Dynamical Mathematical Models for Plates and Numerical Solution of Boundary Value and Cauchy Problems for Ordinary Differential Equations

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*Abstrac***t**:-in the first part there are created and justified new 2D with respect to spatial coordinates nonlinear dynamical mathematical models *von Kármán-Mindlin-Reissner(KMR)* type systems of partial differential equations for anisotropic porous, piezo, viscous elastic prismatic shells. *Truesdell-Ciarlet* unsolved(even in case of isotropic elastic plates) problem about physical soundness respect to *von Kármán* system is decided. There is find also new dynamical summand $\partial_{tt}\Delta\Phi$ (Φ is Airy stress function) in the another equation of *von Kármán* type systems too. Thus the corresponding systems in this case contains *Rayleigh-Lamb* wave processes not only in the vertical, but also in the horizontal direction. For comlpleteness we also lead 2D *Kirchhoff-Mindlin-Reissner* type models for elastic plates of variable thickness.
Then if KMR type systems are 1D one respect to spatial coordinates at first part for numerical solution of corresponding initial-boundary value problems we consider the finite-element method using new class of B-type splain-functions. The exactness of such schemes depends from differential properties of unknown solutions: it has an arbitrary order of accuracy respect to a mesh width in case of sufficiently smoothness functions and *Sard* type best coefficients characterizing remainder proximate members on less smoothing class of admissible solutions.
Corresponding dynamical systems represent evolutionary equations for which the methods of Harmonic Analyses are nonapplicable. In this connection for Cauchy problem suggests new schemes having arbitrary order of accuracy and based on Gauss-Hermite processes. This processes are new even for ordinary differential equations.

Key-Words: **-** Elasticity, Poro-viscosity, Plate, Physical soundness, Finite-difference scheme, Gauss quadrature and Hermite interpolation formula,Mesh width.

1. Nonlinear dynamical mathematical models of *von Kármán-Mindlin- Reissner* type systems

One of the most principal objects in development of mechanics and mathematics is a system of nonlinear differential equations for elastic isotropic plate constructed by *von Kármán.* This system with corresponding boundary conditions represents the most essential part of the main manuals in elasticity theory. In spite of this in 1978 *Truesdell* expressed an idea about neediness of “Physical Soundness”of *von Kármán* system. This circumstance generated the problem of justification of *von Kármán* system. Afterwards this problem is studied by many authors, but with most attention it was investigated by Ciarlet [1].In particular, he wrote:“the *von Kármán* equations may be given a full justification by means of the leading term of a formal asymptotic expansion”[1,p.368]. This result obviously is not suffice for a justification of “Physical Soundness” of *von Kármán* system as representations by asymptotic expansions is dissimilar: leading terms are only coefficients of power series without any physical meaning.

Based on the [2], the method of constructing such anisotropic inhomogeneous 2D nonlinear models of *von Kármán--Mindlin-Reissner(KMR)* type for binary mixture of poro,piezo and viscous elastic thin-walled structures with variable thickness is given, by means of which terms take quite determined "Physical Soundness". The corresponding variables are quantities with certain physical meaning: averaged components of the displacement vector, bending and twisting moments, shearing forces, rotation of normals, surface efforts. In addition the corresponding equations are constructed taking into account the conditions of equality of the main vector and moment to zero. By choosing parameters in the isotropic case from *KMR* type system (having a continuum power) the system as one of the possible models is obtained. The given method differs from the classical one by the fact, that according to the classical method, one of the equations of *von Kármán* system represents one of *Saint-Venant*'s compatibility conditions, i.e. it's obtained on the basis of geometry and not taking into account the equilibrium equations. This remark is essential for dynamical problems. Further for isotropic and generalized transversal elastic plates in linear case from *KMR* the unified representation for all 2D BVP (considered in terms of planar expansions and rotations) is obtained. So this report is devoted to problems of constructing the *KMR t*ype 2D BVP with respect to spatial variables for binary mixture of viscous-porous-elastic and piezo-electric and electrically conductive elastic thin-walled structures. At first will be introduced the nonlinear dynamic 3D (with respect to spatial variables) mathematical model for porous, piezo and viscous elastic media. At last we shall report the new iterative methods and numerical schemes for solving the corresponding BVP for 2D nonlinear systems of differential equations of *KMR* type.

Below we consider some simple (for obviousness) cases arising in the nonlinear problems of continuum mechanics and typical for seismology and structural mechanics too.

Using methodology of [2], from ch.1 (in the case when thin-walled structure is an elastic isotropic homogeneous plate with constant thickness) we have the following nonlinear systems of PDEs of *KMR* type:

$$D\Delta^2 \bar{u}_3 = \left(1 - \frac{h^2(1+2\gamma)(2-\nu)}{3(1-\nu)}\Delta\right)\left(g_3^+ - g_3^-\right) + 2h\left(1 - \frac{2h^2(1+2\gamma)}{3(1-\nu)}\Delta\right)[\bar{u}_3, \Phi^*] + h\left(g_{3,\alpha}^+ - g_{3,\alpha}^-\right) - \int_{-h}^{h}\left(zf_{\alpha,\alpha} - (1 - \frac{1}{1-\nu}\Delta\left(h^2 - z^2\right)f_3)\right)dz + R_1[\bar{u}_3;\gamma] \quad (1)$$

$$\Delta^2\Phi^* = -\frac{E}{2}[\bar{u}_3, \bar{u}_3] + \frac{\nu}{2}\Delta\left(g_3^+ + g_3^-\right) + \frac{1+\nu}{2h}\bar{f}_{\alpha,\alpha} + R_2\left[\Phi^*\right] \quad (2)$$

$$Q_{\alpha 3} - \frac{1+2\gamma}{3}h^2\Delta Q_{\alpha 3} = -D\Delta\bar{u}_{3,\alpha} + \frac{h^2(1+2\gamma)}{3(1-\nu)}\partial_\alpha\left(g_3^+ - g_3^- + 2h(1+\nu)\right)[\bar{u}_3, \Phi^*] + h\left(g_\alpha^+ - g_\alpha^-\right) - \int_{-h}^{h} zf_\alpha dz + \frac{1+\nu}{2(1-\nu)}\int_{-h}^{h}\left(h^2 - z^2\right)f_{3,\alpha}dz + R_{2+\alpha}\left[Q_{\alpha 3};\gamma\right]. \quad (3)$$

The system (1) - (3) without reminder terms R gives 2D system of refined theories with control parameters γ.By choosing γ we got all well-known refined theories and from other γ some new ones.

Let us consider (1) equation underling the main members:

$D'\Delta[w,\varphi] = D'\left([\Delta w,\varphi] + [w,\Delta\varphi] + 2\left[\partial_\alpha w, \partial_\alpha \varphi\right]\right)$

($D' = 4h^3(1+2\gamma)/3(1-\nu)$), $D\Delta^2 w$.

By using for simplicity the typical relations as $\partial_{11}\varphi = \bar{\sigma}_{12}$, $\partial_{12}\varphi = -\bar{\sigma}_{12}$, $\partial_{22}\varphi = \bar{\sigma}_{11}$, the last expression may be rewritten in the following form:

$$\begin{aligned} &D'\Delta[w,\varphi] = \\ &= D'[(\bar{\sigma}_{11}\partial_{11}\Delta w + 2\bar{\sigma}_{12}\partial_{12}\Delta w + \bar{\sigma}_{22}\partial_{22}\Delta w) + \\ &(\partial_{11}w\Delta\bar{\sigma}_{11} + 2\partial_{12}w\Delta\sigma_{12} + \partial_{22}w\Delta\sigma_{22}) + \\ &2\left(\bar{\sigma}_{11,\alpha}\partial_{11}w_{,\alpha} + 2\bar{\sigma}_{12,\alpha}\partial_{12}w_{,\alpha} + \bar{\sigma}_{22,\alpha}\partial_{22}w_{,\alpha}\right)]. \end{aligned} \quad (4)$$

The calculate and analysis by these expressions of a symbolical determinant show that the characteristic form of systems type (1) and (2) may be positive, negative or zero numbers as well as an arbitrary continuous function of x, y. Here we must remark that $ED' = 4(1+2\gamma)(1+\nu)D$, as so if $\{f\}$ denotes physical dimension of value f, it's evident $\{\Delta^2 w\} = \{\Delta[w,\Phi/E]\}$.

Thus, the first and second summands of (4) are defining the nonlinear wave processes for static cases. The structure of the third summand obviously corresponds to 2D soliton type solutions of Corteveg-de Vries or Kadomtsev-Petviashvili kind.

Analogous three-dimensional nonlinear model for anisotropic binary mixtures are presented in the works [3,4], which generalizes previously known model for poro–viscous-elastic binary mixtures. The constructed models together with certain independent scientific interest represent such form of spatial models, which allow not only to construct, but also to justify von KMR type systems as in the stationary, as well in nonstationary cases. Under justification we mean assumption of "Physical Soundness" to these models in view of *Truesdell-Ciarlet* (see for example details in [1, ch.5],[5]). As is known, even in case of isotropic elastic plate with constant thickness the subject of justification constituted an unsolved problem. The point is that *von Kármán, Love, Timoshenko, Landau & Lifshits* and others considered one of the compatibility conditions of *Saint-Venant-Beltrami* as one of the equations of the corresponding system of differential equations. This fact was verified also by *Podio-Guidugli* recently.

In the presented model we demonstrated a correct equation that is especially important for dynamic problems. The corresponding system in this case contains wave processes not only in the vertical, but also in the horizontal direction. The equations has the following form:

$$\left(\Delta^2-\frac{1-\nu^2}{E}\rho\Delta\partial_{tt}\right)\Phi=$$
$$-\frac{E}{2}[w,w]+\frac{\nu}{2}\left(\Delta-\frac{2\rho}{E}\partial_{tt}\right)\left(g_3^++g_3^-\right)+\frac{1+\nu}{2h}f_{\alpha,\alpha} \qquad (5)$$

The first dynamical equation respect to w has the following form:

$$\left(D\Delta^2+2h\rho\partial_{tt}-2DE^{-1}(1+\upsilon)\rho\partial_{tt}\Delta\right)w=$$
$$\left(1-\frac{h^2(1+2\gamma)(2-\upsilon)}{3(1-\upsilon)}\Delta\right)\left(g_3{}^+-g_3{}^-\right)+$$
$$2h\left(1-\frac{2h^2(1+2\gamma)}{3(1-\upsilon)}\Delta\right)\left[u_3{}^*,F_*\right]+h\left(g_{\alpha,\alpha}{}^+-g_{\alpha,\alpha}{}^-\right) \qquad (6)$$
$$-\int_{-h}^{+h}\left(tf_{\alpha,\alpha}-\left(1-\frac{1}{1-\upsilon}\Delta\left(h^2-t^2\right)\right)f_3\right)dt$$

The precision of the presented mathematical model is also conditioned by a new quantity, introduced in [2,ch.1] , which describes an effect of boundary layer. Existence of this member not only explains a set of paradoxes in the two-dimensional elasticity theory (*Babushka, Lukasievicz, Mazia, Saponjan*), but also is very important for example for process of generating cracks and holes (details see in [2], ch.1, par. 3.3). Further, let us note that in works [4] equations of (5) type are constructed with respect to certain components of stress tensor by differentiation and summation of two differential equations. Also other equations of KMR type, which differ from (5) type equation, are equivalent to the system, where the order of each equation is not higher than two. For example, in the isotropic case, obviously, for coefficients we have $c_{\alpha\alpha}=\lambda^*+2\mu$, $c_{66}=2\mu$, $c_{12}=\lambda^*$, $c_{\alpha 6}=0$, $\lambda^*=2\lambda\mu(\lambda+2\mu)^{-1}$, λ and μ -are the Lame coefficients. Then the system (1.7) of [4] is presented in a form:

$$\left(\lambda^*+2\mu\right)\partial_1\tau+\mu\partial_2\omega=\frac{1}{2h}\bar{f}_1+\mu(\partial_1(\bar{u}_{3,2})^2$$
$$-\partial_2(\bar{u}_{3,1}\bar{u}_{3,2}))-\frac{\lambda}{2h(\lambda+2\mu)}\int_{-h}^{h}\sigma_{33,1}dz$$
$$(6)$$
$$-\mu\partial_1\omega+\left(\lambda^*+2\mu\right)\partial_2\tau=\frac{1}{2h}\bar{f}_2+\mu(\partial_2(\bar{u}_{3,1})^2$$
$$-\partial_1(\bar{u}_{3,1}\bar{u}_{3,2}))-\frac{\lambda}{2h(\lambda+2\mu)}\int_{-h}^{h}\sigma_{33,2}dz,$$

where the functions: $\tau=\bar{\varepsilon}_{\alpha\alpha}$, $\omega=\bar{u}_{1,2}-\bar{u}_{2,1}$ correspond to plane expansion and rotation.

For variable thickness of refined theories we have (see details [2],ch.II,point 4):

$$\frac{1}{h^3}D\partial_\alpha\left(h^3\Delta_{3,\alpha}\left(x_1,x_2,\hbar\right)\right)$$
$$=\sigma_{33}^+-\sigma_{33}^--\int_{h_1}^{h_2}f_3dt+\partial_\alpha\left[h\left(\sigma_{\alpha 3}^++\sigma_{\alpha 3}^-\right)-\int_{h_1}^{h_2}(t-\hbar)f_\alpha dt\right]$$
$$-\left(h_{2,\alpha}\sigma_{\alpha 3}^+-h_{1,\alpha}\sigma_{\alpha 3}^-\right)-\frac{h}{1-\upsilon}\left(\sigma_{33,\alpha}^+h_{2,\alpha}+\sigma_{33,\alpha}^-h_{1,\alpha}\right)$$
$$+\frac{1}{1-\upsilon}\int_{h_1}^{h_2}\left(h^2-(t-\hbar)^2\right)\Delta f_3dt$$
$$-\frac{1+\upsilon}{1-\upsilon}\int_{h_1}^{h_2}\left(h\partial_\alpha h+(t-\hbar)\partial_\alpha\hbar\right)f_{3,\alpha}dt$$
$$-\frac{h^2(2-\upsilon)}{3(1-\upsilon)}(1+2\gamma)\left(\Delta\sigma_{33}^+-\Delta\sigma_{33}^-\right)$$
$$+\frac{h^2}{3}\left[\partial_\alpha(h+\hbar)\Delta\sigma_{\alpha 3}^++\partial_\alpha(h-\hbar)\Delta\sigma_{\alpha 3}^-\right]$$
$$+\frac{4h^2\partial_\alpha h}{3}\Delta\sigma_{\alpha 3}\left(x_1,x_2,\hbar\right)+R_1\left[u_3\left(x_1,x_2,\hbar\right);\gamma\right] \qquad (7)$$

where

$$R_1[u_3,\gamma]=-\frac{3D}{4h^3}\partial_\alpha\rho s_m\left[\left(h^2-(t-\hbar)^2\right)\Delta u_{3,\alpha}\right]$$
$$+\rho s_m\left[\left(h\partial_\alpha h+(t-\hbar)\partial_\alpha\hbar\right)\Delta\sigma_{\alpha 3}\right]$$
$$-\frac{2-\upsilon}{1-\upsilon}r_1\left[(t-\hbar)\Delta\sigma_{33};\lambda\right]$$

Further

$$\frac{4h}{3}\sigma_{\alpha 3}(x_1,x_2,\hbar)-\frac{2h^3}{3}\Delta\sigma_{\alpha 3}(x_1,x_2,\hbar)$$
$$=-D\Delta u_{3,\alpha}(x_1,x_2,\hbar)+\frac{2}{3}h\left(\sigma^+_{\alpha 3}+\sigma^-_{\alpha 3}\right)$$
$$-\int_{h_1}^{h_2}(t-\hbar)f_\alpha dt-\frac{(1+2\gamma)h^2}{3(1-\upsilon)}\left(\sigma^-_{33,\alpha}-\sigma^+_{33,\alpha}\right)$$
$$+\frac{1+\upsilon}{2(1-\upsilon)}\int_{h_1}^{h_2}\left(h^2-(t-\hbar)^2\right)f_{3,\alpha}dt+R_{1+\alpha}[Q_\alpha;\gamma] \qquad (8)$$

(7-8) expressions without remainder terms R_i present 2D mathematical models of *Germen-Reissner* type. They give any refined theory choosing an arbitrary parameter γ.For example,if $\gamma=0.1$ we have *Reissner's* theory; for $\gamma=-0.5$ (7) gives immediately Germen's equation for variable thickness without any physical or geometrical hypotheses

Using Dirichlet's formula (for repeated integrals, containing v) and quadrature formula of trapezoid, after some calculations the last relation will take the form

$$\mu\Delta u_\alpha(x_1,x_2,\hbar)+\left(\lambda^*+\mu\right)\partial_\alpha u_{\beta,\beta}(x_1,x_2,\hbar)=f^h_\alpha + R_\alpha[u_\alpha]. \qquad (9)$$

Here

$$f^h_\alpha(x_1,x_2)$$
$$=\frac{1}{2h}\left(\int_{h_1}^{h_1} f_\alpha(x_1,x_2,\hbar)dt-\sigma^+_{\alpha 3}+\sigma^-_{\alpha 3}\right)$$
$$-\frac{\lambda}{2(\lambda+2\mu)}\left(\sigma^+_{33,\alpha}+\sigma^-_{33,\alpha}\right)$$
$$-\mu\left[\Delta\hbar v_\alpha(x_1,x_2,\hbar)+2\partial_\beta\hbar v_{\alpha,\beta}(x_1,x_2,\hbar)\right]-\left(\lambda^*+\mu\right)\times$$
$$\left[\partial_{\alpha\beta}\hbar v_\beta(x_1,x_2,\hbar)+\partial_\alpha\hbar v_{\beta,\beta}(x_1,x_2,\hbar)+\partial_\beta\hbar v_{\beta,\alpha}(x_1,x_2,\hbar)\right]$$

$$R_\alpha[u_\alpha(x_1,x_2,\hbar)]$$
$$=\frac{1}{2h}\left[\mu\left(\rho_{tr}[(h_1-t)\Delta v_\alpha]+\rho_{tr}[(h_2-t)\Delta v_\alpha]\right)\right]$$
$$+\left(\lambda^*+\mu\right)\left[\rho_{tr}[(h_1-t)v_{\beta,\alpha\beta}]+\rho_{tr}[(h_2-t)v_{\beta,\alpha\beta}]\right]$$
$$-\frac{\lambda}{2h(\lambda+2\mu)}\rho_{tr}[\sigma_{33,\alpha}] .$$

We must remarked also, that v_α unknown functions are defining from preliminary processes solving bending problem.

Thus, the system of differential equations (9) without remainder members present Filon's type equations

Thus, we intend to obtain the following results:

1. Nonlinear mathematical models for porous-viscous-elastic and elastic (with piezo-electric and electrically conductive processes) binary mixtures will be created and justified;
2. Questions of solvability of stationary and thermo-dynamical models (spatial case) will be investigated both in the linear and nonlinear anisotropic cases.
3. New two-dimensional with respect to spatial coordinates mathematical models of KMR type will be created and justified for poro-viscous-elastic binary mixtures when it represents a thin-walled structure; These models even in isotropic elastic case contain and justify (in sense of physical soundness) the well-known von Kármán system of DE for elastic plates;
4. Optimal models especially for nonhomogeneous systems of KMR type will be created and chosen without contracting a class of admissible solutions even in classical case;
5. Effective numerical methods will be constructed and justified; questions of convergence and error estimate will be studied for problems for thermo-poro elastic structures;
6 .Questions of influence of new terms in the equation of form (5) will be investigated. Presence of these terms are very important, especially for seismic problems: in nonstationary problems these terms are of type $\partial_{tt}\Delta\Phi$, in stationary problems there are of type $\frac{\nu}{2}\Delta(q_3^+ + q_3^-)$;

2. Generalized Factorization Method

According to the article of Vashakmadze [1972] below for the numerical solution of BVP:

$$-\left(Au+qu\right)=\frac{d}{dt}\left(k(t)u'(t)\right)-q(t)\cdot u(t)$$
$$= f(t), k>0, q\geq 0, 0<t<1, \qquad (1)$$

$$u(0)-k_1u'(0)=\alpha,\ u(1)+k_2u'(1)=\beta,\ \left(k_\alpha\geq 0,\right) \qquad (2\ )$$

the method of any order of accuracy, depending on the order of the smooth of the unknown solution $u(t)$ will be given. These numerical schemes are created and published in 1972 [5] and contain as particular case the corresponding results presented by Marchuk[6,ch.2,point 2.2].

Preliminarily we shall put the auxiliary formulae. They are generalized (P) and (Q) formulae of [2,subsection 13.1]. Thus we suppose that $u(t)\in C^{p+1}(0,1), p=2s+1$.

(P) formulae have a form (the notation here and below are borrowed from [2,section 13]):

$$u(t_i) = \alpha_i^{p,1}(k)u(t_1) + \beta_i^{p,1}(k)u(t_{i+s})$$
$$-\sum_{j=2}^{p-1} b_{ij}^{p,1}(k)\left[Au(t_j) - R_{p-1}(t_i)\right], \quad i = 2,3,...,p-1, \qquad (3)$$

where

$$\alpha_i^{p,1}(k) = \left(\int_{t_1}^{t_p} k^{-1}(t)dt\right)^{-1} \cdot \int_{t_i}^{t_p} k^{-1}(t)dt,$$
$$\beta_i^{p,1}(k) = 1 - \alpha_i^{p,1}(k)$$

(Q) formulae are presented as follows:

$$u'(t_i) = \gamma_i^{p,1}(k)\left[u(t_p) - u(t_1)\right] + \sum_{j=2}^{p-1} c_{i,j}^{p,1}(k)Au(t_j)$$
$$+\int_{\tau_1}^{\tau_p} dt \int_{\tau_i}^{t} k(t)Ar_{p-3}(t)dt, \quad i = 1,2,...,p, \qquad (4)$$

$$b_{i,j}^{p,1}(\tau) =$$
$$\frac{1}{\tau_p - \tau_1}\left[(\tau_p - \tau_1)\int_{\tau_1}^{\tau_i} dt \int_{\tau_1}^{t} l_j(t)dt - (\tau_i - \tau_1)\int_{\tau_1}^{\tau_p} dt \int_{\tau_1}^{t} l_j(t)dt\right] \qquad (5)$$

$i, j = 2,3,...,p-1,$

$$c_{i,j}^{p,1}(\tau) = \int_{\tau_1}^{\tau_p} dt \int_{\tau_i}^{t} l_j(t)dt, \ (i = 1,2,...,p,\ j = 2,3,...,p-1),$$

$$l_j(t) = \prod_{\substack{i=2 \\ j\neq i}}^{p-1} \frac{t - t_i}{t_j - t_i}$$

Now let ω_h designate the net area, determined as follows: $\omega_h = \{0 = t_1, t_2, ..., t_n, t_{n+1} = 1; h_i = t_i - t_{i-1}\}$.

As bounding points of the net ω_h we shall name those t_i knots, for which or $i \le s+1$, or $i \ge n-s+1$

For relation (3) for bounding points it follows that

$$u(t_i) = \alpha_i^{i+s,1}(k)u(0) + \beta_i^{i+s,1}(k)u(t_{i+s})$$
$$-\sum_{j=2}^{2s} b_{i,j}^{i+s,1}(k)k(t_j)Au(t_j) + O(h^{2s+1}), \quad i \le s+1;$$
$$u(t_i) = \alpha_i^{n+1,i-s}(k)u(t_{i-s}) + \beta_i^{n+1,i-s}(k)u(1) \qquad (6)$$
$$-\sum_{j=2}^{2s} b_{i,j}^{n+1,i-s}(k)k(t_j)Au(t_j) + O(h^{2s+1}), \quad i \ge n-s+1;$$

The above relations permit to receive expressions of a following form:

$$u(t_i) = \frac{\alpha_i}{1+k_1\gamma_1}\left[u(0) - k_1u'(0)\right] + \frac{\beta_i + k_1\gamma_1}{1+k_1\gamma_1}u(t_{i+s})$$

$$-\sum_{j=2}^{2s}\left(b_{i,j}^{i+s,1}(k) - \frac{k_1\alpha_i}{1+k_1\gamma_1}c_{i,j}\right)k(t_j)Au(t_j) + O(h^{2s+1})$$
$$i \le s+1 \qquad (7)$$
$$u(t_i) = \frac{\beta_i}{1+k_2\gamma_{n+1}}\left[u(1) + k_2u'(1)\right] + \frac{\alpha_i + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}}u(t_{i-s})$$
$$-\sum_{j=n-2s+1}^{n}\left(b_{i,j}^{i+s,1}(k) - \frac{k_2\beta_i\gamma_{n+1}}{1+k_2\gamma_{n+1}}c_{n+1,j}\right)k(t_j)Au(t_j),$$
$$+O(h^{2s+1}), i \ge n-s+1.$$

Here and below, in the coefficients the top indexes and the dependence of factors on the function $k(t)$ are omitted. In addition the designation $h = \max_i(t_{i+1} - t_i)$ is entered. A feature of the formulae (7) is that the right parts contain same expression, (from conditions (3)), as data of initial problem. Obviously, the approach of construction of the formulae of a type (7) allows generalization for other conditions.

Let $t_i - t_{i-j} = t_{i+j} - t_i (s+2 \le i \le n-s)$. Then the residual member of the formula (3) allows the valuation:

$$\left|\sum_{j=i-s+1}^{i+s-1} b_{i,j}(k)AR_{p-1}(t_j)\right| < c_1M_{p+1}h^{p+1},$$
$$M_{p+1} = \max_{(0,1)}\left|u^{(p+1)}(t)\right|.$$

For interior knots $t_i \in \omega_h$ from expression (3) there follows:

$$u(t_i) = \alpha_iu(t_{i-s}) + \beta_iu(t_{i+s}) - \sum_{j=i-s+1}^{i+s-1} b_{i,j}(k)Au(t_j) \qquad (8)$$
$$+O(h^{2s+2}).$$

If now in the formulae (8) we replace the expression Au by $qu + f$ and then omit the remainder term, we shall obtain algebraic system of linear equations, the solution of which shall designate through u_i, $(i = 2,3,...,n)$.

The matrix appropriate to this system is a multi-diagonal matrix depending on s. For the solution of such systems it's easy applied the classical factorization method .

A system of equations concerning the values u_i, received from (7), for convenience we shall rewrite as:

$$u_i = \frac{\beta_i + k_1\gamma_1}{1+k_1\gamma_1}u_{i+s} + \sum_{j=2}^{2s} d_{ij}u_j + F_i, \quad i = 2,3,...,s+1, \qquad (9)$$
$$u_i = \alpha_iu_{i-s} + \beta_iu_{i+s} + \sum_{j=i-s+1}^{i+s-1} d_{ij}u_j + F_j, i = s+2,...,n-s$$
$$u_i = \frac{\alpha_i + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}}u_{i-s} + \sum_{j=n-2s+1}^{n} d_{ij}u_j + F_i, \quad i = n-s+1,...,n$$

where, for example,

$$F_i = \frac{\alpha_i}{1+k_1\gamma_{1,i}}\alpha + \sum_{j=2}^{2s} d_{ij}f(t_j), \quad i \le s+1.$$

The first s of the formulae give the following recurrence expression:

$$u_i = A_i u_{i+s} + \sum_{\substack{j=i+1 \\ j\neq i+s}}^{2s} A_{ij} u_j + B_i \,,\ i = 2,3,...,s+1\,, \qquad (10)$$

where

$$A_{ij} = \frac{e_{ij}}{1-e_{ij}}\,,\ j = i+1,...,2s, \qquad j \neq i+s\,,$$

$$A_i = A_{i,i+s} = \frac{\beta_i + k_1\gamma_1}{(1-e_{ii})(1+k_1\gamma_1)} \qquad (11)$$

$$e_{ii} = d_{ij} + \sum_{k=2}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{lj} \prod_{m=k}^{l-1} A_{m,m+1}\,,\ \prod_{m=k}^{l-1} \cdot = 1\,,\ k > l-1,$$

$$B_i = \frac{F_i + \sum_{k=2}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1}}{1-e_{ii}}\,,\ i = 2,3,....,s+1\,.$$

Let i be the number of any internal point of the net area ω_h. Then from expressions (9-11) follows:

$$u_i = A_i u_{i+s} + \sum_{j=i+1}^{i+s-1} A_{ij} u_j + B_j\,, \quad i = s+2,...,n-s\,, \qquad (12)$$

where

$$A_{ij} = \frac{e_{ij}}{1-e_{ij}}\,,\ j = i+1,...,2s, \qquad j \neq i+s\,,$$

$$A_i = A_{i,i+s} = \frac{\beta_i}{(1-e_{ii})}$$

$$e_{ii} = d_{ij} + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{lj} \prod_{m=k}^{l-1} A_{m,m+1} + d_i A_{i-s,j}\,, \qquad (13)$$

$$B_i = \frac{F_i + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1} + d_i B_{i-s}}{1-e_{ii}}\,,$$

$$i = 2,3,....,s+1\,.$$

The values $u_i, i = n-s+1,...,n$ satisfy the following equalities:

$$u_i = \sum_{j=i+1}^{i+s-1} A_{ij} u_j + B_i\,, \quad i = n-s+1,...,n-1\,, \qquad (14)$$

where

$$A_{ij} = \frac{e_{ij}}{1-e_{ij}}\,,$$

$$A_i = A_{i,i+s} = \frac{\beta_i}{(1-e_{ii})} \qquad (15)$$

$$e_{ii} = d_{ij} + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{lj} \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_i + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}} A_{i-s,j}\,,$$

$$B_i = \frac{F_i + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_2 + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}} A_{i-s,j}}{1-e_{ii}}$$

.

At last, the value u_n defines explicitly:

$$u_n = B_n\,, \qquad (16)$$

$$B_n = \frac{F_n + \frac{\alpha_n + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}} B_{n-s} + \sum_{k=n-s+1}^{n-1} d_{nk} \sum_{l=k}^{n-1} B_l \prod_{m=k}^{l-1} A_{m,m+1}}{1-e_{ii}}\,, \qquad (17)$$

$$e_{nn} = d_{nn} + \sum_{k=n-s+1}^{n-1} d_{nk} \sum_{l=k}^{n-1} A_{ln} \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_n + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}} A_{n-s,n}$$

Let α_i and β_i satisfy to following bilateral inequalities:

$$\frac{1}{s} < \beta_i\,, \quad \alpha_{n+1-i} < \frac{1}{2}\,, \quad i = 2,3,...,s+1\,,$$

$$1 - c_3 h^2 < \alpha_i \beta_i^{-1} < 1 + c_4 h^2\,, \quad c_3, c_4 > 0\,,$$

$$i = s+2,...,n-s\,,$$

where c_3, c_4 are constants. Obviously, it is possible bye the appropriate choice h (see expressions for α and β from the formula (3)).

Then from (11), (13) and (15) follows:

$$A_{ij} < (1 - c_5 h)\max_{i\leq s+1}\{A_i, A_{s+2}\}\,,$$

$$B_i < c_6\alpha + c_7\beta + c_8 \max_j \left| f(t_j) \right|\,, \qquad (18)$$

where nonnegative constants c_5, c_6, c_7, c_8 do not depent on h.

The conditions (18) are the definition of stability of computing process by formulae (11), (13) and (15) concerning initial data and right part accordingly.

The stability of process (10), (12) and (14) for calculation of values u_i is also obvious, as the operator appropriate these expressions is an operator of compression. From the above stated formulae it follows that the method of generalized factorization is optimum, as the number of arithmetic operations necessary for calculation of approximate solution u_i is directly proportional to the number of points of the net areas ω_h.

3. Nonlinear Case with Newton's Boundary Conditions

We shall consider a nonlinear boundary value problem:

$$u''(x) = f(x, u(x), u'(x)), \quad -M < u\,,\ u' < M\,, \qquad (1)$$

$$k_1 u(0) - u'(0) = \alpha, \quad k_2 u(1) + u'(1) = \beta,\ k_1^2 + k_2^2 > 0\,. (2)$$

With the basis of generalized (P) and (Q) formulae (see [2],subsection 13.1) in this part we shall begin the construction of one-parametrical computing schemes, to an equivalent nonlinear problem (1)-(2).

Let is given uniform or Gaussian (in a sense of [2,subsection 13.2]) lying in the interval [0,1]. We shall make out the formulae for central knots x_{tz+1}:

$$u_{tz+1}=\frac{1}{2}u_{(t-1)z+1}\frac{1}{2}u_{(t+1)z+1}+A_t\,,\ t=2,3,...,2k-2\,, \qquad (3)$$

where

$$A_t=\sum_{j=2}^{2z}b_{z+1,j}u''_{(t-1)z+1}+O\left(h_{z-8}^{p+1}\right)$$

To these formulae we shall attach expression similar to the formulae (2.7)(for premilinary section):

$$u_{z+1}=\frac{1}{2}\frac{1}{k+k_1}\left(k_1u(0)-u'(0)\right)+\frac{1}{2}\frac{2k+k_1}{k+k_1}u_{2z+1}+A_1\,,$$

$$u_{(2k-1)z+1}= \qquad (4)$$

$$\frac{1}{2}\frac{1}{k+k_1}\left(k_2u(1)-u'(1)\right)+\frac{1}{2}\frac{2k+k_1}{k+k_1}u_{(2k-2)z+1}+A_{2k-1}\,,$$

where

$$A_i=\sum_{j=2}^{2z}\left(b_{z+1,j}-k^2\frac{x_{z+1}}{k+k_2}c_{1,j}\right)u''_j+O\left(h_{z-8}^{p+1}\right)$$

$$A_{2k-1}=\sum_{j=2(k-2)z+2}^{2kz}\left(b_{z+1,j}+k^2\frac{x_{z+1}}{k+k_2}c_{2z+1,j}\right)u''_j+O\left(h_{z-8}^{p+1}\right)$$

The formula (3) multiplies accordingly on the uncertain multipliers $\alpha_i(i=1,2,...,2k-1)$ and selects these numbers so that ratios were executed:

$$u_{kz+1}=\frac{2+k_2}{2(k_1+k_2+k_1k_2)}\alpha+\frac{2+k}{2(k_1+k_2+k_1k_2)}\beta+\sigma_{kz+1}$$

$$\begin{aligned}\sigma_{kz+1}=&\left(k_1+k_2+k_1k_2\right)^{-1}[(2+k_1)(k+k_1)A_1 \qquad (5)_k\\&+(2+k_2)\sum_{i=2}^{k-1}(2k+ik_1)A_i+k(2+k_1)(2+k_2)A_k\\&+(2+k_1)\sum_{i=2}^{k-1}(2k+ik_2)A_{2k-i}+(2+k_1)(k+k_2)A_{2k-1}]\,,\end{aligned}$$

$$u_{tz+1}=\frac{\alpha}{2k+(t+1)k_1}+\frac{2k+tk_1}{2k+(t+1)k_1}u_{(t+1)z+1}+\Sigma^{[t]}\,,$$

$$t=1,2,...,k-1, \qquad (5)_t$$

where

$$\Sigma^{[t]}=\frac{2\left[(k+k_1)A_1+\sum_{i=2}^{t}(2k+ik_1)A_i\right]}{2k+(t+1)k_1}\,,$$

$$u_{(2k-t)z+1}=\frac{\beta}{2k+(t+1)k_2}+\frac{2k+tk_1}{2k+(t+1)k_2}u_{(2k-t+1)z+1},\ (5)_{2k-t}$$

$$+\Sigma^{[2k-t]},t=1,2,...,k-i,$$

where

$$\Sigma^{[2k-t]}=\frac{2\left[(k+k_2)A_{2k-1}+\sum_{i=2}^{t}(2k+ik_2)A_{2k-i}\right]}{2k+(t+1)k_2}\,,$$

From expressions (5) after some calculations, follows

$$u_{tz+1}=\frac{2k+(2k-t)k_2}{2k(k_1+k_2+k_1k_2)}\alpha+\frac{2k+tk_1}{2k(k_1+k_2+k_1k_2)}\beta+\sigma_{tz+1}\,,$$

$$t=1,2,...,2k-1, \qquad (6)$$

where

$$\sigma_{tz+1}=\frac{2k+tk_1}{2k+kk_1}\sigma_{kz+1}+\sum_{j=t}^{k-1}\frac{2k+tk_1}{2k+jk_1}\Sigma^{[j]}\,,$$

$$\sigma_{(2k-t)z+1}=\frac{2k+tk_2}{2k+kk_2}\sigma_{kz+1}+\sum_{j=t}^{k-1}\frac{2k+tk_2}{2k+jk_2}\Sigma^{[2k-j]}\,.$$

From the formulae (2.3) and (6) we can easily receive expressions, similar to (6), appropriate to the other net points of $\omega_h:x_{(t-1)z+i}(i\neq z)$. We have:

$$u_{(t-1)z+i}=\frac{2k+(2k-2kx_i-t+1)k_2}{2k(k_1+k_2+k_1k_2)}\alpha$$

$$+\frac{2k+(2kx_i+t-1)k_1}{2k(k_1+k_2+k_1k_2)}\beta+\sigma_{(t-1)z+i} \qquad (5.a)$$

where

$$\sigma_{(t-1)z+i}=(1-kx_i)\sigma_{(t-1)z+1}+kx_i\sigma_{(t+1)z+1}+\sum_{j=2}^{2s}b_{ij}\Phi_{(t-1)z+h}$$

$$(t=2,3,...,2k-1,i=2,3,...,z+1).$$

If we use the formulae of type:

$$u_i=k\frac{x_{2z+1}-x_i}{k+k_1}\alpha+k\frac{1+x_ik_1}{k+k_1}y_{2z+1}$$

$$+\sum_{j=2}^{2s}\left(b_{ij}-k^2\frac{x_{2z+1}-x_i}{k+k_1}c_{1j}\right)u''_j+O\left(h_{z-8}^{p}\right)$$

and

$$u_{2kz+1-i}=k\frac{x_{2z+1}-x_i}{k+k_2}\beta+k\frac{1+x_ik_2}{k+k_2}y_{(2k-1)z+1}$$

$$+\sum_{j=2(k-1)z+2}^{2kz}\left(b_{2z+2-i,j}+k^2\frac{x_{2z+1}-x_i}{k+k_2}c_{2z+1,j}\right)u''_j+O\left(h_{z-8}^{p}\right)$$

for bounding points x_i and $1-x_i$ $(i=2,3,...,z)$, analogously to the last formulae we will have:

$$u_i=\frac{1+(1-x_i)k_2}{k_1+k_2+k_1k_2}\alpha+\frac{1+x_ik_1}{k_1+k_2+k_1k_2}\beta+\sigma_i\,, \qquad (5.b)$$

$$u_{2kz+1-i}=\frac{1+x_ik_2}{k_1+k_2+k_1k_2}\alpha+\frac{1+(1-x_i)k_1}{k_1+k_2+k_1k_2}\beta+\sigma_{2kz+1-i}\,,$$

where

$$\sigma_i=\frac{k+x_ikk_1}{k+k_1}\sigma_{2z+1}+\sum_{j=2}^{2s}\left(b_{ij}-k^2\frac{x_{2z+1}-x_i}{k+k_1}c_{ij}\right)u''_j$$

$$+O\left(h_{z-8}^{p}\right)$$

and

$$\sigma_{2kz+1-i}=\frac{k+x_ikk_2}{k+k_2}\sigma_{2(k-1)z+1}$$

$$+\sum_{j=2(k-1)z+2}^{2kz}\left(b_{2z+2-i,j}+k^2\frac{x_{2z+1}-x_i}{k+k_2}c_{2z+1,j}\right)u_j''+O\left(h_{z-8}^p\right).$$

We therefore will attach the last formulae (5.a) and (5.b) to the expressions (5) and shall name such set as the formulae of a type (5).

The formulae of type (5) are difference analogue of Green's function any arbitrary (fixed) degree of exactly concerning ordinates of unknown solution (compare with Berezin, Zgidkov [7] or Schröder[8] To (5) should add the difference formulae respect derivatives of first order if the right hand function f depends of $u'(x)$.

It is evident that for these purposes use of the formulae of numerical differentiation there is inconvenient. However, if to take advantage generalized (Q) formulae (2.4) for the points $x_{(i-1)z+i}(i=1,2,...,2z+1)$ and (6) (at $t=k-1,k+1$), for the derivative we receive the following expressions:

$$u'_{(k-1)z+1}=\frac{k_1\beta-k_2\alpha}{k_1+k_2+k_1k_2}+\sigma'_{(k-1)z+i}[f] \tag{7}$$

where

$$\sigma'_{(k-1)z+i}=\frac{2}{k_1+k_2+k_1k_2}$$

$$\times\left\{k_1\left[\frac{1}{2}(2+k_2)A_k+\sum_{i=2}^{k-1}(2k+ik_2)A_{2k-i}+(k+k_2)A_{2k-1}\right]\right.$$

$$\left.-k_2\left[\frac{1}{2}k(2+k_1)A_k+\sum_{i=2}^{k-1}(2k+ik_1)A_i+(k+k_1)A_1\right]\right\}$$

$$-k\sum_{j=2}^{2z}c_{i,j}y''_{(k-1)z+j}+O\left(h^{p-1}\right).$$

The construction of the one-parameter schemes will be completed, if to expressions of a type (6) and (7) we attach two Cauchy (initial) problems:

$$u_1'(x)=f(x,\lambda(x),u_1(x)),\ l_1\le x\le 1,$$
$$u_1(l_1)=\gamma,\quad l_1=x_{(k-1)z+1},$$
$$u_1'(x)=f(x,\mu(x),u_1(x)),\ l_2\ge x\ge 0,$$
$$u_1(l_2)=\delta,\quad l_2=x_{(k+1)z+1}.$$

Now we return to study the problem (1)-(2) and introduce the following values:

$$\omega_1=\frac{1}{8}+\frac{1}{4(k_1+k_2+k_1k_2)}\left(4+k_1+k_2+\frac{(k_2-k_1)^2}{k_1+k_2+k_1k_2}\right),$$

$$\omega_2=\frac{1}{2(k_1+k_2+k_1k_2)}\left(k_1k_2+2\max\{k_1,k_2\}\right), \tag{8}$$

$$\omega_2'=\frac{1}{2}-\frac{k_1k_2}{4(k_1+k_2+k_1k_2)},\ \omega=\max\{\omega_2,\omega_2'\}.$$

The following theorem is true:

Theorem 1. *Let the function $f(x,u(x),u'(x))$ be continuous with respect to x, satisfy a Lipschitz's condition relative to u and u' with constant L and L' respectively; in addition, let one of two conditions be executed:*

$$\omega(L+L')<1,\ \omega_1L+\omega_2L'<1. \tag{9}$$

At these restrictions the initial problem has the unique solution which can be constructed by an iterative method.

The proof of this theorem coincides with the scheme of the proofs of the theorems 13.2 and 13.3 [2].

Now in the formulae of th type (6) and (7), we omit the remainder terms. We get the expressions for construction of the initial table. The Cauchy problem we shall replace by the multistage methods . We shall name the resulting system as the difference scheme.

Following theorem is true:

Theorem .2. *For the problem (1)-(2) let one of conditions (9) be true. Then:*

1) the difference scheme has a unique solution and the iteration method converges;

2) as in the case of the uniform grid $(p=3,5,7)$, as in the case of Gaussian grid $(p>3)$ convergence of the solution of the algebraic analogue to the solution of a problem (1)-(2) and its derivative has $(p-1)$-degree respect h.

Proof this theorem is similar to the proof of the theorem 13.2 [2].

The following theorem is true:

Theprem 3. *The number of arithmetic operations which is necessary for the calculation of approximate solution $\bar{u}(x)$ and its derivative $\bar{u}'(x)$ has the order $k\cdot\ln k$.*

A proof of this theorem is based on the specific character of sums σ_{tz+1}. If we calculate σ_{kz+1}, then σ_{tz+1} $\forall t\ne k$ will be calculated, as it is contained in σ_{kz+1} as subsums.

The practical convenience of a generalized factorization method constructing algebraic analogue is the following: in difference from other high accuracy methods (Tichonov, Samarski [1961]; Volkov [1971]) is not present necessity to make up the table of multiple integrals or derivatives from of initial data.

4. Numerical realizations of some difference schemes for boundary

value problems for second order ordinary differential equations

Let us consider boundary value problem for II order linear ordinary differential equations when the main part has self-conjugate form:

$$-\left(Au+qu\right)=\frac{d}{dt}\left(k(t)u'(t)\right)-q(t)\cdot u(t)=f(t), \qquad (1)$$
$$k>0, q\geq 0, 0<t<1,$$

$$k_1 u(0)-u'(0)=\alpha,\ k_2 u(1)+u'(1)=\beta,\ \left(k_\alpha \geq 0,\right) \quad (2)$$

Denote by $\omega_h=\left\{0=t_1,t_2,...,t_n,t_{n+1}=1,t_i=h\cdot(i-1),h=\frac{1}{n}\right\}$ a network, when n is arbitrary integer. Using methodology of previous points we have the following high order difference schemes, corresponding to (1)-(2) boundary value problems:

$$u_{i+m}=\alpha_{i+m}^{p+m,1+m}(k)\times u_{1+m}+\beta_{i+m}^{p+m,1+m}(k)\times u_{p+m}$$
$$+\sum_{j=2}^{p-1}\left(f_{j+m}+q_{j+m}\cdot u_{j+m}\right)$$
$$\times\left(\gamma_{i+m,j+m}(k)-\beta_{i+m}^{p+m,1+m}(k)\times\gamma_{p+m,j+m}(k)\right), \qquad (3)$$

where p is arbitrary parameter defining class of difference schemes and exactness of approximation, $\alpha_{i+m}^{p+m,1+m}(k)$, $\beta_{i+m}^{p+m,1+m}(k)$, $\gamma_{i+m,j+m}(k)$ are known coefficients

$$\alpha_{i+m}^{p+m,1+m}(k)=\frac{\int_{t_{i+m}}^{t_{p+m}}\frac{dx}{k(x)}}{\int_{t_{1+m}}^{t_{p+m}}\frac{dx}{k(x)}},\ \beta_{i+m}^{p+m,1+m}(k)=\frac{\int_{t_{1+m}}^{t_{i+m}}\frac{dx}{k(x)}}{\int_{t_{1+m}}^{t_{p+m}}\frac{dx}{k(x)}},$$

$$\gamma_{i+m,j+m}(k)=\int_{t_{1+m}}^{t_{i+m}}\frac{1}{k(x)}\int_0^x l_{j+m}(\tau)d\tau dx. \qquad (4)$$

Below we consider the case when in the schemes (3) the main parameter $p=5$. Thus we got the concrete algorithms by which we created a package of applied program. It was using for operations for numerical realizations of some typical nontrivial examples.
Let in (3): when $m=0$, $i=2$; if $m=0,1,...,n-4$, $i=3$ and if $m=n-4$, $i=4$. For $m=0$ and $m=n-4$ (3) is written for boundary points, and for other m _ for midpoints . Thus we have the algebraic system of linear $(n-1)$ equations of $(n-1)$ unknown values with five diagonal matrix. For simplicity if we introduce the values

$$q_2 b_{2,0,2}-1\equiv c_2,\ q_3 b_{2,0,3}\equiv d_2,$$
$$q_4 b_{2,0,4}\equiv e_2,\ \beta_{2,0}\equiv X,$$
$$-\left(f_2 b_{2,0,2}+f_3 b_{2,0,3}+f_4 b_{2,0,4}\right)-\alpha\times\left(\alpha_{2,0}\right)\equiv g_2$$
$$q_2 b_{3,0,2}\equiv b_3,\ q_3 b_{3,0,3}-1\equiv c_3,$$
$$q_4 b_{3,0,4}\equiv d_3,\ \beta_{3,0}\equiv e_3,$$
$$-\left(f_2 b_{3,0,2}+f_3 b_{3,0,3}+f_4 b_{3,0,4}\right)-\alpha\times\left(\alpha_{3,0}\right)\equiv g_3,$$
$$\alpha_{3,i-3}\equiv a_i,\ q_{i-1} b_{3,i-3,2}\equiv b_i,$$
$$q_i b_{3,i-3,3}-1\equiv c_i,\ q_{i+1} b_{3,i-3,4}\equiv d_i,\ \beta_{3,i-3}\equiv e_i,$$
$$-\left(f_{2+m} b_{3,m,2}+f_{3+m} b_{3,m,3}+f_{4+m} b_{3,m,4}\right)\equiv g_i\ i=4,...,n-2$$
$$\alpha_{3,n-4}\equiv a_{n-1},\ q_{n-2} b_{3,n-4,2}\equiv b_{n-1},$$
$$q_{n-1} b_{3,n-4,3}-1\equiv c_{n-1},\ q_n b_{3,n-4,4}\equiv d_{n-1},$$
$$-\left(f_{n-2} b_{3,n-4,2}+f_{n-1} b_{3,n-4,3}+f_n b_{3,n-4,4}\right)-\beta\times\left(\beta_{3,n-4}\right)\equiv g_{n-1}$$
$$\alpha_{4,n-4}\equiv Y,\ q_{n-2} b_{4,n-4,2}\equiv a_n,$$
$$q_{n-1} b_{4,n-4,3}\equiv b_n,\ q_n b_{4,n-4,4}-1\equiv c_n,$$
$$-\left(f_{n-2} b_{4,n-4,2}+f_{n-1} b_{4,n-4,3}+f_n b_{4,n-4,4}\right)-\beta\times\left(\beta_{4,n-4}\right)\equiv g_n,$$

where

$$\alpha_{i,m}=\alpha_{i+m}^{5+m,1+m}(k),\ \beta_{i,m}=\beta_{i+m}^{5+m,1+m}(k),$$
$$b_{i,m,j}=\gamma_{i+m,j+m}(k)-\beta_{i+m}^{5+m,1+m}(k)\times\gamma_{5+m,j+m}(k),$$

the system (3) will the following form:

$$u_2\times c_2+u_3\times d_2+u_4\times e_2+u_5\times X=g_2$$
$$u_2\times b_3+u_3\times c_3+u_4\times d_3+u_5\times e_3=g_3$$
$$u_{i-2}\times a_i+u_{i-1}\times b_i+u_i\times c_i+u_{i+1}\times d_i+u_{i+2}\times e_i=g_i$$
$$i=4,...,n-2 \qquad (5)$$
$$u_{n-3}\times a_{n-1}+u_{n-2}\times b_{n-1}+u_{n-1}\times c_{n-1}+u_n\times d_{n-1}=g_{n-1}$$
$$u_{n-3}\times Y+u_{n-2}\times a_n+u_{n-1}\times b_n+u_n\times c_n=g_n,$$

or

$$\begin{pmatrix} c_2 & d_2 & e_2 & X & \cdot & \cdot & 0 & 0 \\ b_3 & c_3 & d_3 & e_3 & \cdot & \cdot & 0 & 0 \\ a_4 & b_4 & c_4 & d_4 & \cdot & \cdot & 0 & 0 \\ 0 & a_5 & b_5 & c_5 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & c_{n-2} & d_{n-2} & e_{n-2} \\ 0 & 0 & 0 & \cdot & \cdot & b_{n-1} & c_{n-1} & d_{n-1} \\ 0 & 0 & 0 & \cdot & Y & a_n & b_n & c_n \end{pmatrix}\times\begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ \cdot \\ u_{n-2} \\ u_{n-1} \\ u_n \end{pmatrix}=\begin{pmatrix} g_2 \\ g_3 \\ g_4 \\ g_5 \\ \cdot \\ g_{n-2} \\ g_{n-1} \\ g_n \end{pmatrix} \qquad (6)$$

Using the corresponding representations of [1], ch.3, p.14, the system (5) was solving evidently by stable schemes.
For control of quality of above algorithms and created the standard program package below we reduce tables

of some typical test examples for $n = 10$ and for $n = 100$.

5. Standard package program (SPP)

SPP are divided into 3 main parts.

In the first part entering parameters are functions $k(t), q(t), f(t)$ and numbers $\alpha,\ \beta,\ k_1,\ k_2,\ n$. These values are defining in SPP as functions or from file *income.in*. Here also are entering functions by which we compute (4) type integral sums. $\alpha_{i,m}$, $\beta_{i,m}$ coefficients are preserved in *alfabeta.out* file, numbers $b_{i,m,j}$ keep in *bimj.out* file. Matrix of system (5) is preserved in *abcdef.out* file. In the second part the entering parameter is matrix of system (5) which must be inverted by data from *abcdef.out* file. By general factorization scheme we find this inverse matrix and construct the solution of initial system (5) preserving it in *amo.out* file. In the third part of SPP (using data from second part and the corresponding standard program from "Matlab 6.5") there are formed tables and diagrams of approximate and exact solutions, an error.

The I and II parts of SPP are using "Turbo Pascal 7.0", III part is written by "Matlab 6.5".

Problem 1. $k(t) = 1$, $q(t) = \dfrac{1}{1 + t + t^2}$, $f(t) = 1$, $\alpha = 1, \beta = 3$.

Approximate solution	Exact solution $u(t) = 1 + t + t^2$	Error
u[1]=1.00000000000	1.00000000000	0.00000000000
u[2]=1.11000000000	1.11000000000	0.00000000000
u[3]=1.24000000000	1.23999999999	0.00000000000
u[4]=1.38999999999	1.38999999999	0.00000000000
u[5]=1.56000000000	1.55999999999	0.00000000000
u[6]=1.75000000000	1.75000000000	0.00000000000
u[7]=1.96000000000	1.96000000000	0.00000000000
u[8]=2.19000000000	2.18999999999	0.00000000000
u[9]=2.44000000000	2.43999999999	0.00000000000
u[10]=2.71000000000	2.70999999999	0.00000000000
u[11]=3.00000000000	3.00000000000	0.00000000000

Problem 2: $k(t) = e^{-t}$, $q(t) = e^t$, $f(t) = e^{2t}$, $\alpha = -1$, $\beta = -2.71828182845905$.

Approximate solution	Exact solution $u(t) = -e^t$	Error
u[1]=-1.00000000000	-1.00000000000	0.00000000000
u[2]=-1.10517091807	-1.10517091807	0.00000000000
u[3]=-1.22140275815	-1.22140275815	0.00000000000
u[4]=-1.34985880757	-1.34985880757	0.00000000000
u[5]=-1.49182469763	-1.49182469764	0.00000000000
u[6]=-1.64872127070	-1.64872127069	0.00000000000
u[7]=-1.82211880038	-1.82211880039	0.00000000000
u[8]=-2.01375270747	-2.01375270746	0.00000000000
u[9]=-2.22554092848	-2.22554092849	0.00000000000
u[10]=-2.45960311115	-2.45960311115	0.00000000000
u[11]=-2.71828182845	-2.71828182845	0.00000000000

Problem 3. $k(t) = e^t + 1$, $q(t) = 2 \cdot e^t$, $f(t) = e^t$, $\alpha = 1$, $\beta = 2.71828182845905$.

Approximate solution	Exact solution $u(t) = e^t$	Error
u[45]=1.5527072232423	1.5527072185122	0.0000000047300
u[46]=1.5683121884748	1.5683121854890	0.0000000029858
u[47]=1.5840739898717	1.5840739849936	0.0000000048781
u[48]=1.5999941963404	1.5999941932168	0.0000000031235
u[49]=1.6160744072600	1.6160744021926	0.0000000050674
u[50]=1.6323162232362	1.6323162199551	0.0000000032811
u[51]=1.6487212758441	1.6487212706997	0.0000000051444
u[52]=1.6652911984283	1.6652911949458	0.0000000034824
u[53]=1.6820276549625	1.6820276497001	0.0000000052624
u[54]=1.6989323123795	1.6989323086181	0.0000000037613
u[55]=1.7160068675928	1.7160068621851	0.0000000054077

Problem 4: $k(t) = e^t$, $q(t) = \dfrac{e^t \cdot 6t^5}{1 + t^6}$, $f(t) = e^t \cdot 30t^4$, $\alpha = 1$, $\beta = 2$.

Approximate solution	Exact solution $u(t) = 1 + t^6$	Error
u[45]=1.0072563879500	1.0072563138564	0.0000000740935
u[46]=1.0083037950666	1.0083037656240	0.0000000294426
u[47]=1.0094743726131	1.0094742968958	0.0000000757172
u[48]=1.0107792447327	1.0107792153285	0.0000000294041
u[49]=1.0122306676875	1.0122305904642	0.0000000772232
u[50]=1.0138413164828	1.0138412872001	0.0000000292826
u[51]=1.0156250785456	1.0156250000000	0.0000000785456
u[52]=1.0175963169126	1.0175962878001	0.0000000291125
u[53]=1.0197706894134	1.0197706096641	0.0000000797492
u[54]=1.0221643899408	1.0221643611293	0.0000000288114
u[55]=1.0247949921024	1.0247949112963	0.0000000808061

6. Finite-difference scheme of numerical solution Cauchy problem by Gauss-Hermite processes

Let us consider Cauchy problem for ordinary differential equations

$$y'(x) = f(x, y(x)), \qquad y(0) = y_0, \quad 0 \le x \le l. \qquad (1)$$

Below we consider the problem of numerical solution of (1) by finite-difference method basing on applications of Gauss theory of quadrature folmula and Hermite interpolation process. By such way it's possible investigate Adam's type finite-difference schemes .

For simplicity and clearness we consider detailed the schemes having sixth order of accuracy respect to net step. At consideration of schemes having arbitrary order of accuracy we investigated too the processes connected with their numerical realizations. Gauss quadrature formula with 3 knots in interval $[0,h]$ has the following form:

$$\int_0^h f(x)dx = \frac{h}{18}[5f(x_{0,0}) + 8f(x_{0,1}) + 9f(x_{0,2})]$$
$$+E_{G_S,2}[f,0,h]],$$
$$x_{0,0} = \frac{h}{2}\left(1-\sqrt{0.6}\right),\ x_{0,1} = \frac{h}{2},\quad x_{0,2} = \frac{h}{2}\left(1+\sqrt{0.6}\right),$$
$$E_{G_S,2}[f;0,h] = \frac{h^7}{504000} f^{(6)}(\xi).$$

We also consider the following network ($l = nh$):

$$\omega_h = \{x_{0,0}, x_{0,1}, x_{0,2}; x_{1,0}, x_{1,1}, \ldots, x_{2n,0}, x_{2n,1}, x_{2n,2}\}$$
$$x_{k,i} = x_{0,k} + k\frac{h}{2},\ k = 0,1,\ldots,2n, i = 0,1,2.$$

Hermite interpolation formula with two knots and by ordinates and slopes as is well known has the such form:

$$f(t) = H_5(t) + R[f;t] =$$
$$\sum_{i=0}^{2}\{y_i(1-\frac{\varpi_3''(t_i)}{\varpi_3'(t_i)}(t- \ t_i)) + (t-t_i)y_i'\}L_{2i}^2(t),$$
$$+ f^{(6)}(\xi)\omega_3^{\ 2}(t)/6!,$$
$$a \le t \le b),$$

where $\omega_3(t) = \prod_{i=0}^{2}(t-t_i),\quad t\in(a,b),$

$$a < t_0 < t_1 < t_2 < b.$$

By simple calculations If $t = x + a$ we have.

$$\omega_{n+1}(t) = \omega_{n+1}(x+a) = \prod_i (x + a - x_i - a) = \omega_{n+1}(x)$$
$$\omega_{n+1}'(t) = \omega_{n+1}'(x).$$
$$\omega_{n+1}''(t) = \omega_{n+1}''(x).$$
$$L_{ni}(t) = \prod_{i\ne j}\frac{t+t_j}{t_i - t_j} = \prod_{i\ne j}\frac{x+a-x_j-a}{x_i+a-x_j-a} = L_{ni}(x).$$

We must calculate $H_5\left(x_{0,0} + h/2\right)$ by $y^{(\alpha)}\left(x_{0,i}\right), i = 0,1 y^{(\alpha)}\left(x_{0,2} - h/2\right), \alpha = 0,1$ values.

Thus if $H_5(t)$ is defining in $(k\frac{h}{2};(k+2)\frac{h}{2})$ then by (4) the ordinates in knot points of the following interval are calculated if known values we multiply on $\frac{h}{2}$. We calculate also the values $L_{2i}\left(x_{0,0} + h/2\right)$.

$$L_{20}\left(x_{0,0} + h/2\right) = -\frac{2x^2_{\ 0,0}}{\left(2x^2_{\ 0,0} - h/2\right)x_{0,0}}$$
$$= -\frac{2\left(1-\sqrt{0.6}\right)^2}{\sqrt{0.6}\left(1-2\sqrt{0.6}\right)} = 1 - \frac{\sqrt{0.6}}{0.9} \approx 0.139337,$$
$$L_{21}\left(x_{0,0} + h/2\right) == \frac{1}{1-2\sqrt{0.6}}$$
$$\approx -1.820852,$$
$$L_{22}\left(x_{0,0} + h/2\right) == \frac{hx_{0,0}}{\left(h/2 - x_{0,0}\right)x_{0,0}}$$
$$= \frac{2}{\sqrt{0.6}} \approx 2.581989.$$

These values must calculate on $\frac{h}{2}$.

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