investigation of boundary value problems, considered on graphs (see, for example, [4], [2] and the literature, mentioned there. Certainly, this list is incomplete).

In the present work there are given some generalizations of the above mentioned problem: in the three-dimensional space on some two-dimensional structures with one-dimensional common part the boundary-value problem for Poisson's equation is stated and investigated. This technique of investigation can be easily applied to the more complex initial data and equations. Obviously such problems have practical sense and they can be used for mathematical modeling of specific problems of physics, engineering, ecology and so on ([5]-[11]). Certainly, this list is incomplete.

2 Ordinary differential equations of the second order on graphs

Let us consider a graph $G=(V,E)$, where $V=(a_0,a_1,\dots,a_n)$ is a set of tops of this graph, a_0 is a node of the graph and E is a set of ribs of the graph $\{\overline{a_0a_1},\overline{a_0a_2},\cdots,\overline{a_0a_n}\}$. Denote the rib $\overline{a_0a_i}$ by Γ_i. On each rib introduce a local coordinate system with the origin in the node a_0 and the coordinate $x_\alpha\in(0,l_\alpha)$, where l_α is length of the curve $\Gamma_\alpha(\alpha=1,2,\dots,n)$.

Let us state the following problem: find the functions $u_\alpha(x_\alpha)(\alpha=1,2,\dots,n)$, which satisfies the differential equations

$$\frac{d}{dx_\alpha}\left(K_\alpha(x_\alpha)\frac{du_\alpha(x_\alpha)}{dx_\alpha}\right)-q_\alpha(x_\alpha)u_\alpha(x_\alpha)= = f_\alpha(x_\alpha),\quad \alpha=\overline{1,n},\quad x_\alpha\in(0,l_\alpha) \tag{1}$$

boundary conditions

$$u_\alpha(l_\alpha)=u^{(\alpha)},\quad \alpha=\overline{1,n}, \tag{2}$$

and conditions of conjunctions

$$u_\alpha(a_0)=u_\beta(a_0),\quad \alpha,\beta=\overline{1,n}, \tag{3}$$

$$\sum_{\alpha=1}^{n}K_\alpha(x_\alpha)\frac{du_\alpha(x_\alpha)}{dx_\alpha}\bigg|_{x_\alpha=0}=b. \tag{4}$$

where

$K_\alpha(x_\alpha)\in C^1[0,l_\alpha],\ K_\alpha(x_\alpha)>C_0=const>0,$
$q_\alpha(x_\alpha)\in C^1[0,l_\alpha],\quad q_\alpha(x_\alpha)\ge 0,$
$f_\alpha(x_\alpha)\in C^0[0,l_\alpha]$ are the given functions and $b,\ u^{(\alpha)}\left(\alpha=\overline{1,n}\right)$ are the given numbers.

Theorem 1. *There exists a unique regular solution of problem* (1)-(4), *i.e. exists unique functions*
$u_\alpha(x_\alpha)\in C^2]0,l_\alpha[\cap C^1[0,l_\alpha],\quad \left(\alpha=\overline{1,n}\right),$
which satisfies equations (1), *boundary conditions* (2) *and conditions of conjunctions* (3), (4).

The proof of this theorem see in [1].

3 Difference scheme for numerical solution of problem (1)-(4)

On $\Gamma_\alpha(\alpha=1,2,\dots,n)$ we introduce a uniform mesh with step h_α:

$$\overline{\omega}_h^{(\alpha)}=\{x_\alpha^{(i_\alpha)}=i_\alpha h_\alpha,\ i_\alpha=0,1,2,\dots,N_\alpha;\ x_\alpha^{(0)}=0;\quad h_\alpha N_\alpha=l_\alpha\},\quad \alpha=\overline{1,n}.$$

If on the mesh $\overline{\omega}_h^{(\alpha)}$ we substitute differential operator by the difference operator, we obtain the following difference scheme:

$$\left(\overline{K}_\alpha y_{\alpha,\overline{x}_\alpha}\right)_{x_\alpha}^{(i_\alpha)}-q_\alpha^{(i_\alpha)}y_\alpha^{(i_\alpha)}=f_\alpha^{(i_\alpha)},\quad i_\alpha=\overline{1,N_\alpha-1},\ \ \alpha=\overline{1,n}., \tag{5}$$

$$y_\alpha^{(N_\alpha)}=u^{(\alpha)},\quad \alpha=\overline{1,n}, \tag{6}$$

$$y_\alpha^{(0)}=y_\beta^{(0)},\quad \alpha,\beta=\overline{1,n}, \tag{7}$$

$$\sum_{\alpha=1}^{n} \overline{K}_{\alpha}^{(1)} \frac{y_{\alpha}^{(1)} - y_{\alpha}^{(0)}}{h_{\alpha}} = 0. \qquad (8)$$

where

$$y_{\alpha}^{(i_{\alpha})} = y_{\alpha}(i_{\alpha} h_{\alpha}), \quad y_{\alpha,\bar{x}_{\alpha}} = \frac{y_{\alpha}^{(i_{\alpha})} - y_{\alpha}^{(i_{\alpha}-1)}}{h_{\alpha}},$$

$$y_{\alpha,x_{\alpha}} = \frac{y_{\alpha}^{(i_{\alpha}+1)} - y_{\alpha}^{(i_{\alpha})}}{h_{\alpha}}, \quad \overline{K}_{\alpha}^{(i_{\alpha})} = K_{\alpha}^{(i_{\alpha}-\frac{1}{2})}.$$

Theorem 2. *There exist no more then one solution of the difference scheme* (5)-(8) .

Theorem 3. *Let* $u_{\alpha} \in C^{3}[0,l_{\alpha}], \left(\alpha = \overline{1, n}\right)$. *Then the solution of the difference scheme* (5)-(8) *uniformly converges to the solution of the problem* (1)-(4) *at the rate of O*(*h*), *when* $h \to 0$, *where* $h = \max_{1 \le \alpha \le n} h_{\alpha}$.

The proof of the theorems 2, 3 see in [1].

Remark. Let $u_{\alpha} \in C^{3}[0,l_{\alpha}], \left(\alpha = \overline{1, n}\right)$. Then

$$\left\|\Psi_{\alpha}^{(i_{\alpha})}\right\| = O(h), \quad |\Theta_0| = O(h),$$
$$i_{\alpha} = \overline{1, N_{\alpha}-1}, \quad \alpha = \overline{1, n}.$$

where

$$\Psi_{\alpha}^{(i_{\alpha})} = \left(K_{\alpha} u_{\alpha,\bar{x}_{\alpha}}\right)_{x_{\alpha}}^{(i_{\alpha})} - q_{\alpha}^{(i_{\alpha})} u_{\alpha}^{(i_{\alpha})}$$

and

$$\Theta_0 = \sum_{\alpha=1}^{n} \frac{K_{\alpha}^{(1)}}{h_{\alpha}} \left(u_{\alpha}^{(1)} - u_{\alpha}^{(0)}\right).$$

Let $u_{\alpha} \in C^{4}[0,l_{\alpha}], \left(\alpha = \overline{1, n}\right)$. Then, if instead of condition (8) we consider the following approximation of the conjunction conditions:

$$\sum_{\alpha=1}^{n} K_{\alpha}(0.5h_{\alpha}) \frac{y_{\alpha}^{(1)} - y_{\alpha}^{(0)}}{h_{\alpha}} - 0.5 \sum_{\alpha=1}^{n} h_{\alpha}\left[q_{\alpha}^{(0)} y_{\alpha}^{(0)} + f_{\alpha}^{(0)}\right] = 0, \qquad (9)$$

Then the error of approximation

$$\Theta_0 = \sum_{\alpha=1}^{n} K_{\alpha}(0.5h_{\alpha}) \frac{u_{\alpha}^{(1)} - u_{\alpha}^{(0)}}{h_{\alpha}} - 0.5 \sum_{\alpha=1}^{n} h_{\alpha}\left[q_{\alpha}^{(0)} y_{\alpha}^{(0)} + f_{\alpha}^{(0)}\right]$$

will have the order $O(h^2)$.
Indeed,

$$\Theta_0 = \sum_{\alpha=1}^{n} \left\{ K_{\alpha}(0) \frac{du_{\alpha}}{dx_{\alpha}}\bigg|_{x_{\alpha}=0} - \right.$$

$$- 0.5h_{\alpha}\left(K'_{\alpha}(0)u'_{\alpha}(0) + K_{\alpha}(0)u''_{\alpha}(0)\right) -$$

$$\left. - 0.5h_{\alpha}\left(q_{\alpha}^{(0)} u_{\alpha}^{(0)} + f_{\alpha}^{(0)}\right) \right\} + O\left(h^{2}\right) = O\left(h^{2}\right),$$

as $\sum_{\alpha=1}^{n} K_{\alpha}(0) \frac{du_{\alpha}}{dx_{\alpha}}\bigg|_{x_{\alpha}=0} = 0$

(conjunction condition) and

$$K'_{\alpha}(0)u'_{\alpha}(0) + K_{\alpha}(0)u''_{\alpha}(0) - q_{\alpha}^{(0)} u_{\alpha}^{(0)} - f_{\alpha}^{(0)} +$$
$$= \frac{\partial}{\partial x}\left(K_{\alpha} \frac{du_{\alpha}}{dx_{\alpha}}\right)\bigg|_{x_{\alpha}=0} - q_{\alpha} u_{\alpha}^{(0)} - f_{\alpha} = 0.$$

Therefore,

$$\left\|\Psi_{\alpha}\right\| = O\left(h^{2}\right), \quad \alpha = \overline{1, n}, \quad |\Theta_0| = O\left(h^{2}\right),$$

if $u_{\alpha} \in C^{4}[0,l_{\alpha}]$.

So, in the case of difference scheme (5)-(7), (9) the following theorem is true.

Theorem 4. *Let* $u_{\alpha} \in C^{4}[0,l_{\alpha}], \left(\alpha = \overline{1, n}\right)$. *Then the solution of the difference scheme* (5)-(7), (9) *uniformly converges to the solution of the problem* (1)-(4) *at the rate of O*(h^2), *when* $h \to 0$, *where* $h = \max_{1 \le \alpha \le n} h_{\alpha}$.

4 Variant of double-sweep method for difference equations (5)-(8)

Let us write the difference scheme (5)-(8) as a system of the linear algebraic equations:

$$\frac{K_\alpha^{(i_\alpha-1)}}{h_\alpha^2} y_\alpha^{(i_\alpha-1)} - \left(\frac{K_\alpha^{(i_\alpha-1)} + K_\alpha^{(i_\alpha)}}{h_\alpha^2} + q_\alpha^{(i_\alpha)} \right) \times$$
$$\times y_\alpha^{(i_\alpha)} + \frac{K_\alpha^{(i_\alpha)}}{h_\alpha^2} y_\alpha^{(i_\alpha+1)} = f_\alpha^{(i_\alpha)},$$
$$i_\alpha = \overline{1, N_\alpha - 1}, \quad \alpha = \overline{1, n}.$$

$$y_\alpha^{(N_\alpha)} = u^{(\alpha)}, \quad \alpha = \overline{1, n},$$

$$y_\alpha^{(0)} = y_\beta^{(0)}, \quad \alpha, \beta = \overline{1, n}, \tag{10}$$

$$\sum_{\alpha=1}^{n} \frac{K_\alpha^{(1)}}{h_\alpha} \left(y_\alpha^{(1)} - y_\alpha^{(0)} \right) = 0.$$

Introduce the following denotations:

$$a_\alpha^{(i_\alpha)} = \frac{K_\alpha^{(i_\alpha-1)}}{h_\alpha^2}, \quad b_\alpha^{(i_\alpha)} = \frac{K_\alpha^{(i_\alpha)}}{h_\alpha^2},$$

$$c_\alpha^{(i_\alpha)} = \frac{K_\alpha^{(i_\alpha-1)} + K_\alpha^{(i_\alpha)}}{h_\alpha^2}, \quad m_\alpha = \frac{K_\alpha^{(0)}}{h_\alpha}.$$
$$i_\alpha = \overline{1, N_\alpha - 1}, \quad \alpha = \overline{1, n}.$$

Then the system of equations (10) can be rewritten in the following form:

$$a_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha-1)} - c_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha)} + b_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha+1)} = f_\alpha^{(i_\alpha)} \tag{11}$$

$$i_\alpha = \overline{1, N_\alpha - 1}, \quad \alpha = \overline{1, n}.$$

$$y_\alpha^{(N_\alpha)} = u^{(\alpha)}, \quad \alpha = \overline{1, n}, \tag{12}$$

$$y_\alpha^{(0)} = y_\beta^{(0)}, \quad \alpha, \beta = \overline{1, n}, \tag{13}$$

$$\sum_{\alpha=1}^{n} m_\alpha \left(y_\alpha^{(1)} - y_\alpha^{(0)} \right) = 0. \tag{14}$$

Suppose, that for the solution of difference equation (11) the relation holds:

$$y_\alpha^{(i_\alpha+1)} = \xi_\alpha^{(i_\alpha+1)} y_\alpha^{(i_\alpha)} + \eta_\alpha^{(i_\alpha+1)},$$
$$i_\alpha = \overline{0, N_\alpha - 1}, \quad \alpha = \overline{1, n}. \tag{15}$$

then

$$y_\alpha^{(i_\alpha)} = \xi_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha-1)} + \eta_\alpha^{(i_\alpha)}, \tag{16}$$

Substituting expression (15) in the equation (11) we obtain

$$a_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha-1)} - c_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha)} +$$
$$+ b_\alpha^{(i_\alpha)} \left(\xi_\alpha^{(i_\alpha+1)} y_\alpha^{(i_\alpha)} + \eta_\alpha^{(i_\alpha+1)} \right) = f_\alpha^{(i_\alpha)}.$$

From this equation we define $y_\alpha^{(i_\alpha)}$:

$$y_\alpha^{(i_\alpha)} = \frac{a_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)}} y_\alpha^{(i_\alpha-1)} +$$
$$+ \frac{b_\alpha^{(i_\alpha)} \eta_\alpha^{(i_\alpha+1)} - f_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)}}$$
$$\left(c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)} \neq 0 \right).$$

Comparing this equality with the equality (16) we obtain:

$$\xi_\alpha^{(i_\alpha)} = \frac{a_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)}},$$
$$\eta_\alpha^{(i_\alpha)} = \frac{b_\alpha^{(i_\alpha)} \eta_\alpha^{(i_\alpha+1)} - f_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)}}, \tag{17}$$
$$i_\alpha = N_\alpha - 1, N_\alpha - 2, \ldots, 1, \quad \alpha = \overline{1, n}.$$

Using the boundary conditions (12) to define $\xi_\alpha^{(N_\alpha)}$ and $\eta_\alpha^{(N_\alpha)}$, we obtain:

$$\xi_\alpha^{(N_\alpha)}=0, \quad \eta_\alpha^{(N_\alpha)}=u^{(\alpha)}, \quad \alpha=\overline{1,n}. \tag{18}$$

Recurrent relations (17), (18) allow to define coefficients

$$\xi_\alpha^{(i_\alpha)} \text{ and } \eta_\alpha^{(i_\alpha)}$$

$(i_\alpha = N_\alpha - 1, N_\alpha - 2, \ldots, 1, \ \alpha=\overline{1,n}.)$,

if $c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)}\xi_\alpha^{(i_\alpha+1)} \neq 0$.

As $\left|c_\alpha^{(i_\alpha)}\right| \geq \left|a_\alpha^{(i_\alpha)}\right| + \left|b_\alpha^{(i_\alpha)}\right|, \quad i_\alpha = \overline{1, N_\alpha - 1},$ $\alpha=\overline{1,n}.$, therefore repeating the reasoning from [12], it can be proved, that $\left|\xi_\alpha^{(i_\alpha)}\right| < 1$ and $\left|c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)}\xi_\alpha^{(i_\alpha+1)}\right| \geq \left|a_\alpha^{(i_\alpha)}\right|$.

Thus, we have proved that by means of recurrent formulas (17), (18) uniquely can be defined values of the coefficient

$\xi_\alpha^{(i_\alpha)}, \ \eta_\alpha^{(i_\alpha)} \ (i_\alpha = N_\alpha - 1, N_\alpha - 2, \ldots, 1,$ $\alpha=\overline{1,n}.)$.

Write out formulas (15) in case of i_α=0:

$$y_\alpha^{(1)} = \xi_\alpha^{(1)} y_\alpha^{(0)} + \eta_\alpha^{(1)}, \quad \alpha=\overline{1,n}.$$

Insert these equalities in (14) and take into account relations (13), then we obtain:

$$\sum_{\alpha=1}^{n} m_\alpha \left(\xi_\alpha^{(1)} y_\alpha^{(0)} - \eta_\alpha^{(1)} - y_\alpha^{(0)}\right) = 0.$$

As $\left|\xi_\alpha^{(1)}\right| < 1$, from the last equality we obtain:

$$y_1^{(0)} = y_\alpha^{(0)} = \frac{\sum_{\alpha=1}^{n} m_\alpha \eta_\alpha^{(1)}}{\sum_{\alpha=1}^{n} m_\alpha - \sum_{\alpha=1}^{n} m_\alpha \xi_\alpha^{(1)}}, \tag{19}$$

$\alpha=\overline{2,n}.$

Collect all formulas of double-sweep method and write them down in order of application:

$$\xi_\alpha^{(i_\alpha)} = \frac{a_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)}\xi_\alpha^{(i_\alpha+1)}},$$

$$\eta_\alpha^{(i_\alpha)} = \frac{b_\alpha^{(i_\alpha)}\eta_\alpha^{(i_\alpha+1)} - f_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)}\xi_\alpha^{(i_\alpha+1)}}$$

$$i_\alpha = N_\alpha - 1, N_\alpha - 2, \ldots, 0, \quad \alpha=\overline{1,n}.$$

$$\xi_\alpha^{(N_\alpha)}=0, \quad \eta_\alpha^{(N_\alpha)}=u^{(\alpha)}, \quad \alpha=\overline{1,n}.$$

$$y_\alpha^{(i_\alpha+1)} = \xi_\alpha^{(i_\alpha+1)} y_\alpha^{(i_\alpha)} + \eta_\alpha^{(i_\alpha+1)},$$

$$i_\alpha = \overline{0, N_\alpha - 1}, \quad \alpha=\overline{1,n}.$$

$$y_\alpha^{(0)} = \frac{\sum_{\alpha=1}^{n} m_\alpha \eta_\alpha^{(1)}}{\sum_{\alpha=1}^{n} m_\alpha - \sum_{\alpha=1}^{n} m_\alpha \xi_\alpha^{(1)}}, \quad \alpha=\overline{1,n}.$$

5 On one generalization of the problem (1)-(4)

Let us consider one generalization of problem (1)-(4). Instead of graph let's consider n-half-planes in R^3, which are bounded by common boundary line $\gamma = (0, x_0)$. On these planes we consider the local coordinate systems: (x_0, x_1), (x_0, x_2),…, (x_0, x_n).

Consider the following problem: find functions $u_\alpha(x_0, x_\alpha)$, $\alpha=\overline{1,n}.$, which satisfy the differential equations

$$\frac{\partial^2 u_\alpha}{\partial x_0^2}+\frac{\partial^2 u_\alpha}{\partial x_\alpha^2}=f_\alpha(x_0,x_\alpha),$$
$$0<x_0<l_0,\quad 0<x_\alpha<l_\alpha,\quad \alpha=\overline{1,n}. \qquad (20)$$

boundary conditions

$$u_\alpha(0,x_\alpha)=a_\alpha(x_\alpha),$$
$$\alpha=\overline{1,n},\quad 0\le x_\alpha\le l_\alpha, \qquad (21)$$

$$u_\alpha(l_0,x_\alpha)=b_\alpha(x_\alpha),\quad \alpha=\overline{1,n},$$
$$0\le x_\alpha\le l_\alpha, \qquad (22)$$

$$u_\alpha(x_0,l_\alpha)=\varphi_\alpha(x_0),\quad \alpha=\overline{1,n},$$
$$0\le x_0\le l_0, \qquad (23)$$

and conjunction conditions

$$u_\alpha(x_0,0)=u_\beta(x_0,0),\quad \alpha,\beta=\overline{1,n}, \qquad (24)$$

$$\sum_{\alpha=1}^{n}\left.\frac{\partial u_\alpha(x_0,x_\alpha)}{\partial x_\alpha}\right|_{x_\alpha=0}=0,\quad x_0\in[0,l_0], \qquad (25)$$

where

$$a_0,b_0\in C^1[0,l_\alpha],\ \varphi_\alpha\in C^1[0,l_\alpha],$$
$$f_\alpha\in C^1([0,l_0]\times[0,l_\alpha]),\quad \alpha=\overline{1,n},$$

are given functions.

Theorem 5. *There exists no more then one regular solution of problem* (20)-(25).

Proof. It is sufficient to prove that the homogeneous problem with the homogeneous boundary conditions corresponding to the problem (20)-(25) has only trivial solution. Consider this problem:

$$\frac{\partial^2 u_\alpha}{\partial x_0^2}+\frac{\partial^2 u_\alpha}{\partial x_\alpha^2}=0,\quad 0<x_0<l_0,$$
$$0<x_\alpha<l_\alpha,\quad \alpha=\overline{1,n}. \qquad (26)$$

$$u_\alpha(0,x_\alpha)=0,\quad \alpha=\overline{1,n},$$
$$0\le x_\alpha\le l_\alpha, \qquad (27)$$

$$u_\alpha(l_0,x_\alpha)=0,\quad \alpha=\overline{1,n},$$
$$0\le x_\alpha\le l_\alpha, \qquad (28)$$

$$u_\alpha(x_0,l_\alpha)=0,\quad \alpha=\overline{1,n},$$
$$0\le x_0\le l_0, \qquad (29)$$

$$u_\alpha(x_0,0)=u_\beta(x_0,0),\quad \alpha,\beta=\overline{1,n}, \qquad (30)$$

$$\sum_{\alpha=1}^{n}\left.\frac{\partial u_\alpha(x_0,x_\alpha)}{\partial x_\alpha}\right|_{x_\alpha=0}=0,\quad x_0\in[0,l_0]. \qquad (31)$$

Multiply the equation (23) on function $u_\alpha(x_0,x_\alpha)$ and integrate it at first on the interval $[0,l_\alpha]$ with respect to variable x_0 and then on the interval $[0,l_\alpha]$ with respect to variable x_α. Further sum up these equalities by α from 1 to n:

$$\sum_{\alpha=1}^{n}\int_0^{l_0}\int_0^{l_\alpha}\left(\frac{\partial^2 u_\alpha}{\partial x_0^2}+\frac{\partial^2 u_\alpha}{\partial x_\alpha^2}\right)u_\alpha(x_0,x_\alpha)dx_0dx_\alpha=0$$
$$. \qquad (32)$$

To change the left hand-side of this equality first we need to be sure in the fairness of the following equalities (thereat take into account conditions (27)-(31)):

$$\sum_{\alpha=1}^{n}\int_0^{l_\alpha}\int_0^{l_0}\frac{\partial^2 u_\alpha}{\partial x_\alpha^2}u_\alpha(x_0,x_\alpha)dx_0dx_\alpha=$$
$$=-\sum_{\alpha=1}^{n}\int_0^{l_\alpha}\int_0^{l_0}\left(\frac{\partial u_\alpha}{\partial x_\alpha}\right)^2dx_0dx_\alpha+$$
$$+\sum_{\alpha=1}^{n}\left.\left(\int_0^{l_0}\frac{\partial u_\alpha(x_0,x_\alpha)}{\partial x_\alpha}u_\alpha(x_0,x_\alpha)dx_0\right)\right|_0^{l_\alpha}=$$

$$=-\sum_{\alpha=1}^{n}\int_0^{l_\alpha}\int_0^{l_0}\left(\frac{\partial u_\alpha}{\partial x_\alpha}\right)^2 dx_0 dx_\alpha .$$

Analogously:

$$\sum_{\alpha=1}^{n}\int_0^{l_\alpha}\int_0^{l_0}\frac{\partial^2 u_\alpha}{\partial x_0^2}u_\alpha(x_0,x_\alpha)dx_0dx_\alpha =$$

$$=-\sum_{\alpha=1}^{n}\int_0^{l_\alpha}\int_0^{l_0}\left(\frac{\partial u_\alpha}{\partial x_0}\right)^2 dx_0 dx_\alpha +$$

$$+\sum_{\alpha=1}^{n}\left(\int_0^{l_\alpha}\frac{\partial u_\alpha(x_0,x_\alpha)}{\partial x_0}u_\alpha(x_0,x_\alpha)dx_\alpha\right)\Bigg|_0^{l_0} =$$

$$=-\sum_{\alpha=1}^{n}\int_0^{l_\alpha}\int_0^{l_0}\left(\frac{\partial u_\alpha}{\partial x_0}\right)^2 dx_0 dx_\alpha .$$

Taking these equalities into account, from the (32) we obtain:

$$\sum_{\alpha=1}^{n}\int_0^{l_0}\int_0^{l_0}\left[\left(\frac{\partial u_\alpha}{\partial x_0}\right)^2+\left(\frac{\partial u_\alpha}{\partial x_\alpha}\right)^2\right]dx_0dx_\alpha = 0 . \quad (33)$$

From the last equation we obtain, that $\frac{\partial u_\alpha(x_0,x_\alpha)}{\partial x_0}=0,\ \frac{\partial u_\alpha(x_0,x_\alpha)}{\partial x_\alpha}=0$, i.e. $u_\alpha(x_0,x_\alpha)\equiv const\ (\alpha=\overline{1,n})$. Taking into account boundary conditions (27)-(29), we obtain that $u_\alpha(x_0,x_\alpha)\equiv 0$.

The theorem is proved.

Introduce the denotation

$$\Omega_\alpha=(0,l_0)\times(0,l_\alpha),\quad \overline{\Omega}_\alpha=[0,l_0]\times[0,l_\alpha],\quad \alpha=\overline{1,n} .$$

Let us consider the problem (20)-(25), but supposing that $a_0(x_\alpha)\equiv b_0(x_\alpha)\equiv\varphi_0(x_\alpha)\equiv 0$ and $f_\alpha\in L_2(\Omega_\alpha)$.

Introduce the following set of functions:

$$\tilde{H}_0^1(\Omega_\alpha)=\{u_\alpha(x_0,x_\alpha): u_\alpha\in L_2(\Omega_\alpha),$$
$$\frac{\partial u_\alpha}{\partial x_0},\frac{\partial u_\alpha}{\partial x_\alpha}\in L_2(\Omega_\alpha),$$
$$u_\alpha(0,x_\alpha)=u_\alpha(l_0,x_\alpha)=u_\alpha(x_0,l_\alpha)=0\}$$

i.e. $\tilde{H}_0^1$ is Sobolev space of the first order on $(0,l_0)\times(0,l_\alpha)$ and equalize to zero on the boundary is meant in sense of trace (see for example [13]).

Let us define functions $v[x_0,x]$ on the set $\Omega\equiv\bigcup_{\alpha=1}^{n}\Omega_\alpha\subset R^3$ in the following way:

$v[x_0,x]=u_\alpha(x_0,x_\alpha)$, if

$$(x_0,x_\alpha)\in\overline{\Omega}_\alpha,\quad u_\alpha(x_0,x_\alpha)\in\tilde{H}_0^1(\Omega_\alpha),\quad \alpha=\overline{1,n}.$$

Denote

$$\tilde{H}_0^1(\Omega)=\{v[x_0,x]: u_\alpha(x_0,x_\alpha)\in\tilde{H}_0^1(\Omega_\alpha),$$
$$u_\alpha(x_0,0)=u_\beta(x_0,0),\ \alpha,\beta=\overline{1,n},\ x_\alpha\in[0,l_\alpha]\}.$$

Introduce in this space the scalar product and the norm induced by this product:

$$(v_1[x_0,x],v_2[x_0,x])_{\tilde{H}_0^1(\Omega)}=$$
$$=\sum_{\alpha=1}^{n}(v_{1\alpha}(x_0,x_\alpha),v_{2\alpha}(x_0,x_\alpha))_{\tilde{H}_0^1(\Omega_\alpha)},$$

$$v_1[\,,\,],v_2[\,,\,]\in\tilde{H}_0^1(\Omega),$$
$$\|v[x_0,x]\|_{\tilde{H}_0^1(\Omega)}=(v[x_0,x],v[x_0,x])^{1/2}_{\tilde{H}_0^1(\Omega)} .$$

The generalized solution of the problem (20)-(25) with the homogeneous boundary conditions we call the function $v[x_0,x]\in\tilde{H}_0^1(\Omega)$, for which the equality takes place:

$$a[v[x_0,x],u[x_0,x]] = \\ = -(F[x_0,x],u[x_0,x])_{\tilde{H}_0^1(\Omega)} \qquad (34)$$

for any function $u[x_0,x] \in \tilde{H}_0^1(\Omega)$, where

$$a[v[x_0,x],u[x_0,x]] = \\ = \sum_{\alpha=1}^{n} \int_{\Omega_\alpha} \left(\frac{\partial u_\alpha}{\partial x_0}\frac{\partial v_\alpha}{\partial x_0} + \frac{\partial v_\alpha}{\partial x_\alpha}\frac{\partial u_\alpha}{\partial x_\alpha} \right), \qquad (35)$$

and $F[x_0,x]=f_\alpha(x_0,x_\alpha)$, if

$(x_0,x_\alpha) \in \overline{\Omega}_\alpha, \quad \alpha = \overline{1,n}.$ (see [6], [7]).

Theorem 6. *There exists a unique generalized solution* $v \in \tilde{H}_0^1(\Omega)$ *of the problem* (20)-(25).

Proof. It can be easily shown, that the $a[v[x_0,x],u[x_0,x]]$ is continuous and coercive on $\tilde{H}_0^1(\Omega)$ [13]. From this, on the basis of Lax-Milgram Theorem the fairness of statement of the theorem immediately follows.

6 Difference scheme for numerical solution of the problem (20)-(25)

Let's define the meshes:

$$\omega_\alpha = \{(x_0,x_\alpha) \mid x_0^{i_0} = i_0 h_0, \quad x_\alpha^{i_\alpha} = i_\alpha h_\alpha, \\ i_0 = 1,\dots,N_0-1; \quad i_\alpha = 0,\dots,N_\alpha - 1\}$$

$$\overline{\omega}_\alpha = \omega_\alpha \bigcup \{x_0^0; x_\alpha^{N_\alpha)}\},$$

$$\gamma_1 = \{(x_0,x_\alpha) \mid (x_0^0; x_\alpha^{i_\alpha}), i_\alpha = \overline{0,N_\alpha}\},$$

$$\gamma_2 = \{(x_0,x_\alpha) \mid (x_0^{N_0}; x_\alpha^{i_\alpha}), i_\alpha = \overline{0,N_\alpha}\},$$

$$\gamma_3 = \{(x_0,x_\alpha) \mid (x_0^{i_0}; x_\alpha^{0}), i_0 = \overline{0,N_0}\},$$

$$\gamma_4 = \{(x_0,x_\alpha) \mid (x_0^{i_0}; x_\alpha^{N_\alpha}), i_0 = \overline{0,N_0}\},$$

$$(\alpha = \overline{1,n}). \quad \omega = \bigcup_{\alpha=1}^{n} \omega_\alpha, \quad \overline{\omega} = \bigcup \overline{\omega}_\alpha .$$

Introduce also the following denotations:

$$y_\alpha = y_\alpha^{(i_0,i_\alpha)} = y_\alpha(x_0^{i_0}, x_\alpha^{i_\alpha}),$$
$$y_\alpha^{(\pm 1_0)} = y_\alpha(x_0^{i_0 \pm 1}, x_\alpha^{i_\alpha}),$$
$$y_\alpha^{(\pm 1_\alpha)} = y_\alpha(x_0^{i_0}, x_\alpha^{i_\alpha \pm 1}),$$
$$y_{\alpha x_0} = (y_\alpha^{(+1_0)} - y_\alpha)/h_0,$$
$$y_{\alpha \bar{x}_0} = (y_\alpha - y_\alpha^{(-1_0)})/h_0,$$
$$y_{\alpha x_\alpha} = (y_\alpha^{(+1_\alpha)} - y_\alpha)/h_\alpha,$$
$$y_{\alpha \bar{x}_\alpha} = (y_\alpha - y_\alpha^{(-1_\alpha)})/h_\alpha.$$

Let's put in conformity to the problem (20)-(25) the difference scheme

$$\Delta_{h_\alpha} y_\alpha = f_\alpha, \quad \overline{x} = (x_0,x_\alpha) \in \omega_\alpha, \qquad (36)$$

$$y_\alpha(\overline{x}) = a_\alpha(\overline{x}), \quad \overline{x} \in \gamma_1, \qquad (37)$$

$$y_\alpha(\overline{x}) = b_\alpha(\overline{x}), \quad \overline{x} \in \gamma_2, \qquad (38)$$

$$y_\alpha(\overline{x}) = \varphi_\alpha(\overline{x}), \quad \overline{x} \in \gamma_3, \qquad (39)$$

$$y_\alpha(x_0,0) = y_\beta(x_0,0), \quad \alpha,\beta = \overline{1.n}, \qquad (40)$$

$$\sum_{\alpha=1}^{n} y_{\alpha x_\alpha} = 0, \quad \overline{x} \in \gamma_4. \qquad (41)$$

where $\Delta_{h_\alpha} y \equiv y_{\bar{x}_0 x_0} + y_{\bar{x}_\alpha x_\alpha}$.

It is easy to show that if initial problem (20)-(25) has sufficiently smooth solution, then (36) approximates the equation (20) with error $O(h_0^2 + h_\alpha^2)$ and the condition (41) approximates the condition (25) with error $O\left(\sum_{\alpha=1}^{n} h_\alpha\right)$.

So, the scheme (36)-(41) approximates the problem (20)-(25) with error $O\left(\sum_{\alpha=0}^{n} h_\alpha\right)$.

For the numerical solution of the problem (20)-(25) it is possible to construct the scheme with error $O\left(\sum_{\alpha=0}^{n} h_\alpha^2\right)$.

For this purpose the condition (25) can be approximated as follows:

$$\sum_{\alpha=1}^{n}\left[\frac{2}{h_\alpha} y_{\alpha x_\alpha} + y_{\alpha \bar{x}_0 x_0} - f_{\alpha h}\right] = 0. \quad (42)$$

According to (40), $y_\alpha = y_\beta$ at $\left(x_0^{(i_0)}, 0\right)$; therefore (42) can be rewritten as follows:

$$2\sum_{\alpha=1}^{n}\frac{1}{h_\alpha} y_{\alpha x_\alpha} + n y_{\bar{x}_0 \bar{x}_0} - F = 0,$$

where

$$y \equiv y_\alpha; \quad \bar{x} = (x_0, 0) \quad \left(\alpha = \overline{1,n}\right), \quad F = \sum_{\alpha=1}^{n} f_{\alpha h}.$$

Further, for simplicity of the statemen,t we will consider that

$$a_\alpha(\cdot) = b_\alpha(\cdot) \equiv 0, \text{ also } f_\alpha(\,\cdot\,,\,\cdot\,) \equiv 0. \quad (43)$$

The problem (20)-(25) at the accepted assumptions (43) can be solved formally in the form of a row

$$u_\alpha(x_0, x_\alpha) = \sum_{k=1}^{\infty}\left(c_{\alpha k} e^{-k\pi x_\alpha} + d_{\alpha k} e^{k\pi x_\alpha}\right)\sin k\pi x_\alpha, \quad (44)$$

where

$$c_{\alpha k} = \frac{2}{\pi}\frac{e^{k\pi}\sum_{\alpha=1}^{n}\varphi_{\alpha k}}{\left(e^{2k\pi} - e^{-2k\pi}\right)} - \frac{\varphi_{\alpha k}}{e^{k\pi} - e^{-k\pi}},$$

$$d_{\alpha k} = \frac{2}{\pi}\frac{e^{-k\pi}\sum_{\alpha=1}^{n}\varphi_{\alpha k}}{\left(e^{-2k\pi} - e^{2k\pi}\right)} + \frac{\varphi_{\alpha k}}{e^{k\pi} - e^{-k\pi}},$$

$\left(\alpha = \overline{1,n}\right)$;

$\varphi_{\alpha k}$ k-th coefficient of Fure row for function $\varphi_\alpha(x_\alpha)$.

It is easy to notice, that the row (44) represents the solution of the problem (20)-(25), if $\varphi_{\alpha k}(x_\alpha) \in C^3([0,1])$.

Similarly it is possible to construct the solution of the scheme (36) - (41) at assumptions (43).

It is possible to search the solution of (36)-(41) in the form of finite sum on mesh area ω. The technology of construction of the solution of difference scheme repeats the reasonings, applied when formula (44) is obtained. Simple generalisation of methods for estimation of accuracy of difference schemes for rectangular area in case of mixed boundary conditions [12] (the consideration on one of the parties of the rectangle Neumann's condition) allows to prove the convergence of solution of the difference scheme (36)-(41) to sufficiently smooth solution of initial problem with a speed $O\left(\sum_{\alpha=1}^{n} h_\alpha\right)$.

The questions of convergence for specified difference scheme (36)-(40)-(42) by us are not investigated.

References

[1] D.Gorgeziani, T.Davitashvili, M.Kupreishvili, H.Meladze. On the Solution of Boundary Value Problem for Differential Equations Given in Graphs // Applied Mathematics, Informatics and Mechanics – Tbilisi – 2008 –v.13, №.1 – pp.1-14

[2] Pokornyi Y.V., Penkin O.M., Borovskikh A.V., Pryadiev V.L., Lazarev K.P., SHabrov S.A. Differential equations on geometric graphs (Russian). M.: Fizmatlit, 2004, p.272.

[3] Samarskii A.A. Theory of difference schemes. – The third edition – M.: Nauka, 1989, p.612.

[4] Pokornyi Yu.V., Pryadiev V.L. On transmission conditions in the Sturm-Liouville problem on a network. (Russian) Sovrem. Mat. Prilozh. No. 12, Differ. Uravn. Chast. Proizvod. (2004), 107-137; translation in J. Math. Sci. (N. Y.) 130 (2005), no. 5, 5013-5045.

[5] Karaushev A.B. Hydraulic of rivers and water reservoirs. – M.: Publ. “River Transport”, 1955, p.291.
[6] Losev A.K. Theory of linear electric chains. – M.: “High School”, 1987, p.267.
[7] Kuchment P. Graphs models for waves in thin struktures // Waves Random Media – 2002 – 12 – pp.1-24.
[8] Akkermans E., Comtet A., Desbois J., Montambaux G., Texier C. Spectral determinant on quantum graphs // Ann. Phys. – 2000 – 284 – pp.10-51.
[9] Nikos E. Mastorakis, Existence, Uniqueness and Finite Difference Solution for the Dirichlet problem of the Schrodinger-Maxwell equations, WSEAS Transactions on Mathematics, Issue 4, Volume 8, April 2009, pp.177-183.
[10] Nikos E. Mastorakis, Hassan Fathabadi, On the Solution of p-Laplacian for non-Newtonian fluid flow, WSEAS Transactions on Mathematics, Issue 6, Volume 8, June 2009, pp.239-245.
[11] Alexandre Grebennikov, Fast Algorithms and MATLAB Software for Solution of the Dirichlet Boundary Value Problems for Elliptic Partial Differential Equations in Domains with Complicated Geometry, WSEAS Transactions on Mathematics, Issue 4, Volume 7, April 2008, pp.173-182.
[12] Samarskii A.A., Andreev B.G. Difference methods for elliptic equations. – M.: Nauka, 1976, p.352.
[13] Lions J.-L., Magenes E. Problèmes aux limites non homogènes et applications. – Dunod. Paris, 1968, p.372.