

Dynamical Mathematical Models for Plates and Numerical Solution of Boundary Value and Cauchy Problems for Ordinary Differential Equations

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Abstract:-in the first part there are created and justified new 2D with respect to spatial coordinates nonlinear dynamical mathematical models *von Kármán-Mindlin-Reissner(KMR)* type systems of partial differential equations for anisotropic porous, piezo, viscous elastic prismatic shells. *Truesdell-Ciarlet* unsolved(even in case of isotropic elastic plates) problem about physical soundness respect to *von Kármán* system is decided. There is find also new dynamical summand $\partial_{,i}\Delta\Phi$ (Φ is Airy stress function) in the another equation of *von Kármán* type systems too. Thus the corresponding systems in this case contains *Rayleigh-Lamb* wave processes not only in the vertical, but also in the horizontal direction. For completeness we also lead 2D *Kirchhoff-Mindlin-Reissner* type models for elastic plates of variable thickness.

Then if KMR type systems are 1D one respect to spatial coordinates at first part for numerical solution of corresponding initial-boundary value problems we consider the finite-element method using new class of B-type spline-functions. The exactness of such schemes depends from differential properties of unknown solutions: it has an arbitrary order of accuracy respect to a mesh width in case of sufficiently smoothness functions and *Sard* type best coefficients characterizing remainder proximate members on less smoothing class of admissible solutions.

Corresponding dynamical systems represent evolutionary equations for which the methods of Harmonic Analyses are nonapplicable. In this connection for Cauchy problem suggests new schemes having arbitrary order of accuracy and based on Gauss-Hermite processes. This processes are new even for ordinary differential equations.

Key-Words: - Elasticity, Poro-viscosity, Plate, Physical soundness, Finite-difference scheme, Gauss quadrature and Hermite interpolation formula,Mesh width.

1. Nonlinear dynamical mathematical models of *von Kármán-Mindlin-Reissner* type systems

One of the most principal objects in development of mechanics and mathematics is a system of nonlinear differential equations for elastic isotropic plate constructed by *von Kármán*. This system with corresponding boundary conditions represents the most essential part of the main manuals in elasticity

theory. In spite of this in 1978 *Truesdell* expressed an idea about neediness of “Physical Soundness” of *von Kármán* system. This circumstance generated the problem of justification of *von Kármán* system. Afterwards this problem is studied by many authors, but with most attention it was investigated by *Ciarlet* [1].In particular, he wrote:“the *von Kármán* equations may be given a full justification by means of the leading term of a formal asymptotic expansion”[1,p.368]. This result obviously is not suffice for a justification of “Physical Soundness” of *von Kármán* system as representations by asymptotic

expansions is dissimilar: leading terms are only coefficients of power series without any physical meaning.

Based on the [2], the method of constructing such anisotropic inhomogeneous 2D nonlinear models of *von Kármán--Mindlin-Reissner(KMR)* type for binary mixture of poro,piezo and viscous elastic thin-walled structures with variable thickness is given, by means of which terms take quite determined "Physical Soundness". The corresponding variables are quantities with certain physical meaning: averaged components of the displacement vector, bending and twisting moments, shearing forces, rotation of normals, surface efforts. In addition the corresponding equations are constructed taking into account the conditions of equality of the main vector and moment to zero. By choosing parameters in the isotropic case from *KMR* type system (having a continuum power) the system as one of the possible models is obtained. The given method differs from the classical one by the fact, that according to the classical method, one of the equations of *von Kármán* system represents one of *Saint-Venant's* compatibility conditions, i.e. it's obtained on the basis of geometry and not taking into account the equilibrium equations. This remark is essential for dynamical problems. Further for isotropic and generalized transversal elastic plates in linear case from *KMR* the unified representation for all 2D BVP (considered in terms of planar expansions and rotations) is obtained. So this report is devoted to problems of constructing the *KMR* type 2D BVP with respect to spatial variables for binary mixture of viscous-porous-elastic and piezo-electric and electrically conductive elastic thin-walled structures. At first will be introduced the nonlinear dynamic 3D (with respect to spatial variables) mathematical model for porous, piezo and viscous elastic media. At last we shall report the new iterative methods and numerical schemes for solving the corresponding BVP for 2D nonlinear systems of differential equations of *KMR* type.

Below we consider some simple (for obviousness) cases arising in the nonlinear problems of continuum mechanics and typical for seismology and structural mechanics too.

Using methodology of [2], from ch.1 (in the case when thin-walled structure is an elastic isotropic homogeneous plate with constant thickness) we have the following nonlinear systems of PDEs of *KMR* type:

$$D\Delta^2 \bar{u}_3 = \left(1 - \frac{h^2(1+2\gamma)(2-\nu)}{3(1-\nu)}\Delta\right) (g_3^+ - g_3^-) + 2h \left(1 - \frac{2h^2(1+2\gamma)}{3(1-\nu)}\Delta\right) [\bar{u}_3, \Phi^*] + h(g_{3,\alpha}^+ - g_{3,\alpha}^-)$$

$$-\int_{-h}^h \left(z f_{\alpha,\alpha} - (1 - \frac{1}{1-\nu}) \Delta (h^2 - z^2) f_3 \right) dz + R_1[\bar{u}_3; \gamma] \tag{1}$$

$$\Delta^2 \Phi^* = -\frac{E}{2} [\bar{u}_3, \bar{u}_3] + \frac{\nu}{2} \Delta (g_3^+ + g_3^-) + \frac{1+\nu}{2h} f_{\alpha,\alpha} + R_2[\Phi^*]$$

$$Q_{\alpha 3} - \frac{1+2\gamma}{3} h^2 \Delta Q_{\alpha 3} = -D \Delta \bar{u}_{3,\alpha} + \frac{h^2(1+2\gamma)}{3(1-\nu)} \partial_\alpha (g_3^+ - g_3^- + 2h(1+\nu)) [\bar{u}_3, \Phi^*] + h(g_\alpha^+ - g_\alpha^-) - \int_{-h}^h z f_\alpha dz + \frac{1+\nu}{2(1-\nu)} \int_{-h}^h (h^2 - z^2) f_{3,\alpha} dz + R_{2+\alpha} [Q_{\alpha 3}; \gamma] \tag{3}$$

The system (1) - (3) without reminder terms *R* gives 2D system of refined theories with control parameters γ . By choosing γ we got all well-known refined theories and from other γ some new ones.

Let us consider (1) equation underling the main members:

$$D' \Delta [w, \varphi] = D' ([\Delta w, \varphi] + [w, \Delta \varphi] + 2[\partial_\alpha w, \partial_\alpha \varphi]) (D' = 4h^3(1+2\gamma)/3(1-\nu)), \quad D \Delta^2 w.$$

By using for simplicity the typical relations as $\partial_{11} \varphi = \bar{\sigma}_{12}$, $\partial_{12} \varphi = -\bar{\sigma}_{12}$, $\partial_{22} \varphi = \bar{\sigma}_{11}$, the last expression may be rewritten in the following form:

$$D' \Delta [w, \varphi] = D' [(\bar{\sigma}_{11} \partial_{11} w + 2\bar{\sigma}_{12} \partial_{12} w + \bar{\sigma}_{22} \partial_{22} w) + (\partial_{11} w \Delta \bar{\sigma}_{11} + 2\partial_{12} w \Delta \bar{\sigma}_{12} + \partial_{22} w \Delta \bar{\sigma}_{22}) + 2(\bar{\sigma}_{11,\alpha} \partial_{11} w_{,\alpha} + 2\bar{\sigma}_{12,\alpha} \partial_{12} w_{,\alpha} + \bar{\sigma}_{22,\alpha} \partial_{22} w_{,\alpha})]. \tag{4}$$

The calculate and analysis by these expressions of a symbolical determinant show that the characteristic form of systems type (1) and (2) may be positive, negative or zero numbers as well as an arbitrary continuous function of x, y . Here we must remark that $ED' = 4(1+2\gamma)(1+\nu)D$, as so if $\{f\}$ denotes physical dimension of value f , it's evident $\{\Delta^2 w\} = \{\Delta[w, \Phi/E]\}$.

Thus, the first and second summands of (4) are defining the nonlinear wave processes for static cases. The structure of the third summand obviously corresponds to 2D soliton type solutions of Cortevge-Vries or Kadomtsev-Petviashvili kind.

Analogous three-dimensional nonlinear model for

anisotropic binary mixtures are presented in the works [3,4], which generalizes previously known model for poro-viscous-elastic binary mixtures. The constructed models together with certain independent scientific interest represent such form of spatial models, which allow not only to construct, but also to justify von KMR type systems as in the stationary, as well in nonstationary cases. Under justification we mean assumption of “Physical Soundness” to these models in view of *Truesdell-Ciarlet* (see for example details in [1, ch.5],[5]). As is known, even in case of isotropic elastic plate with constant thickness the subject of justification constituted an unsolved problem. The point is that *von Kármán, Love, Timoshenko, Landau & Lifshits* and others considered one of the compatibility conditions of *Saint-Venant-Beltrami* as one of the equations of the corresponding system of differential equations. This fact was verified also by *Podio-Guidugli* recently.

In the presented model we demonstrated a correct equation that is especially important for dynamic problems. The corresponding system in this case contains wave processes not only in the vertical, but also in the horizontal direction. The equations has the following form:

$$\left(\Delta^2 - \frac{1-\nu^2}{E} \rho \Delta \partial_{tt}\right) \Phi = -\frac{E}{2} [w, w] + \frac{\nu}{2} \left(\Delta - \frac{2\rho}{E} \partial_{tt}\right) (g_3^+ + g_3^-) + \frac{1+\nu}{2h} f_{\alpha,\alpha} \quad (5)$$

The first dynamical equation respect to w has the following form:

$$\begin{aligned} & (D\Delta^2 + 2h\rho\partial_{tt} - 2DE^{-1}(1+\nu)\rho\partial_{tt}\Delta)w = \\ & \left(1 - \frac{h^2(1+2\nu)(2-\nu)}{3(1-\nu)}\Delta\right)(g_3^+ - g_3^-) + \\ & 2h\left(1 - \frac{2h^2(1+2\nu)}{3(1-\nu)}\Delta\right)[u_3^*, F_*] + h(g_{\alpha,\alpha}^+ - g_{\alpha,\alpha}^-) \\ & - \int_{-h}^{+h} \left(t f_{\alpha,\alpha} - \left(1 - \frac{1}{1-\nu}\Delta(h^2 - t^2)\right) f_3 \right) dt \end{aligned} \quad (6)$$

The precision of the presented mathematical model is also conditioned by a new quantity, introduced in [2, ch.1], which describes an effect of boundary layer. Existence of this member not only explains a set of paradoxes in the two-dimensional elasticity theory (*Babushka, Lukasiewicz, Mazia, Saponjan*), but also is very important for example for process of generating cracks and holes (details see in [2], ch.1, par. 3.3). Further, let us note that in works [4] equations of (5) type are constructed with respect to certain

components of stress tensor by differentiation and summation of two differential equations. Also other equations of KMR type, which differ from (5) type equation, are equivalent to the system, where the order of each equation is not higher than two. For example, in the isotropic case, obviously, for coefficients we have $c_{\alpha\alpha} = \lambda^* + 2\mu$, $c_{66} = 2\mu$, $c_{12} = \lambda^*$, $c_{\alpha 6} = 0$, $\lambda^* = 2\lambda\mu(\lambda + 2\mu)^{-1}$, λ and μ - are the Lamé coefficients. Then the system (1.7) of [4] is presented in a form:

$$\begin{aligned} & (\lambda^* + 2\mu)\partial_1\tau + \mu\partial_2\omega = \frac{1}{2h} \bar{f}_1 + \mu(\partial_1(\bar{u}_{3,2})^2 \\ & - \partial_2(\bar{u}_{3,1}\bar{u}_{3,2})) - \frac{\lambda}{2h(\lambda + 2\mu)} \int_{-h}^h \sigma_{33,1} dz \\ & - \mu\partial_1\omega + (\lambda^* + 2\mu)\partial_2\tau = \frac{1}{2h} \bar{f}_2 + \mu(\partial_2(\bar{u}_{3,1})^2 \\ & - \partial_1(\bar{u}_{3,1}\bar{u}_{3,2})) - \frac{\lambda}{2h(\lambda + 2\mu)} \int_{-h}^h \sigma_{33,2} dz, \end{aligned} \quad (6)$$

where the functions: $\tau = \bar{\varepsilon}_{\alpha\alpha}$, $\omega = \bar{u}_{1,2} - \bar{u}_{2,1}$ correspond to plane expansion and rotation.

For variable thickness of refined theories we have (see details [2], ch.II, point 4):

$$\begin{aligned} & \frac{1}{h^3} D\partial_{\alpha} (h^3 \Delta_{3,\alpha} (x_1, x_2, \bar{h})) \\ & = \sigma_{33}^+ - \sigma_{33}^- - \int_{h_1}^{h_2} f_3 dt + \partial_{\alpha} \left[h(\sigma_{\alpha 3}^+ + \sigma_{\alpha 3}^-) - \int_{h_1}^{h_2} (t - \bar{h}) f_{\alpha} dt \right] \\ & - (h_{2,\alpha} \sigma_{\alpha 3}^+ - h_{1,\alpha} \sigma_{\alpha 3}^-) - \frac{h}{1-\nu} (\sigma_{33,\alpha}^+ h_{2,\alpha} + \sigma_{33,\alpha}^- h_{1,\alpha}) \\ & + \frac{1}{1-\nu} \int_{h_1}^{h_2} (h^2 - (t - \bar{h})^2) \Delta f_3 dt \\ & - \frac{1+\nu}{1-\nu} \int_{h_1}^{h_2} (h\partial_{\alpha} h + (t - \bar{h})\partial_{\alpha} \bar{h}) f_{3,\alpha} dt \\ & - \frac{h^2(2-\nu)}{3(1-\nu)} (1+2\nu) (\Delta\sigma_{33}^+ - \Delta\sigma_{33}^-) \\ & + \frac{h^2}{3} [\partial_{\alpha} (h + \bar{h}) \Delta\sigma_{\alpha 3}^+ + \partial_{\alpha} (h - \bar{h}) \Delta\sigma_{\alpha 3}^-] \\ & + \frac{4h^2\partial_{\alpha} h}{3} \Delta\sigma_{\alpha 3} (x_1, x_2, \bar{h}) + R_1[u_3(x_1, x_2, \bar{h}); \gamma] \end{aligned} \quad (7)$$

where

$$\begin{aligned} R_1[u_3, \gamma] = & -\frac{3D}{4h^3} \partial_{\alpha} \rho s_m [(h^2 - (t - \bar{h})^2) \Delta u_{3,\alpha}] \\ & + \rho s_m [(h\partial_{\alpha} h + (t - \bar{h})\partial_{\alpha} \bar{h}) \Delta\sigma_{\alpha 3}] \\ & - \frac{2-\nu}{1-\nu} r_1 [(t - \bar{h}) \Delta\sigma_{33}; \lambda] \end{aligned}$$

Further

$$\begin{aligned} & \frac{4h}{3}\sigma_{\alpha 3}(x_1, x_2, \hbar) - \frac{2h^3}{3}\Delta\sigma_{\alpha 3}(x_1, x_2, \hbar) \\ &= -D\Delta u_{3,\alpha}(x_1, x_2, \hbar) + \frac{2}{3}h(\sigma_{\alpha 3}^+ + \sigma_{\alpha 3}^-) \\ & - \int_{h_1}^{h_2} (t - \hbar) f_{\alpha} dt - \frac{(1+2\gamma)h^2}{3(1-\nu)}(\sigma_{33,\alpha}^- - \sigma_{33,\alpha}^+) \\ & + \frac{1+\nu}{2(1-\nu)} \int_{h_1}^{h_2} (h^2 - (t - \hbar)^2) f_{3,\alpha} dt + R_{1+\alpha}[Q_{\alpha}; \gamma] \end{aligned} \quad (8)$$

(7-8) expressions without remainder terms R_i present 2D mathematical models of *Germen-Reissner* type. They give any refined theory choosing an arbitrary parameter γ . For example, if $\gamma = 0.1$ we have *Reissner's* theory; for $\gamma = -0.5$ (7) gives immediately *Germen's* equation for variable thickness without any physical or geometrical hypotheses

Using *Dirichlet's* formula (for repeated integrals, containing ν) and quadrature formula of trapezoid, after some calculations the last relation will take the form

$$\mu\Delta u_{\alpha}(x_1, x_2, \hbar) + (\lambda^* + \mu)\partial_{\alpha} u_{\beta,\beta}(x_1, x_2, \hbar) = f_{\alpha}^h + R_{\alpha}[u_{\alpha}] \quad (9)$$

Here

$$\begin{aligned} & f_{\alpha}^h(x_1, x_2) \\ &= \frac{1}{2h} \left(\int_{h_1}^{h_2} f_{\alpha}(x_1, x_2, \hbar) dt - \sigma_{\alpha 3}^+ + \sigma_{\alpha 3}^- \right) \\ & - \frac{\lambda}{2(\lambda + 2\mu)} (\sigma_{33,\alpha}^+ + \sigma_{33,\alpha}^-) \\ & - \mu [\Delta \hbar v_{\alpha}(x_1, x_2, \hbar) + 2\partial_{\beta} \hbar v_{\alpha,\beta}(x_1, x_2, \hbar)] - (\lambda^* + \mu) \times \\ & [\partial_{\alpha\beta} \hbar v_{\beta}(x_1, x_2, \hbar) + \partial_{\alpha} \hbar v_{\beta,\beta}(x_1, x_2, \hbar) + \partial_{\beta} \hbar v_{\beta,\alpha}(x_1, x_2, \hbar)] \end{aligned}$$

$$\begin{aligned} & R_{\alpha}[u_{\alpha}(x_1, x_2, \hbar)] \\ &= \frac{1}{2h} [\mu(\rho_{tr}[(h_1 - t)\Delta v_{\alpha}] + \rho_{tr}[(h_2 - t)\Delta v_{\alpha}])] \\ & + (\lambda^* + \mu) [\rho_{tr}[(h_1 - t)v_{\beta,\alpha\beta}] + \rho_{tr}[(h_2 - t)v_{\beta,\alpha\beta}]] \\ & - \frac{\lambda}{2h(\lambda + 2\mu)} \rho_{tr}[\sigma_{33,\alpha}]. \end{aligned}$$

We must remarked also, that v_{α} unknown functions are defining from preliminary processes solving bending problem.

Thus, the system of differential equations (9) without remainder members present *Filon's* type equations

Thus, we intend to obtain the following results:

1. Nonlinear mathematical models for porous-viscous-elastic and elastic (with piezo-electric and electrically conductive processes) binary mixtures will be created and justified;
2. Questions of solvability of stationary and thermodynamical models (spatial case) will be investigated both in the linear and nonlinear anisotropic cases.
3. New two-dimensional with respect to spatial coordinates mathematical models of *KMR* type will be created and justified for poro-viscous-elastic binary mixtures when it represents a thin-walled structure; These models even in isotropic elastic case contain and justify (in sense of physical soundness) the well-known von *Kármán* system of DE for elastic plates;
4. Optimal models especially for nonhomogeneous systems of *KMR* type will be created and chosen without contracting a class of admissible solutions even in classical case;
5. Effective numerical methods will be constructed and justified; questions of convergence and error estimate will be studied for problems for thermo-poro elastic structures;
6. Questions of influence of new terms in the equation of form (5) will be investigated. Presence of these terms are very important, especially for seismic problems: in nonstationary problems these terms are of type $\partial_{tt}\Delta\Phi$, in stationary problems there are of type $\frac{\nu}{2}\Delta(q_3^+ + q_3^-)$;

2. Generalized Factorization Method

According to the article of *Vashakmadze* [1972] below for the numerical solution of BVP:

$$-(Au + qu) = \frac{d}{dt}(k(t)u'(t)) - q(t) \cdot u(t) \quad (1)$$

$$= f(t), k > 0, q \geq 0, 0 < t < 1,$$

$$u(0) - k_1 u'(0) = \alpha, u(1) + k_2 u'(1) = \beta, (k_{\alpha} \geq 0) \quad (2)$$

the method of any order of accuracy, depending on the order of the smooth of the unknown solution $u(t)$ will be given. These numerical schemes are created and published in 1972 [5] and contain as particular case the corresponding results presented by *Marchuk*[6,ch.2,point 2.2].

Preliminarily we shall put the auxiliary formulae. They are generalized (P) and (Q) formulae of [2,subsection 13.1]. Thus we suppose that $u(t) \in C^{p+1}(0,1), p = 2s + 1$.

(P) formulae have a form (the notation here and below are borrowed from [2,section 13]):

$$u(t_i) = \alpha_i^{p,1}(k)u(t_1) + \beta_i^{p,1}(k)u(t_{i+s}) - \sum_{j=2}^{p-1} b_{ij}^{p,1}(k)[Au(t_j) - R_{p-1}(t_i)], \quad i = 2, 3, \dots, p-1, \quad (3)$$

where

$$\alpha_i^{p,1}(k) = \left(\int_{t_1}^{t_p} k^{-1}(t) dt \right)^{-1} \cdot \int_{t_i}^{t_p} k^{-1}(t) dt, \quad \beta_i^{p,1}(k) = 1 - \alpha_i^{p,1}(k)$$

(Q) formulae are presented as follows:

$$u'(t_i) = \gamma_i^{p,1}(k)[u(t_p) - u(t_1)] + \sum_{j=2}^{p-1} c_{ij}^{p,1}(k)Au(t_j) + \int_{\tau_1}^{\tau_p} \int_{\tau_i}^t k(t) Ar_{p-3}(t) dt, \quad i = 1, 2, \dots, p, \quad (4)$$

$$b_{i,j}^{p,1}(\tau) = \frac{1}{\tau_p - \tau_1} \left[(\tau_p - \tau_1) \int_{\tau_1}^{\tau_j} \int_{\tau_1}^t l_j(t) dt - (\tau_i - \tau_1) \int_{\tau_1}^{\tau_p} \int_{\tau_1}^t l_j(t) dt \right] \quad (5)$$

$i, j = 2, 3, \dots, p-1,$

$$c_{i,j}^{p,1}(\tau) = \int_{\tau_1}^{\tau_p} \int_{\tau_i}^t l_j(t) dt, \quad (i = 1, 2, \dots, p, j = 2, 3, \dots, p-1),$$

$$l_j(t) = \prod_{\substack{i=2 \\ j \neq i}}^{p-1} \frac{t - t_i}{t_j - t_i}$$

Now let ω_h designate the net area, determined as follows: $\omega_h = \{0 = t_1, t_2, \dots, t_n, t_{n+1} = 1; h_i = t_i - t_{i-1}\}$.

As bounding points of the net ω_h we shall name those t_i knots, for which or $i \leq s+1$, or $i \geq n-s+1$

For relation (3) for bounding points it follows that

$$u(t_i) = \alpha_i^{i+s,1}(k)u(0) + \beta_i^{i+s,1}(k)u(t_{i+s}) - \sum_{j=2}^{2s} b_{i,j}^{i+s,1}(k)k(t_j)Au(t_j) + O(h^{2s+1}), \quad i \leq s+1; \quad (6)$$

$$u(t_i) = \alpha_i^{n+1,i-s}(k)u(t_{i-s}) + \beta_i^{n+1,i-s}(k)u(1) - \sum_{j=2}^{2s} b_{i,j}^{n+1,i-s}(k)k(t_j)Au(t_j) + O(h^{2s+1}), \quad i \geq n-s+1;$$

The above relations permit to receive expressions of a following form:

$$u(t_i) = \frac{\alpha_i}{1+k_1\gamma_1} [u(0) - k_1u'(0)] + \frac{\beta_i + k_1\gamma_1}{1+k_1\gamma_1} u(t_{i+s}) - \sum_{j=2}^{2s} \left(b_{i,j}^{i+s,1}(k) - \frac{k_1\alpha_i}{1+k_1\gamma_1} c_{i,j} \right) k(t_j)Au(t_j) + O(h^{2s+1}) \quad (7)$$

$i \leq s+1$

$$u(t_i) = \frac{\beta_i}{1+k_2\gamma_{n+1}} [u(1) + k_2u'(1)] + \frac{\alpha_i + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}} u(t_{i-s})$$

$$- \sum_{j=n-2s+1}^n \left(b_{i,j}^{i+s,1}(k) - \frac{k_2\beta_i\gamma_{n+1}}{1+k_2\gamma_{n+1}} c_{n+1,j} \right) k(t_j)Au(t_j), \quad + O(h^{2s+1}), \quad i \geq n-s+1.$$

Here and below, in the coefficients the top indexes and the dependence of factors on the function $k(t)$ are omitted. In addition the designation $h = \max_i(t_{i+1} - t_i)$ is entered. A feature of the formulae (7) is that the right parts contain same expression, (from conditions (3)), as data of initial problem. Obviously, the approach of construction of the formulae of a type (7) allows generalization for other conditions.

Let $t_i - t_{i-j} = t_{i+j} - t_i (s+2 \leq i \leq n-s)$. Then the residual member of the formula (3) allows the valuation:

$$\left| \sum_{j=i-s+1}^{i+s-1} b_{i,j}(k)AR_{p-1}(t_j) \right| < c_1 M_{p+1} h^{p+1}, \quad M_{p+1} = \max_{(0,1)} |u^{(p+1)}(t)|.$$

For interior knots $t_i \in \omega_h$ from expression (3) there follows:

$$u(t_i) = \alpha_i u(t_{i-s}) + \beta_i u(t_{i+s}) - \sum_{j=i-s+1}^{i+s-1} b_{i,j}(k)Au(t_j) + O(h^{2s+2}) \quad (8)$$

If now in the formulae (8) we replace the expression Au by $qu + f$ and then omit the remainder term, we shall obtain algebraic system of linear equations, the solution of which shall designate through u_i , ($i = 2, 3, \dots, n$).

The matrix appropriate to this system is a multi-diagonal matrix depending on s . For the solution of such systems it's easy applied the classical factorization method.

A system of equations concerning the values u_i , received from (7), for convenience we shall rewrite as:

$$u_i = \frac{\beta_i + k_1\gamma_1}{1+k_1\gamma_1} u_{i+s} + \sum_{j=2}^{2s} d_{ij} u_j + F_i, \quad i = 2, 3, \dots, s+1, \quad (9)$$

$$u_i = \alpha_i u_{i-s} + \beta_i u_{i+s} + \sum_{j=i-s+1}^{i+s-1} d_{ij} u_j + F_j, \quad i = s+2, \dots, n-s$$

$$u_i = \frac{\alpha_i + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}} u_{i-s} + \sum_{j=n-2s+1}^n d_{ij} u_j + F_i, \quad i = n-s+1, \dots, n$$

where, for example,

$$F_i = \frac{\alpha_i}{1+k_1\gamma_{1,i}} \alpha + \sum_{j=2}^{2s} d_{ij} f(t_j), \quad i \leq s+1.$$

The first s of the formulae give the following recurrence expression:

$$u_i = A_i u_{i+s} + \sum_{\substack{j=i+1 \\ j \neq i+s}}^{2s} A_{ij} u_j + B_i, \quad i = 2, 3, \dots, s+1, \quad (10)$$

where

$$A_{ij} = \frac{e_{ij}}{1 - e_{ij}}, \quad j = i+1, \dots, 2s, \quad j \neq i+s, \quad (11)$$

$$A_i = A_{i,i+s} = \frac{\beta_i + k_1 \gamma_1}{(1 - e_{ii})(1 + k_1 \gamma_1)}$$

$$e_{ii} = d_{ij} + \sum_{k=2}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{ij} \prod_{m=k}^{l-1} A_{m,m+1}, \quad \prod_{m=k}^{l-1} \cdot = 1, \quad k > l-1,$$

$$B_i = \frac{F_i + \sum_{k=2}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1}}{1 - e_{ii}}, \quad i = 2, 3, \dots, s+1.$$

Let i be the number of any internal point of the net area ω_h . Then from expressions (9-11) follows:

$$u_i = A_i u_{i+s} + \sum_{j=i+1}^{i+s-1} A_{ij} u_j + B_j, \quad i = s+2, \dots, n-s, \quad (12)$$

where

$$A_{ij} = \frac{e_{ij}}{1 - e_{ij}}, \quad j = i+1, \dots, 2s, \quad j \neq i+s, \quad (13)$$

$$A_i = A_{i,i+s} = \frac{\beta_i}{(1 - e_{ii})}$$

$$e_{ii} = d_{ij} + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{ij} \prod_{m=k}^{l-1} A_{m,m+1} + d_i A_{i-s,j},$$

$$B_i = \frac{F_i + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1} + d_i B_{i-s}}{1 - e_{ii}},$$

$$i = 2, 3, \dots, s+1.$$

The values $u_i, i = n-s+1, \dots, n$ satisfy the following equalities:

$$u_i = \sum_{j=i+1}^{i+s-1} A_{ij} u_j + B_i, \quad i = n-s+1, \dots, n-1, \quad (14)$$

where

$$A_{ij} = \frac{e_{ij}}{1 - e_{ij}}, \quad (15)$$

$$A_i = A_{i,i+s} = \frac{\beta_i}{(1 - e_{ii})}$$

$$e_{ii} = d_{ij} + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{ij} \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_i + k_2 \gamma_{n+1}}{1 + k_2 \gamma_{n+1}} A_{i-s,j},$$

$$B_i = \frac{F_i + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_2 + k_2 \gamma_{n+1}}{1 + k_2 \gamma_{n+1}} A_{i-s,j}}{1 - e_{ii}}$$

At last, the value u_n defines explicitly:

$$u_n = B_n, \quad (16)$$

$$B_n = \frac{F_n + \frac{\alpha_n + k_2 \gamma_{n+1}}{1 + k_2 \gamma_{n+1}} B_{n-s} + \sum_{k=n-s+1}^{n-1} d_{nk} \sum_{l=k}^{n-1} B_l \prod_{m=k}^{l-1} A_{m,m+1}}{1 - e_{ii}}, \quad (17)$$

$$e_{nn} = d_{nn} + \sum_{k=n-s+1}^{n-1} d_{nk} \sum_{l=k}^{n-1} A_{ln} \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_n + k_2 \gamma_{n+1}}{1 + k_2 \gamma_{n+1}} A_{n-s,n}$$

Let α_i and β_i satisfy to following bilateral inequalities:

$$\frac{1}{s} < \beta_i, \quad \alpha_{n+1-i} < \frac{1}{2}, \quad i = 2, 3, \dots, s+1,$$

$$1 - c_3 h^2 < \alpha_i \beta_i^{-1} < 1 + c_4 h^2, \quad c_3, c_4 > 0,$$

$$i = s+2, \dots, n-s,$$

where c_3, c_4 are constants. Obviously, it is possible by the appropriate choice h (see expressions for α and β from the formula (3)).

Then from (11), (13) and (15) follows:

$$A_{ij} < (1 - c_5 h) \max_{i \leq s+1} \{A_i, A_{s+2}\},$$

$$B_i < c_6 \alpha + c_7 \beta + c_8 \max_j |f(t_j)|, \quad (18)$$

where nonnegative constants c_5, c_6, c_7, c_8 do not depend on h .

The conditions (18) are the definition of stability of computing process by formulae (11), (13) and (15) concerning initial data and right part accordingly.

The stability of process (10), (12) and (14) for calculation of values u_i is also obvious, as the operator appropriate these expressions is an operator of compression. From the above stated formulae it follows that the method of generalized factorization is optimum, as the number of arithmetic operations necessary for calculation of approximate solution u_i is directly proportional to the number of points of the net areas ω_h .

3. Nonlinear Case with Newton's Boundary Conditions

We shall consider a nonlinear boundary value problem:

$$u''(x) = f(x, u(x), u'(x)), \quad -M < u, \quad u' < M, \quad (1)$$

$$k_1 u(0) - u'(0) = \alpha, \quad k_2 u(1) + u'(1) = \beta, \quad k_1^2 + k_2^2 > 0. \quad (2)$$

With the basis of generalized (P) and (Q) formulae (see [2], subsection 13.1) in this part we shall begin the construction of one-parametrical computing schemes, to an equivalent nonlinear problem (1)-(2).

Let is given uniform or Gaussian (in a sense of [2, subsection 13.2]) lying in the interval [0,1]. We

shall make out the formulae for central knots $x_{t\pm 1}$:

$$u_{t\pm 1} = \frac{1}{2}u_{(t-1)\pm 1} + \frac{1}{2}u_{(t+1)\pm 1} + A_t, \quad t = 2, 3, \dots, 2k - 2, \quad (3)$$

where

$$A_t = \sum_{j=2}^{2\tau} b_{z+1,j} u''_{(t-1)\pm 1} + O(h_{z-8}^{p+1})$$

To these formulae we shall attach expression similar to the formulae (2.7)(for preliminary section):

$$u_{z+1} = \frac{1}{2} \frac{1}{k + k_1} (k_1 u(0) - u'(0)) + \frac{1}{2} \frac{2k + k_1}{k + k_1} u_{2z+1} + A_1,$$

$$u_{(2k-1)z+1} = \quad (4)$$

$$\frac{1}{2} \frac{1}{k + k_1} (k_2 u(1) - u'(1)) + \frac{1}{2} \frac{2k + k_1}{k + k_1} u_{(2k-2)z+1} + A_{2k-1},$$

where

$$A_i = \sum_{j=2}^{2\tau} \left(b_{z+1,j} - k^2 \frac{x_{z+1}}{k + k_2} c_{1,j} \right) u''_j + O(h_{z-8}^{p+1})$$

$$A_{2k-1} = \sum_{j=2(k-2)z+2}^{2kz} \left(b_{z+1,j} + k^2 \frac{x_{z+1}}{k + k_2} c_{2z+1,j} \right) u''_j + O(h_{z-8}^{p+1})$$

The formula (3) multiplies accordingly on the uncertain multipliers $\alpha_i (i = 1, 2, \dots, 2k - 1)$ and selects these numbers so that ratios were executed:

$$u_{kz+1} = \frac{2 + k_2}{2(k_1 + k_2 + k_1 k_2)} \alpha + \frac{2 + k}{2(k_1 + k_2 + k_1 k_2)} \beta + \sigma_{kz+1}$$

$$\sigma_{kz+1} = (k_1 + k_2 + k_1 k_2)^{-1} [(2 + k_1)(k + k_1)A_1 \quad (5)_k$$

$$+ (2 + k_2) \sum_{i=2}^{k-1} (2k + ik_1)A_i + k(2 + k_1)(2 + k_2)A_k$$

$$+ (2 + k_1) \sum_{i=2}^{k-1} (2k + ik_2)A_{2k-i} + (2 + k_1)(k + k_2)A_{2k-1}],$$

$$u_{t\pm 1} = \frac{\alpha}{2k + (t+1)k_1} + \frac{2k + tk_1}{2k + (t+1)k_1} u_{(t+1)\pm 1} + \sum^{[t]},$$

$$t = 1, 2, \dots, k - 1, \quad (5)_t$$

where

$$\sum^{[t]} = \frac{2 \left[(k + k_1)A_1 + \sum_{i=2}^t (2k + ik_1)A_i \right]}{2k + (t+1)k_1},$$

$$u_{(2k-t)z+1} = \frac{\beta}{2k + (t+1)k_2} + \frac{2k + tk_1}{2k + (t+1)k_2} u_{(2k-t+1)z+1}, \quad (5)_{2k-t}$$

$$+ \sum^{[2k-t]}, \quad t = 1, 2, \dots, k - i,$$

where

$$\sum^{[2k-t]} = \frac{2 \left[(k + k_2)A_{2k-1} + \sum_{i=2}^t (2k + ik_2)A_{2k-i} \right]}{2k + (t+1)k_2},$$

From expressions (5) after some calculations, follows

$$u_{t\pm 1} = \frac{2k + (2k - t)k_2}{2k(k_1 + k_2 + k_1 k_2)} \alpha + \frac{2k + tk_1}{2k(k_1 + k_2 + k_1 k_2)} \beta + \sigma_{t\pm 1}, \quad t = 1, 2, \dots, 2k - 1, \quad (6)$$

where

$$\sigma_{t\pm 1} = \frac{2k + tk_1}{2k + kk_1} \sigma_{kz+1} + \sum_{j=t}^{k-1} \frac{2k + tk_1}{2k + jk_1} \sum^{[j]},$$

$$\sigma_{(2k-t)z+1} = \frac{2k + tk_2}{2k + kk_2} \sigma_{kz+1} + \sum_{j=t}^{k-1} \frac{2k + tk_2}{2k + jk_2} \sum^{[2k-j]}.$$

From the formulae (2.3) and (6) we can easily receive expressions, similar to (6), appropriate to the other net points of $\omega_h : x_{(t-1)z+i} (i \neq z)$. We have:

$$u_{(t-1)z+i} = \frac{2k + (2k - 2kx_i - t + 1)k_2}{2k(k_1 + k_2 + k_1 k_2)} \alpha + \frac{2k + (2kx_i + t - 1)k_1}{2k(k_1 + k_2 + k_1 k_2)} \beta + \sigma_{(t-1)z+i} \quad (5.a)$$

where

$$\sigma_{(t-1)z+i} = (1 - kx_i) \sigma_{(t-1)z+1} + kx_i \sigma_{(t+1)z+1} + \sum_{j=2}^{2s} b_{ij} \Phi_{(t-1)z+h}$$

$$(t = 2, 3, \dots, 2k - 1, i = 2, 3, \dots, z + 1).$$

If we use the formulae of type:

$$u_i = k \frac{x_{2z+1} - x_i}{k + k_1} \alpha + k \frac{1 + x_i k_1}{k + k_1} y_{2z+1} + \sum_{j=2}^{2s} \left(b_{ij} - k^2 \frac{x_{2z+1} - x_i}{k + k_1} c_{1,j} \right) u''_j + O(h_{z-8}^p)$$

and

$$u_{2kz+1-i} = k \frac{x_{2z+1} - x_i}{k + k_2} \beta + k \frac{1 + x_i k_2}{k + k_2} y_{(2k-1)z+1} + \sum_{j=2(k-1)z+2}^{2kz} \left(b_{2z+2-i,j} + k^2 \frac{x_{2z+1} - x_i}{k + k_2} c_{2z+1,j} \right) u''_j + O(h_{z-8}^p)$$

for bounding points x_i and $1 - x_i (i = 2, 3, \dots, z)$,

analogously to the last formulae we will have:

$$u_i = \frac{1 + (1 - x_i)k_2}{k_1 + k_2 + k_1 k_2} \alpha + \frac{1 + x_i k_1}{k_1 + k_2 + k_1 k_2} \beta + \sigma_i, \quad (5.b)$$

$$u_{2kz+1-i} = \frac{1 + x_i k_2}{k_1 + k_2 + k_1 k_2} \alpha + \frac{1 + (1 - x_i)k_1}{k_1 + k_2 + k_1 k_2} \beta + \sigma_{2kz+1-i},$$

where

$$\sigma_i = \frac{k + x_i k k_1}{k + k_1} \sigma_{2z+1} + \sum_{j=2}^{2s} \left(b_{ij} - k^2 \frac{x_{2z+1} - x_i}{k + k_1} c_{ij} \right) u''_j + O(h_{z-8}^p)$$

and

$$\sigma_{2kz+1-i} = \frac{k + x_i k k_2}{k + k_2} \sigma_{2(k-1)z+1}$$

