MEASURABILITY AND GOULD INTEGRABILITY IN FINITELY PURELY ATOMIC MULTISUBMEASURE SPACES

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Abstract: In this paper, we present some results concerning measurability, Gould integrability and L^p spaces with respect to finitely purely atomic set multifunctions.

Key–words: finitely purely atomic, atom, pseudo-atom, multisubmeasure, measurable, Gould integral, L^p space.

1 Introduction

In the last years, many authors (e.g. Dobrakov [4], Drewnowski [5], Jiang and Suzuki [16], Li [17], Pap [22], Precupanu [24], Sugeno [28], Suzuki [29], Zadeh [30], Wu Congxin and Wu Cong [31], Wu and Sun [32]) investigated the non-additive field of measure theory due to its applications in mathematical economics, statistics, theory of games etc. Fuzzy measures have applications in biology, physics, medicine, theory of probabilities, human decision making, economic mathematics.

It is well-known the importance of non-additive measure theory (such as: continuity, regularity, extensions, decompositions, measures, integrals, (pseudo)atoms, non(pseudo)atomicity, purely atomicity) in fuzzy measures theory. Finiteness is an important point in mathematical research, due to its interesting applications (for example, see Mastorakis [19,20]). Finitely purely atomic measures where studied in literature in different variants (e.g. [1,2], [4], [16], [20]). For instance, Chiţescu [1,2], Leung [18] established interesting results on different classical problems concerning L^p spaces.

In [3], [8-10] and [21] we introduced and studied notions as (pseudo)atom, (non)(pseudo)atomicity, purely atomicity in the set valued case. In this paper, we continue our study, obtaining results concerning measurability and Gould integrability for finitely purely atomic set multifunctions. The Gould integral [14] was extended to the set-valued case (see [25-27], [6,7], [11,12]) and to the non-additive case (see [13]).

2 Preliminaries

$(X, \|\cdot\|)$ will be a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X, $\mathcal{P}_f(X)$ the family of all nonvoid, closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of all nonvoid, closed, bounded subsets of X, $\mathcal{P}_{bfc}(X)$ the family of all nonvoid, closed, bounded, convex subsets of X, $\mathcal{P}_{kc}(X)$ the family of all nonvoid, compact, convex subsets of X and h the Hausdorff pseudometric on $\mathcal{P}_0(X)$, which becomes a metric on $\mathcal{P}_{bf}(X)$.

It is known that

$$h(M,N) = \max\{e(M,N), e(N,M)\},$$

where

$$e(M,N) = \sup_{x\in M} d(x,N),$$

for every $M, N \in \mathcal{P}_0(X)$ is the excess of M over N and $d(x,N)$ is the distance from x to N with respect to the distance induced by the norm of X.

We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_0(X)$, where 0 is the origin of X. We have

$$|M| = h(M, \{0\}) = \sup_{x \in M} \|x\|.$$

On $\mathcal{P}_0(X)$ we consider the Minkowski addition " $\overset{\bullet}{+}$ " [15], defined by:

$$M \overset{\bullet}{+} N = \overline{M + N}, \text{ for every } M, N \in \mathcal{P}_0(X),$$

where $M + N = \{x + y | x \in M, y \in N\}$ is the classical addition of two sets and $\overline{M + N}$ is the closure of $M + N$ with respect to the topology induced by the norm of X.

Let T be an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{C}$ a ring of subsets of T.

By $i = \overline{1, n}$ we mean $i \in \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$, $\overline{\mathbb{R}}_+ = [0, +\infty]$ and $\overline{\mathbb{R}} = [-\infty, +\infty]$, where $\mathbb{R}$ is the set of all reals.

Remark 2.1. It follows from definitions that:

$$h([a, b], [c, d]) = \max(|a - c|, |b - d|),$$

for every $a, b, c, d \in \mathbb{R}$, $a < b, c < d$ (for other properties of h, see Hu and Papageorgiou [15], Petruşel and Moţ [23]).

We now recall the definitions of finitely additive measures, countably additive measures and some types of non-additive set functions.

Definition 2.2. Let $m : \mathcal{C} \to \mathbb{R}_+$ be a set function, so that $m(\emptyset) = 0$. m is said to be:

I) *monotone* if $m(A) \le m(B)$, for every $A, B \in \mathcal{C}$, $A \subseteq B$.

II) *a finitely additive measure* (shortly, *finitely additive*) if

$$m(A \cup B) = m(A) + m(B),$$

for every $A, B \in \mathcal{C}, A \cap B = \emptyset$.

III) *a submeasure* (in the sense of Drewnowski [5]) if m is monotone and

$$m(A \cup B) \le m(A) + m(B),$$

for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$ (or, equivalently, for every $A, B \in \mathcal{C}$).

IV) *null-additive* if $m(A \cup B) = m(A)$, for every $A, B \in \mathcal{C}$, with $m(B) = 0$.

V) *null-null-additive* if $m(A \cup B) = 0$, for every $A, B \in \mathcal{C}$, with $m(A) = m(B) = 0$.

VI) *null-monotone* if for every $A, B \in \mathcal{C}$, with $A \subseteq B$, $m(B) = 0$ implies $m(A) = 0$.

VII) *o-continuous* if $\lim_{n \to \infty} m(A_n) = 0$, for every $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow \emptyset$ (i.e., $A_n \supset A_{n+1}$, for every $n \in \mathbb{N}^*$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$).

VIII) *countably subadditive* if

$$m(A) \le \sum_{n=1}^{\infty} m(A_n),$$

for every sequence of pairwise disjoint sets $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$, so that $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.

IX) *a countably additive measure* (shortly *countably additive*) if

$$m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n),$$

for every sequence of pairwise disjoint sets $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$, so that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.

Definition 2.3. For a set function $m : \mathcal{C} \to \mathbb{R}_+$, with $m(\emptyset) = 0$, we introduce the following set functions:

I) *the variation* of m is the set function $\overline{m} : \mathcal{P}(T) \to \overline{\mathbb{R}}_+$, defined by

$$\overline{m}(A) = \sup\{\sum_{i=1}^{n} m(A_i)\},$$

for every $A \in \mathcal{P}(T)$, where the supremmum is taken over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{n}$, where $A_i \in \mathcal{C}$ and $A_i \subseteq A$, for every $i \in \{1, 2, \ldots, n\}$.

m is said to be *of finite variation* on $\mathcal{C}$ if $\overline{m}(A) < +\infty$, for every $A \in \mathcal{C}$.

II) $\widetilde{m} : \mathcal{P}(T) \to \overline{\mathbb{R}}_+$, defined by

$$\widetilde{m}(A) = \inf\{\overline{m}(B); A \subseteq B, B \in \mathcal{C}\},$$

for every $A \in \mathcal{P}(T)$.

Remark 2.4. Let $m : \mathcal{A} \to [0, +\infty)$ be a submeasure of finite variation. Then:

I) $m(A) \le \overline{m}(A)$, for every $A \in \mathcal{A}$;

II) $\overline{m}$ is finitely additive on $\mathcal{A}$;

III) $\widetilde{m}(A) = \overline{m}(A)$, for every $A \in \mathcal{A}$;

IV) $\widetilde{m}$ is a submeasure on $\mathcal{P}(T)$.

V) If m is σ-subadditive, then $\widetilde{m}$ is σ-subadditive on $\mathcal{P}(T)$.

Remark 2.5. Suppose $m : \mathcal{A} \to [0, +\infty)$ is a submeasure of finite variation. Then the following statements are equivalent:

(i) m is σ-subadditive.
(ii) m is o-continuous.
(iii) $\overline{m}$ is σ-additive on $\mathcal{A}$.

We extended (for instance, in [3], [6]) the concepts of Definitions 2.2 and 2.3 to set-valued case as follows.

Definition 2.6. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

(i) By $|\mu|$ we denote the extended real valued set function defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{C}$.

(ii) μ is said to be:

I) *monotone* if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

II) *a multimeasure* if $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

III) *a multisubmeasure* if μ is monotone and $\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$ (or, equivalently, for every $A, B \in \mathcal{C}$).

IV) *null-additive* if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{C}$, with $\mu(B) = \{0\}$.

V) *null–null-additive* if $\mu(A \cup B) = \{0\}$, for every $A, B \in \mathcal{C}$, with $\mu(A) = \mu(B) = \{0\}$.

VI) *null-monotone* if for every $A, B \in \mathcal{C}$, with $A \subseteq B$, $\mu(B) = \{0\}$ implies $\mu(A) = \{0\}$.

VII) *o-continuous* if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every $(A_n)_n \subset \mathcal{C}$, with $A_n \searrow \emptyset$.

VIII) *countably subadditive* if

$$|\mu(A)| \leq \sum_{n=1}^{\infty} |\mu(A_n)|,$$

for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, so that $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

IX) *a countably additive multimeasure* (shortly *countably additive*) if

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n),$$

for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, so that $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$ (that is, $\lim\limits_{n\to\infty} h(\sum_{k=1}^{n} \mu(A_k), \mu(\bigcup\limits_{n=1}^{\infty} A_n)) = 0$).

Remark 2.7. I) If μ is $\mathcal{P}_f(X)$-valued, then in Definition 2.2-II) and III) it usually appears the Minkowski addition instead of the classical addition (because the sum of two closed sets is not, generally, a closed set).

II) Any monotone multimeasure is, particularly, a multisubmeasure. Any multisubmeasure is null-additive. Any null-additive set multifunction is null-null-additive. If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is monotone, then it is also null-monotone. The converses are not generally valid, as observed in [3].

III) If $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is a multi(sub)measure, then $|\mu|$ is a submeasure.

Definition 2.8. For a set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$, with $\mu(\emptyset) = \{0\}$, we consider *the variation* of μ to be the set function $\overline{\mu} : \mathcal{P}(T) \to \overline{\mathbb{R}}_+$, defined by

$$\overline{\mu}(A) = \sup\{\sum_{i=1}^{n} |\mu(A_i)|\},$$

for every $A \in \mathcal{P}(T)$, where the supremmum is taken over all finite families of pairwise disjoint sets $\{A_i\}_{i=\overline{1,n}}$, where $A_i \in \mathcal{C}$ and $A_i \subseteq A$, for every $i \in \{1, 2, \ldots, n\}$.

μ is said to be *of finite variation on* $\mathcal{C}$ if $\overline{\mu}(A) < +\infty$, for every $A \in \mathcal{C}$.

Remark 2.9. I) $|\mu(A)| \leq \overline{\mu}(A)$, for every $A \in \mathcal{C}$.

II) $\overline{\mu}$ is monotone and super-additive on $\mathcal{P}(T)$. Also (see [6,7]), if $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a multi(sub)measure, then $\overline{\mu}$ is finitely additive on $\mathcal{C}$.

III) ([26]) If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is of finite variation, then μ is $\mathcal{P}_{bf}(X)$-valued.

Remark 2.10. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction, $A \in \mathcal{C}$ and the following statements:

(i) $\mu(A) = \{0\}$.
(ii) $|\mu(A)| = 0$.
(iii) $\overline{\mu}(A) = 0$.

Then (i)⇔(ii) and (iii)⇒(ii).

Moreover, if μ is null-monotone, then (ii)⇒(iii).

Definition 2.11. Let $\mu, \nu : \mathcal{C} \to \mathcal{P}_0(X)$. We say that μ is *absolutely ν-continuous* (denoted by $\mu << \nu$) if $\nu(A) = \{0\} \Longrightarrow \mu(A) = \{0\}$, $A \in \mathcal{C}$.

Remark 2.12. Let $\mu, \nu : \mathcal{C} \to \mathcal{P}_0(X)$ be null-monotone. The following statements are equivalent:

i) $\mu << \nu$;
ii) $|\mu| << |\nu|$;
iii) $\overline{\mu} << \overline{\nu}$.

Remark 2.13. Suppose $T \in \mathcal{C}$ and μ is a multisubmeasure, so that $\overline{\mu}$ is countably additive and $\overline{\mu}(T) > 0$. Then we can generate a system of upper and lower probabilities (with applications in statistical inference - see Dempster [3]) in the following way:

Let $\mathcal{A} = \{E \subset X | \mu^{-1}(E), \mu^{+1}(E) \in \mathcal{C}\}$, where for every $E \subset X$,

$$\mu^{-1}(E) = \{t \in T | \mu(\{t\}) \cap E \neq \emptyset\}$$

and $\mu^{+1}(E) = \{t \in T | \mu(\{t\}) \subset E\}$. For every $E \in \mathcal{A}$, we define *the upper probability* of E to be

$$P^*(E) = \frac{\overline{\mu}(\mu^{-1}(E))}{\overline{\mu}(T)}$$

and *the lower probability* of E to be

$$P_*(E) = \frac{\overline{\mu}(\mu^{+1}(E))}{\overline{\mu}(T)}.$$

We remark that P^*, $P_* : \mathcal{A} \to [0,1]$ and $P_*(E) \leq P^*(E)$, for every $E \in \mathcal{A}$.

One may regard $\overline{\mu}(\mu^{-1}(E))$ as the largest possible amount of probability from the measure $\overline{\mu}$ that can be transferred to outcomes $x \in E$ and $\overline{\mu}(\mu^{+1}(E))$ as the minimal amount of probability that can be transferred to outcomes $x \in E$.

3 Finitely purely atomic set multifunctions

In the sequel, $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a set multifunction, with $\mu(\emptyset) = \{0\}$.

Definition 3.1. [3, 8-10] I) A set $A \in \mathcal{C}$ is said to be an *atom (pseudo-atom*, respectively) of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$ ($\mu(A) = \mu(B)$, respectively).

II) μ is called

i) *finitely purely (pseudo)atomic* if there is a finite disjoint family $(A_i)_{i=\overline{1,n}} \subset \mathcal{C}$ of (pseudo)atoms of μ so that $T = \bigcup_{i=1}^{n} A_i$.

ii) *purely (pseudo)atomic* if there is at most a countable number of (pseudo)atoms $(A_n)_n \subset \mathcal{C}$ of μ so that $\mu(T \backslash \bigcup_{n=1}^{\infty} A_n) = \{0\}$ (here $\mathcal{C}$ is a σ-algebra).

iii) *non-(pseudo)atomic* if it has no (pseudo)atoms.

Remark 3.2. Suppose μ is monotone. Then the following statements hold:

I) μ is non-atomic if and only if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there exists $B \in \mathcal{C}$ such that $B \subseteq A$, $\mu(B) \supsetneq \{0\}$ and $\mu(A \backslash B) \supsetneq \{0\}$.

II) μ is non-pseudo-atomic if and only if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there exists $B \in \mathcal{C}$ such that $B \subseteq A$, $\mu(B) \supsetneq \{0\}$ and $\mu(B) \subsetneq \mu(A)$.

Using Remark 2.6, we easily obtain the following remark.

Remark 3.3. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be monotone with $\mu(\emptyset) = \{0\}$, $A \in \mathcal{C}$ and the following statements:
(i) A is an atom of μ.
(ii) A is an atom of $|\mu|$.
(iii) A is an atom of $\overline{\mu}$.
Then (i)⇔(ii)⇔(iii).

Remark 3.4. Let $\mu, \nu : \mathcal{C} \to \mathcal{P}_0(X)$ be set multifunctions, so that $\mu(\emptyset) = \nu(\emptyset) = \{0\}$. If $\mu \ll \nu$ and $A \in \mathcal{C}$ is an atom of ν with $\mu(A) \supsetneq \{0\}$, then A is an atom of μ, too.

Example 3.5.

I) Let $T = \{a, b, c\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ defined by

$$\mu(A) = \begin{cases} \{0\}, & A = \emptyset \\ [0,1], & A \neq \emptyset \end{cases}$$

for every $A \in \mathcal{C}$. Let $A = \{a, b\}$. There is $B = \{a\} \subseteq A$, so that $\mu(B) = [0,1] \supsetneq \{0\}$ and $\mu(A \backslash B) = \mu(\{b\}) = [0,1] \supsetneq \{0\}$. So, A is not an atom of μ. We prove that A is a pseudo-atom of μ. Indeed, for every $E \in \mathcal{C}$, $E \subseteq A$, we have $\mu(E) = \{0\}$, for $E = \emptyset$ and $\mu(E) = [0,1] = \mu(A)$ for $E \neq \emptyset$, which shows that A is a pseudo-atom of μ.

II) Let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ defined by

$$\mu(A) = \begin{cases} \{0\}, & A = \emptyset \text{ or } A = \{a\} \\ \{1, 8\}, & A = \{b\} \\ \{0, 7, 9\}, & A = \{a, b\} \end{cases}$$

for every $A \in \mathcal{C}$.

We observe that T is not a pseudo-atom of μ because there is $B = \{b\} \subseteq T$, so that $\mu(B) \neq \{0\}$ and $\mu(B) \neq \mu(T)$. We now prove that T is an atom of μ. Let $E \in \mathcal{C}$, $E \subseteq T$.

We have the following situations:
(i) $E = \emptyset \Rightarrow \mu(E) = \{0\}$.
(ii) $E = T \Rightarrow \mu(T \backslash E) = \mu(\emptyset) = \{0\}$.
(iii) $E = \{a\} \Rightarrow \mu(E) = \{0\}$.
(iv) $E = \{b\} \Rightarrow \mu(T \backslash E) = \mu(\{a\}) = \{0\}$.
So, T is an atom of μ.

Proposition 3.6. *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be monotone with $\mu(\emptyset) = \{0\}$ and $A \in \mathcal{C}$. If A is an atom of μ, then $\overline{\mu}(A) = |\mu(A)|$.*

Proof. According to Remark 2.5-I), it is sufficient to prove that $\overline{\mu}(A) \leq |\mu(A)|$. Let $\{B_i\}_{i=1}^n \subset \mathcal{C}$ a partition of A. That is, $A = \bigcup_{i=1}^n B_i$ and $\{B_i\}_{i=1}^n$ are mutually disjoint. We may have the following cases:

I) $\mu(B_i) = \{0\}$, for every $i \in \{1, 2, \ldots, n\}$. Then

$$\sum_{i=1}^n |\mu(B_i)| = 0 \leq |\mu(A)|.$$

II) There is an unique $i_0 \in \{1, 2, \ldots, n\}$, let say $i_0 = 1$, such that $\mu(B_1) \supsetneq \{0\}$ and $\mu(B_2) = \mu(B_3) = \ldots = \mu(B_n) = \{0\}$. It follows $|\mu(B_1)| > 0$ and $\sum_{i=2}^n |\mu(B_i)| = 0$. Since μ is monotone, we have

$$B_1 \subseteq A_1 \Rightarrow \mu(B_1) \subseteq \mu(A) \Rightarrow |\mu(B_1)| \leq |\mu(A)|.$$

This implies

$$\sum_{i=1}^n |\mu(B_i)| = |\mu(B_1)| \leq |\mu(A)|.$$

III) Suppose $\mu(B_1) \supsetneq \{0\}$ and $\mu(B_2) \supsetneq \{0\}$. Since A is an atom of μ, it results $\mu(A \backslash B_1) = \{0\}$. But $B_2 \subseteq A \backslash B_1$ and μ is monotone. So, $\mu(B_2) = \{0\}$, a contradiction.

Finally, we have

$$\sum_{i=1}^n |\mu(B_i)| \leq |\mu(A)|,$$

for every partition $\{B_i\}_{i=1}^n$ of A.

It follows $\overline{\mu}(A) \leq |\mu(A)|$ and the proof is finished. □

Remark 3.7. Suppose $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is monotone and null-additive and $A \in \mathcal{C}$ is an atom of μ. If $\{B_i\}_{i=1}^n \in \mathcal{C}$ is a partition of A, then there is an unique $i_0 \in \{1, 2, \ldots, n\}$, let say $i_0 = 1$, such that $\mu(B_1) = \mu(A)$ and $\mu(B_2) = \mu(B_3) = \ldots = \mu(B_n) = \{0\}$.

Indeed, we may have the following situations:

I) $\mu(B_i) = \{0\}$, for every $i \in \{1, 2, \ldots, n\}$. Since μ is null-additive, we have

$$\mu(A) = \mu(B_1 \cup \ldots \cup B_{n-1}) =$$

$$= \mu(B_1 \cup \ldots \cup B_{n-2}) = \ldots = \mu(B_1) = \{0\},$$

false.

II) Suppose $\mu(B_1) \supsetneq \{0\}$. As in the case III from the proof of Proposition 3.4, it results $\mu(B_2) = \{0\}$, which is false.

III) There exists an unique $i_0 \in \{1, 2, \ldots, n\}$, let say $i_0 = 1$, such that $\mu(B_1) \supsetneq \{0\}$ and $\mu(B_2) = \mu(B_3) = \ldots = \mu(B_n) = \{0\}$. Since A is an atom of μ, it follows $\mu(A \backslash B_1) = \{0\}$.

By the null-additivity of μ, it results

$$\mu(A) = \mu(B_1 \cup (A \backslash B_1)) = \mu(B_1),$$

as claimed.

The same result is valid for a null-additive monotone set function $m : \mathcal{A} \to [0, +\infty)$.

Remark 3.8. I) Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be monotone. The following statements are equivalent:

i) μ is (finitely) purely atomic;

ii) $|\mu|$ is (finitely) purely atomic;

iii) $\overline{\mu}$ is (finitely) purely atomic.

II) [3] If μ is null-additive, then any atom of μ is, particularly, a pseudo-atom of μ. Consequently, any null-additive, (finitely) purely atomic set multifunction is (finitely) purely pseudo-atomic.

III) If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a multimeasure, then $A \in \mathcal{C}$ is an atom of μ if and only if it is a pseudo-atom. Consequently, in this case, μ is (finitely) purely atomic if and only if it is (finitely) purely pseudo-atomic.

IV) If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is finitely purely (pseudo)atomic, then it is also purely (pseudo)atomic.

Theorem 3.9. *Let $m : \mathcal{C} \to \mathbb{R}_+$ be a finitely additive set function and $\mu : \mathcal{C} \to \mathcal{P}_0(\mathcal{L}^\infty(m))$ defined by $\mu(A) = [0, \aleph_A]$, for every $A \in \mathcal{C}$, where $\aleph_A$ is the characteristic function of A and*

$$[f, g] = \{u | u \in \mathcal{L}^\infty(m), f \leq u \leq g\},$$

for every $f, g \in \mathcal{L}^\infty(m)$ so that $f \leq g$. Then μ is countably additive if and only if m is finitely purely atomic.

Proof. Let $\nu : \mathcal{C} \to \mathcal{L}^\infty(m)$, defined for every $A \in \mathcal{C}$ by $\nu(A) = \aleph_A$. We observe that $\nu(\emptyset) = 0$ and

$$\nu(A \cup B) = \chi_{A \cup B} = \chi_A + \chi_B = \nu(A) + \nu(B),$$

for every disjoint sets $A, B \in \mathcal{C}$. So, ν is finitely additive. Then, by [1,2], ν is countably additive if and only if m is finitely purely atomic. We also remark that μ is countably additive if and only if ν is countably additive. Indeed, let $(A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C}$ be a sequence of pairwise disjoint sets such that $\bigcup_{n=1}^\infty A_n \in \mathcal{C}$. Denote $f_n = \chi_{\bigcup_{k=1}^n A_k}$, for every $n \in \mathbb{N}^*$ and $f = \chi_{\bigcup_{n=1}^\infty A_n}$.

Observe that $0 \leq f_n \leq f$, for each $n \in \mathbb{N}^*$. Using Remark 2.1, we have

$$\nu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n) \Leftrightarrow f_n \overset{\mathcal{L}^{\infty}(m)}{\longrightarrow} f \Leftrightarrow$$

$$\Leftrightarrow h([0, \sum_{k=1}^{n} \nu(A_k)], [0, \nu(\bigcup_{n=1}^{\infty} A_n)]) \to 0 \Leftrightarrow$$

$$\Leftrightarrow h(\sum_{k=1}^{n} \mu(A_k), \mu(\bigcup_{n=1}^{\infty} A_n)) \to 0 \Leftrightarrow$$

$$\Leftrightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

So, the conclusion follows. □

4 Measurability and Gould integrability for finitely purely atomic set multifunctions

In what follows, without any special assumptions, suppose $\mathcal{A}$ is an algebra of subsets of an abstract space T, X is a Banach space, $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is a set multifunction of finite variation, with $\mu(\emptyset) = \{0\}$ and $f : T \to \mathbb{R}$ is a bounded function. We recall from [26, 27] the following notions and results.

Definition 4.1. I) A *partition* of T is a finite family $P = \{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup_{i=1}^{n} A_i = T$.

II) Let $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ be two partitions of T. P' is said to be *finer than* P, denoted $P \leq P'$ (or $P' \geq P$), if for every $j = \overline{1,m}$, there exists $i_j = \overline{1,n}$ so that $B_j \subseteq A_{i_j}$.

III) The *common refinement* of two partitions $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ is the partition $P \wedge P' = \{A_i \cap B_j\}_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$.

Obviously, $P \wedge P' \geq P$ and $P \wedge P' \geq P'$.

We denote by $\mathcal{P}$ the class of all partitions of T and if $A \in \mathcal{A}$ is fixed, by $\mathcal{P}_A$, the class of all partitions of A.

Definition 4.2. For a set multifunction $\mu : \mathcal{A} \to \mathcal{P}_0(X)$, we introduce the set function $\widetilde{\mu}$ defined by:

$$\widetilde{\mu}(A) = \inf\{\overline{\mu}(B); A \subseteq B, B \in \mathcal{A}\},$$

for every $A \subseteq T$.

Remark 4.3. Since $\overline{\mu}$ is monotone, then $\widetilde{\mu}(A) = \overline{\mu}(A)$, for every $A \in \mathcal{A}$. Consequently, $\widetilde{\mu}(A) \geq |\mu(A)|$, for every $A \in \mathcal{A}$.

Remark 4.3 implies the following result.

Proposition 4.4. *Let $\mu : \mathcal{A} \to \mathcal{P}_0(X)$ be a set multifunction. If $\widetilde{\mu}$ is o-continuous on $\mathcal{P}(T)$, then μ is o-continuous on $\mathcal{A}$.*

Definition 4.5. I) f is said to be *$\widetilde{\mu}$-totally-measurable on* $(T, \mathcal{A}, \mu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

$$\begin{cases} i)\ \widetilde{\mu}(A_0) < \varepsilon \ \text{ and} \\ ii)\ \sup\limits_{t,s \in A_i} |f(t) - f(s)| = osc(f, A_i) < \varepsilon, \\ \text{for every } i = \overline{1,n}. \end{cases}$$

II) f is said to be *$\widetilde{\mu}$-totally-measurable on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is $\widetilde{\mu}$-totally measurable on $(B, \mathcal{A}_B, \mu_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $\mu_B = \mu|_{\mathcal{A}_B}$.

Remark 4.6. If f is $\widetilde{\mu}$-totally-measurable on T, then f is $\widetilde{\mu}$-totally-measurable on every $A \in \mathcal{A}$.

Definition 4.7. [26, 27] For a bounded function $f : T \to \mathbb{R}$ we denote

$$\sigma_{f,\mu}(P) = {}^{\bullet}\sum_{i=1}^{n} f(t_i)\mu(A_i)$$

(or, if there is no doubt, $\sigma_f(P), \sigma_\mu(P)$ or $\sigma(P)$), for every $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ and every $t_i \in A_i, i = \overline{1,n}$.

I) f is said to be *μ-integrable on* T if the net $(\sigma(P))_{P \in (\mathcal{P}, \leq)}$ is convergent in $(\mathcal{P}_f(X), h)$, where $\mathcal{P}$ is ordered by the relation " $\leq$ " given in Definition 4.1.

If $(\sigma(P))_{P \in (\mathcal{P}, \leq)}$ is convergent, then its limit is called the *integral of f on T with respect to μ*, denoted by $\int_T f d\mu$.

II) If $B \in \mathcal{A}, f$ is said to be *μ-integrable on* B if the restriction $f|_B$ of f to B is μ-integrable on $(B, \mathcal{A}_B, \mu_B)$.

Remark 4.8. [26, 27] I) f is μ-integrable on T if and only if there exists a set $I \in \mathcal{P}_{bf}(X)$ such that for every $\varepsilon > 0$, there exists a partition P_ε of T, so that for every other partition of T, $P = \{A_i\}_{i=\overline{1,n}}$, with

$P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have

$$h(\sigma(P), I) < \varepsilon.$$

II) If μ is $\mathcal{P}_{kc}(X)$-valued, then $\int_T f d\mu \in \mathcal{P}_{kc}(X)$.

III) According to [13], if $m : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation and $f : T \to \mathbb{R}$ is bounded, then f is $\widetilde{m}$-totally-measurable if and only if f is m-integrable.

Theorem 4.9. *Suppose (T, ρ) is a compact metric space, $\mathcal{B}$ is the Borel δ-ring generated by the compact subsets of T, $f : T \to \mathbb{R}$ is continuous on T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is finitely purely atomic, null-additive and monotone. Then f is $\widetilde{\mu}$-totally-measurable on every atom $A_i, i = \overline{1,p}$ (where $T = \bigcup_{i=1}^{p} A_i$).*

Proof. Since μ is monotone and null-additive, by Theorem 5.2-[3], there is an unique $a_1 \in A_1$ so that $\mu(A_1 \backslash \{a_1\}) = \{0\}$.

Consider an arbitrary partition $\{B_1, B_2, ..., B_n\}$ of A_1. According to Remark 3.7, we may suppose that $\mu(B_1) = \mu(A_1)$ and $\mu(B_2) = ... = \mu(B_n) = \{0\}$.

Since f is continuous in a_1, then for every $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ so that for every $t \in A_1$, with $\rho(t, a_1) < \delta_\varepsilon$, we have $|f(t) - f(a_1)| < \frac{\varepsilon}{3}$.

Let $B_\varepsilon = \{t \in A_1; \rho(t, a_1) < \delta_\varepsilon\} \subset A_1$. We observe that $B_\varepsilon \in \mathcal{B}$. Because A_1 is an atom, we have $\mu(B_\varepsilon) = \{0\}$ or $\mu(A_1 \backslash B_\varepsilon) = \{0\}$.

I. If $\mu(B_\varepsilon) = \{0\}$, then since $a_1 \in B_\varepsilon$, we get $\mu(\{a_1\}) = \{0\}$. But $\mu(A_1 \backslash \{a_1\}) = \{0\}$, so $\mu(A_1) = \{0\}$, a contradiction.

II. If $\mu(A_1 \backslash B_\varepsilon) = \{0\}$, then

$$\mu(B_1 \backslash B_\varepsilon) = ... = \mu(B_n \backslash B_\varepsilon) = \{0\}.$$

The partition $P_{A_1} = \{B_\varepsilon, B_1 \backslash B_\varepsilon, ..., B_n \backslash B_\varepsilon\}$ assures the $\widetilde{\mu}$-totally-measurability of f on A_1.

We make similar considerations for any $A_i, i = \overline{2,p}$. □

Proposition 4.10. *Let $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ be a multi(sub)measure and $A, B \in \mathcal{A}$. Then f is $\widetilde{\mu}$-totally-measurable on $A \cup B$ if and only if it is $\widetilde{\mu}$-totally-measurable on A and B.*

Proof. According to Remark 4.6, the *if part* is straightforward. For the *only if part*, suppose first that $A \cap B = \emptyset$. By the $\widetilde{\mu}$-totally-measurability of f on A and B, there are $P_\varepsilon^A = \{A_i\}_{i=\overline{0,n}} \in \mathcal{P}_A$ and $P_\varepsilon^B = \{B_j\}_{i=\overline{0,q}} \in \mathcal{P}_B$ satisfying condition (M). Since $\overline{\mu}$ is additive on $\mathcal{A}$, then $P_\varepsilon^{A \cup B} = \{A_0 \cup B_0, A_1, ..., A_n, B_1, ..., B_q\} \in \mathcal{P}_{A \cup B}$ also satisfies condition (M), so f is $\widetilde{\mu}$-totally-measurable on $A \cup B$.

If A and B are not disjoint, since $A \cup B = (A \backslash B) \cup B$ and $\widetilde{\mu}$-totally-measurability is hereditary, the statement is proved. □

Remark 4.11. Under the assumptions of Proposition 4.10, let $\{A_i\}_{i=\overline{1,p}} \subset \mathcal{A}$. Then f is $\widetilde{\mu}$-totally-measurable on $\bigcup_{i=1}^{p} A_i$ if and only if the same is f on every $A_i, i = \overline{1,p}$.

By Remark 4.11 and Theorem 4.9, we immediately get:

Corollary 4.12. *Suppose T is a compact metric space, $f : T \to \mathbb{R}$ is continuous on T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a finitely purely atomic multisubmeasure. Then f is $\widetilde{\mu}$-totally-measurable on T.*

Theorem 4.13. *Suppose $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is monotone and null-additive. If f is $\widetilde{\mu}$-totally-measurable on T and $A \in \mathcal{A}$ is an atom of μ, then f is μ-integrable on A.*

Proof. First, we observe that, if A is an atom of μ and if $\{A_i\}_{i=\overline{1,n}} \in \mathcal{P}_A$, then, there exists only one set, for instance, without any loss of generality, A_1, so that $\mu(A_1) \supsetneq \{0\}$ and $\mu(A_2) = ... = \mu(A_n) = \{0\}$ (according to Remark 3.7).

Let $A \in \mathcal{A}$ be an atom of μ.

Since f is $\widetilde{\mu}$-totally-measurable on A, then for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of A such that:

$$\begin{cases} i)\ \widetilde{\mu}(A_0) < \frac{\varepsilon}{2M} \text{ (where } M = \sup\limits_{t \in T} |f(t)|\text{) and} \\ ii)\ \sup\limits_{t,s \in A_i} |f(t) - f(s)| < \frac{\varepsilon}{\widetilde{\mu}(T)},\ \forall\ i = \overline{1,n}. \end{cases}$$

Let $\{B_j\}_{j=\overline{1,k}}, \{C_p\}_{p=\overline{1,s}} \in \mathcal{P}_A$ be two arbitrary partitions which are finer than P_ε and consider $s_j \in B_j, j = \overline{1,k}, \theta_p \in C_p, p = \overline{1,s}$.

We prove that

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) < \varepsilon.$$

We have two cases:

I. $\mu(A_0) \supsetneq \{0\}$. Then $\mu(A_1) = ... = \mu(A_n) = \{0\}$.

Suppose, without any loss of generality that $\mu(B_1) \supsetneq \{0\}$, $\mu(C_1) \supsetneq \{0\}$ and $\mu(B_2) = ... = \mu(B_k) = \{0\}, \mu(C_2) = ... = \mu(C_s) = \{0\}$. Then

$B_1 \subseteq A_0$ and $C_1 \subseteq A_0$. Consequently,

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) =$$
$$= h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1)) \leq$$
$$\leq |f(s_1)||\mu(B_1)| + |f(\theta_1)||\mu(C_1)| \leq$$
$$\leq 2M\overline{\mu}(A_0) < \varepsilon.$$

II. $\mu(A_0) = \{0\}$. Then, without any loss of generality, $\mu(A_1) \supsetneq \{0\}$ and $\mu(A_i) = \{0\}$, for every $i = \overline{2, n}$. Suppose that $\mu(B_1) \supsetneq \{0\}$, $\mu(C_1) \supsetneq \{0\}$ and $\mu(B_2) = ... = \mu(B_k) = \{0\}, \mu(C_2) = ... = \mu(C_s) = \{0\}$. Then $B_1 \subseteq A_1$ and $C_1 \subseteq A_1$, and, therefore,

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) =$$

$$= h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1)).$$

Since A is an atom of μ and $\mu(B_1) \supsetneq \{0\}$, then $\mu(A \backslash B_1) = \{0\}$, so $\mu(C_1 \backslash B_1) = \{0\}$. By the null-additivity of μ, we get $\mu(C_1) = \mu(B_1)$. Then

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) =$$
$$= h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1)) =$$
$$= h(f(s_1)\mu(B_1), f(\theta_1)\mu(B_1)).$$

Because, generally, $h(\alpha M, \beta M) \leq |\alpha - \beta||M|$, for every $\alpha, \beta \in \mathbb{R}$ and every $M \in \mathcal{P}_f(X)$, we have

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) \leq$$

$$\leq |\mu(B_1)||f(s_1) - f(\theta_1)| \leq \overline{\mu}(T)\frac{\varepsilon}{\overline{\mu}(T)} = \varepsilon.$$

Therefore, $(\sigma(P))_{P \in \mathcal{P}_A}$ is a Cauchy net in the complete metric space $(\mathcal{P}_{bf}(X), h)$, hence f is μ-integrable on A. □

Theorem 4.14. *Suppose $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is monotone, null-additive and finitely purely atomic. If $f : T \to \mathbb{R}$ is $\widetilde{\mu}$-totally-measurable on T, then f is μ-integrable on T.*

Proof. Since μ is finitely purely atomic, we may write $T = \bigcup_{i=1}^{n} A_i$, where $(A_i)_{i=1}^{n}$ are disjoint atoms of μ. If f is $\widetilde{\mu}$-totally-measurable on T, then f is $\widetilde{\mu}$-totally-measurable on every A_i, $i = \overline{1, n}$. According to Theorem 4.13, f is μ-integrable on every A_i, $i \in \{1, 2, \ldots, n\}$. By the properties of μ-integrable functions (see [25,26,27]), it results f is μ-integrable on $\bigcup_{i=1}^{n} A_i = T$. □

By Corollary 4.12 and Theorem 4.14, we immediately have:

Corollary 4.15. *If T is a compact metric space, $\mathcal{B}$ is the Borel δ-ring generated by the compact subsets of T, $f : T \to \mathbb{R}$ is continuous on T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a finitely purely atomic multisubmeasure, then f is μ-integrable on T.*

5 L^p spaces

In this section, we introduce L^p spaces with respect to a submeasure of finite variation and point out that under suitable assumptions, L^p is a Banach space.

In the sequel, $\mathcal{A}$ will be a σ-algebra of subsets of T and $m : \mathcal{A} \to \mathbb{R}_+$ a submeasure of finite variation.

Theorem 5.1. (Minkowski inequality) [11] *Let $f, g : T \to \mathbb{R}$ be m-integrable functions on T. Then for every $p \in (1, \infty)$, $|f|^p, |g|^p, |f + g|^p$ are m-integrable on T and*

$$(\int_T |f + g|^p dm)^{\frac{1}{p}} \leq (\int_T |f|^p dm)^{\frac{1}{p}} + (\int_T |g|^p dm)^{\frac{1}{p}}.$$

Now, we consider $\mathcal{L}^p = \{f : T \to \mathbb{R};\ f$ is bounded on T and $|f|^p$ is m-integrable on $T\}$.

Theorem 5.2. [11] *$\mathcal{L}^p$ is a linear space and the function $|| \cdot || : \mathcal{L}^p \to \mathbb{R}_+$, defined for every $f \in \mathcal{L}^p$ by*

$$||f|| = (\int_T |f|^p dm)^{\frac{1}{p}},$$

is a semi-norm.

Definition 5.3. We say that a property (P) *holds m-almost everywhere* (shortly $m - ae$) if there is $A \in \mathcal{P}(T)$, so that $\widetilde{m}(A) = 0$ and (P) holds on $T \backslash A$.

Theorem 5.4. *Suppose $\widetilde{m}$ is countably subadditive, $A \in \mathcal{A}$ is an atom of m and $f : T \to \mathbb{R}$ a bounded m-integrable function. If $\int_A f dm = 0$, then $f = 0$ m-ae on A.*

Proof. Since f is m-integrable, for any $n \in \mathbb{N}^*$, there is $\{B_i^n\}_{i=1}^{p_n} \subset \mathcal{A}$, a partition of A, such that

$$|\sum_{i=1}^{p_n} f(t_i)m(B_i^n)| < \frac{1}{n},$$

for every $t_i \in B_i^n$, $i = \overline{1, p_n}$. According to Remark 3.5, there is an unique let say B_1^n such that $m(B_1^n) = m(A)$ and $m(B_2^n) = m(B_3^n) = \ldots = m(B_{p_n}^n) = 0$, for every $n \in \mathbb{N}^*$.

It results $|f(t)| < \frac{1}{m(A)n}$, for every $t \in B_1^n$ and $n \in \mathbb{N}^*$, which implies

$$E_n = \{t \in A; |f(t)| \geq \frac{1}{m(A)n}\} \subseteq A \backslash B_1^n,$$

for every $n \in \mathbb{N}^*$. By Remarks 2.6 and 5.3, since $A\backslash B_1^n \in \mathcal{A}$ and $m(A\backslash B_1^n) = 0$, we have $\widetilde{m}(A\backslash B_1^n) = \overline{m}(A\backslash B_1^n) = 0$.

This implies $\widetilde{m}(E_n) = 0$, for every $n \in \mathbb{N}^*$. Since $\widetilde{m}$ is countably subadditive on $\mathcal{P}(T)$ and

$$\{t \in A; |f(t)| > 0\} = \bigcup_{n=1}^{\infty} E_n,$$

we obtain $f = 0$ m-ae on A. $\square$

Corollary 5.5. *Suppose m is finitely purely atomic so that $\widetilde{m}$ is countably subadditive and let $f : T \to \mathbb{R}_+$ be a bounded m-integrable function. If $\int_T f dm = 0$, then $f = 0$ $m - ae$ on T.*

Remark 5.6. In the hypothesis of Corollary 5.5, it results that the semi-norm $\|\cdot\|$, introduced in Theorem 5.2, becomes a norm on L^p, the space of all equivalence classes of $\mathcal{L}^p$, with respect to the equivalence relation " $\sim$ " defined by

$$f \sim g \text{ if } f = g \ \ m - ae \text{ on } T.$$

Theorem 5.7. [11] *Suppose $m : \mathcal{A} \to \mathbb{R}_+$ is o-continuous. If for every $n \in \mathbb{N}^*, f_n : T \to \mathbb{R}$ is $\widetilde{m}$-totally-measurable on T and $(f_n)_n$ is uniformly bounded and pointwise converges to $f : T \to \mathbb{R}$, then f is $\widetilde{m}$-totally-measurable on T.*

Theorem 5.8. (Fatou lemma) [11] *Suppose $\widetilde{m}$ is o-continuous on $\mathcal{P}(T)$. Let $(f_n)_n$ be a sequence of uniformly bounded, $\widetilde{m}$-totally-measurable functions $f_n : T \to \mathbb{R}$. Then*

$$\int_T \liminf_n f_n dm \leq \liminf_n \int_T f_n dm.$$

Following a classical reasoning, by Theorems 5.2, 5.4, 5.7, 5.8 and Corollary 5.5, we get:

Theorem 5.9. [12] *Let $m : \mathcal{A} \to \mathbb{R}_+$ be finitely purely atomic, so that $\widetilde{m}$ is o-continuous on $\mathcal{P}(T)$. Then L^p is a Banach space.*

Conclusion. In this paper, we obtained some properties of finitely purely atomic set multifunctions and some results concerning measurability and Gould integrability of real bounded functions with respect to a finitely purely atomic submeasure. We also pointed out that, in this case, L^p is a Banach space.

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