Some New Remarks about Lotka-Volterra System

Camelia POP
POLITEHNICA University of Timisoara
Department of Mathematics
2, Pta. Victoriei, Timisoara
ROMANIA
cariesanu@yahoo.com

Anania ARON
POLITEHNICA University of Timisoara
Department of Mathematics
2, Pta. Victoriei, Timisoara
ROMANIA
ananiagirban@yahoo.com

Abstract: Lotka-Volterra system is a well-known system used as mathematical model in biology. It was proposed as model by Alfred Lotka (1925) and Vito Volterra (1926). We are interested to study it from the mechanical geometry point of view; more exactly we study the stability problem, the existence of periodic solutions, the Lax formulation and numerical integration via Kahan and Lie-Trotter integrators.

Key–Words: Hamilton-Poisson realization, Lotka-Volterra system, nonlinear stability, Arnold method, Lie-Trotter integrator, Kahan integrator, Lax formulation.

1 Introduction

Three dimensional Lotka-Volterra system of competing species has the following form:

$$\begin{cases} \dot{x} = x\left(Cy + z + \lambda\right) \\ \dot{y} = y\left(Az + x + \mu\right) \\ \dot{z} = z\left(Bx + y + \nu\right) \end{cases} \tag{1}$$

It is known that the above system has a Hamilton-Poisson realization if the following two conditions hold:

$$A \cdot B \cdot C = -1 \tag{2}$$

and

$$\nu = \mu B - \lambda AB. \tag{3}$$

The structure of this paper is as follows. In the second section we consider the specific case

$$A = B = C = -1, \ \ \mu = \nu = \lambda = 0,$$

realizing this system as a Hamiltonian system, and then study it from the Poisson geometry point of view. This means that we are interested to study the Lyapunov stability of equilibria by using energy-Casimir type stability tests, the study of the existence of periodic solutions using the Weinstein-Moser theorem with zero eigenvalue, Lax formulation of the system, numerical integration problems using three methods. Numerical simulation for all the three methods is presented, too.

The third section is dedicated to the study of the spectral and Lyapunov stability of equilibria for the general case of the system (1).

In the fourth section of the paper we discuss some numerics associated with the Hamilton- Poisson geometrical structure of the system.

First of all, let us recall briefly the most important notions from Hamilton-Poisson geometry theory used in our paper.

Definition 1 *Let M be a smooth manifold and let $C^\infty(M)$ denote the set of the smooth real functions on M. A* **Poisson bracket on** *M is a bilinear map from $C^\infty(M) \times C^\infty(M)$ into $C^\infty(M)$, denoted as:*

$$(F, G) \mapsto \{F, G\} \in C^\infty(M), F, G \in C^\infty(M)$$

which verifies the following properties:
- skew-symmetry:

$$\{F, G\} = -\{G, F\};$$

- Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0;$$

- Leibniz rule:

$$\{F, G \cdot H\} = \{F, G\} \cdot H + G \cdot \{F, H\}.$$

Proposition 2 *Let $\{\cdot, \cdot\}$ a Poisson structure on $\mathbb{R}^n$. Then for any $f, g \in C^\infty(\mathbb{R}^n, \mathbb{R})$ the following relation holds:*

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Let the matrix given by:

$$\Pi = [\{x_i, x_j\}].$$

Proposition 3 *Any Poisson structure* $\{\cdot,\cdot\}$ *on* $\mathbb{R}^n$ *is completly determined by the matrix* Π *via the relation:*

$$\{f,g\} = (\nabla f)^t \Pi (\nabla g).$$

Definition 4 *A* **Hamilton-Poisson system** *on* $\mathbb{R}^n$ *is the triple* $(\mathbb{R}^n, \{\cdot,\cdot\}, H)$, *where* $\{\cdot,\cdot\}$ *is a Poisson bracket on* $\mathbb{R}^n$ *and* $H \in C^\infty(\mathbb{R}^n, \mathbb{R})$ *is the energy (Hamiltonian). Its dynamics is described by the following differential equations system:*

$$\dot{x} = \Pi \cdot \nabla H$$

where $x = (x_1, x_2, ...x_n)^t$.

Definition 5 *Let* $\{\cdot,\cdot\}$ *a Poisson structure on* $\mathbb{R}^n$. *A* **Casimir** *of the configuration* $(\mathbb{R}^n, \{\cdot,\cdot\})$ *is a smooth function* $C \in C^\infty(\mathbb{R}^n, \mathbb{R})$ *which satisfies:*

$$\{f, C\} = 0, \forall f \in C^\infty(\mathbb{R}^n, \mathbb{R}.).$$

2 The Poisson geometry of Lotka-Volterra system for the specific case $A = B = C = -1$ and $\nu = \mu = \lambda = 0$

For the specific case $A = B = C = -1$ and $\nu = \mu = \lambda = 0$ the system (1) takes the following form:

$$\begin{cases} \dot{x} = -xy + zx \\ \dot{y} = -yz + xy \\ \dot{z} = -zx + yz \end{cases} \tag{4}$$

The dynamics (2) has the following Hamilton-Poisson realization (see [11]):

$$\left(\mathbb{R}^3, \Pi_-, H\right)$$

where

$$\Pi_- = \begin{bmatrix} 0 & -xy & xz \\ xy & 0 & -yz \\ -xz & yz & 0 \end{bmatrix} \tag{5}$$

and

$$H(x,y,z) = x + y + z.$$

Remark 6 *It is not hard to see that the function*

$$C(x,y,z) = ln\ x + ln\ y + ln\ z,$$

$x, y, z > 0$, *is a Casimir of our configuration [7].*

Remark 7 *The phase curves of the dynamics* (2) *are the intersection of the surfaces:*

$$xyz = constant$$

and

$$x + y + z = constant,$$

see the Figures 2.1 and 2.2.

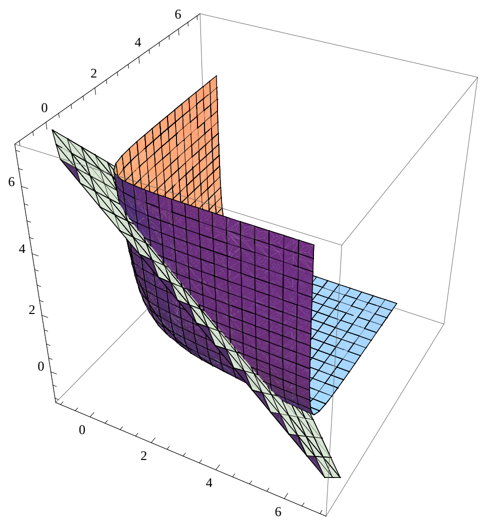

Figure 2.1: The phase curves of the system (2)

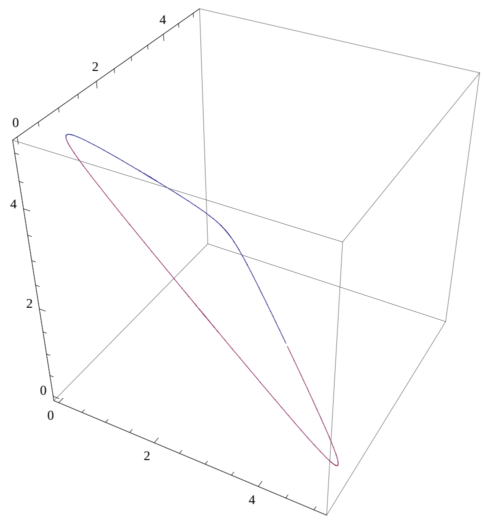

Figure 2.2: The phase curves of the system (2)

Proposition 8 *The dynamics* (2) *has an infinite number of Hamilton-Poisson realizations.*

Proof: An easy computation shows us that the triples:

$$\left(\mathbb{R}^3, \{\cdot,\cdot\}_{ab}, H_{cd}\right)$$

where:

$$\begin{array}{l} \{.,.\}_{ab} : C^\infty(\mathbb{R}^3,\mathbb{R}) \times C^\infty(\mathbb{R}^3,\mathbb{R}) \to C^\infty(\mathbb{R}^3,\mathbb{R}) \\ \quad \{f,g\}_{ab} \overset{def}{=} -\nabla C \cdot (\nabla f \times \nabla g), \\ (\forall)\, f,g \in C^\infty(\mathbb{R}^3,\mathbb{R}) \end{array} \tag{6}$$

$$C_{ab} = aC + bH,$$

$$H_{cd} = cC + dH,$$

$$a,b,c,d \in \mathbb{R}, ad - bc = 1,$$

define Hamilton-Poisson realizations of the dynamics (2), as required.

Remark 9 *The above proposition tells us in fact that the equations* (2) *are unchanged, so the trajectories of motion in* $\mathbb{R}^3$ *remain the same when* H *and* C *are replaced by* $\mathbb{R}^3$ *combinations of* H *and* C.

Let us pass now to discuss the stability problem of the equilibria of the system (2). It is not hard to see that the equilibrium states of the dynamics (2) are:

$$e_1^M = (M,0,0), M \in \mathbb{R}_+,$$

$$e_2^M = (0,M,0), M \in \mathbb{R}_+,$$

$$e_3^M = (0,0,M), M \in \mathbb{R}_+,$$

$$e_4^M = (M,M,M), M \in \mathbb{R}_+.$$

Let A be the matrix of the linear part of our system (2) i.e.

$$A = \begin{bmatrix} -y+z & -x & x \\ y & x-z & -y \\ -z & z & -x+y \end{bmatrix}$$

Then the characteristic roots of $A(e_1^M)$, [resp. $A(e_2^M)$, resp. $A(e_3^M)$] are given by:

$$\lambda_1 = 0, \lambda_{2,3} = \pm M$$

so we can conclude that the equilibrium states e_1^M, e_1^M, e_3^M are unstable.
The characteristic roots of $A(e_4^M)$ are given by:

$$\lambda_1 = 0, \lambda_{2,3} = \pm i\sqrt{3}M$$

so we can conclude that:

Proposition 10 *The equilibrium states* e_4^M *are spectrally stable.*

We can now pass to discuss the nonlinear stability of the equilibrium states $e_4^M, M \in \mathbb{R}_+^*$.

Proposition 11 *The equilibrium states* e_4^M, $M \in \mathbb{R}_+^*$ *are nonlinearlly stable.*

Proof. We shall make the proof using energy-Casimir method [3]. Let

$$H_\varphi = H + \varphi(C) = x + y + z + \varphi(xyz)$$

be the energy-Casimir function, where $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a smooth real valued function defined on $\mathbb{R}_+$. Now, the first variation of H_φ is given by:

$$\delta H_\varphi = \delta x + \delta y + \delta z + yz\,\dot{\varphi}\,\delta x +$$

$$+ xz\,\dot{\varphi}\,\delta y + xy\,\dot{\varphi}\,\delta z,$$

where

$$\dot{\varphi} = \frac{\partial \varphi}{\partial (xyz)}.$$

This equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}(M^3) = -\frac{1}{M^2}.$$

The second variation of H_φ is given by:

$$\delta^2 H_\varphi = y^2 z^2\,\ddot{\varphi}(\delta x)^2 + x^2 z^2\,\ddot{\varphi}(\delta y)^2 +$$

$$+ x^2 y^2\,\ddot{\varphi}(\delta z)^2 + 2x\,\dot{\varphi}\,\delta y \delta z + 2y\,\dot{\varphi}\,\delta x \delta z +$$

$$+ 2z\,\dot{\varphi}\,\delta y \delta x.$$

Choose φ so that:

$$\begin{cases} \dot{\varphi}(M^3) = -\frac{1}{M^2} \\ \\ \ddot{\varphi}(M^3) > -\frac{2}{M^5}, \quad M > 0 \end{cases}$$

we can conclude that the second variation of H_φ at the equilibrium of interest is positivelly defined and thus e_4 is nonlinearly stable.

For the equililibrium states which are nonlinear stable we are able to find the periodic solutions. More exactly, we have:

Proposition 12 *Near to* $e_4^M = (M,M,M)$, $M \in \mathbb{R}_+^*$, *the reduced dynamics has, for each sufficiently small value of the reduced energy, at least 1-periodic solution whose period is close to:*

$$\frac{2\pi}{\sqrt{3}M}.$$

Proof. Indeed, we have successively:

(i) The restriction of our dynamics (2) to the coadjoint orbit:

$$x + y + z = 3M, \; xyz = M^3 \tag{7}$$

gives rise to a classical Hamiltonian system.

(ii) The matrix of the linear part of the reduced dynamics has purely imaginary roots. More exactly:

$$\lambda_{2,3} = \pm i\sqrt{3}M.$$

(iii) $\mathrm{span}(\nabla C(e_4^M)) = V_0$, where

$$V_0 = \ker(A(e_4^M)).$$

(iv) The smooth function $F \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by:

$$F(x, y, z) = x + y + z - \frac{1}{M^2}xyz$$

has the following properties:
• It is a constant of motion for the dynamics (2).
• $\nabla F(e_4^M) = 0$.
• $\nabla^2 F(e_1^M)\big|_{W\times W} > 0$, where

$$W := ker\, dC(e_4^M) = span_{\mathbb{R}}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right).$$

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue, see for details [4].

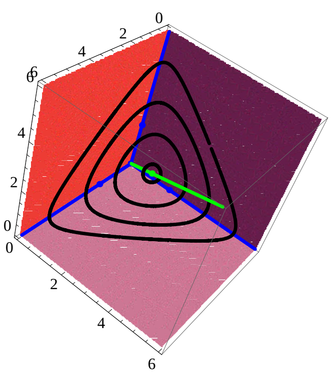

Figure 2.3: The periodic orbits for the equilibrium states e_4^M

A long but straightforward computation or using MATHEMATICA leads us to:

Proposition 13 *The dynamics* (2) *has the following formulation:*

$$\dot{L} = [L, B],$$

where

$$L = \begin{bmatrix} 0 & a & b & 0 & 0 & 0 \\ -a & 0 & c & 0 & 0 & 0 \\ -b & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & -d & 0 & f \\ 0 & 0 & 0 & -e & -f & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & a' & b' & 0 & 0 & 0 \\ -a' & 0 & c' & 0 & 0 & 0 \\ -b' & -c' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d' & e' \\ 0 & 0 & 0 & -d' & 0 & f' \\ 0 & 0 & 0 & -e' & -f' & 0 \end{bmatrix},$$

$$a = -ix - 2y - iz, \; b = 2ix + 2iy + 2iz;$$
$$c = x - 2iy + z, \; d = x + iy + iz;$$
$$e = x + y + z, \; f = -ix + y + z;$$
$$a' = -y - iz, \; b' = iy + 2iz;$$
$$c' = -iy + z, \; d' = y - z;$$
$$e' = -iy + iz, \; f = -iy + iz.$$

Let us pass now to the numerical integration of the equations (2).
It is easy to see that for the equations (2), Kahan's integrator [9] can be written in the following form:

$$\begin{cases} x^{n+1} - x^n = \frac{h}{2}(-x^{n+1}y^n - y^{n+1}x^n + \\ \qquad +z^{n+1}x^n + x^{n+1}z^n) \\ y^{n+1} - y^n = \frac{h}{2}(x^{n+1}y^n + y^{n+1}x^n - \\ \qquad -z^{n+1}y^n + y^{n+1}z^n) \\ z^{n+1} - z^n = \frac{h}{2}(-x^{n+1}z^n - z^{n+1}x^n + \\ \qquad +z^{n+1}y^n + y^{n+1}z^n) \end{cases} \tag{8}$$

A long but straightforward computation or using MATHEMATICA 7 leads us to:

Proposition 14 *Kahan's integrator (6) has the following properties:*

(i) It is not Poisson preserving.

(ii) It does not preserve the Casimir C of our Poisson configuration $(\mathbb{R}^3, \Pi)$.

(iii) *It does not preserve the Hamiltonian H of our system* (2).

We shall discuss now the numerical integration of the dynamics (2) via the Lie-Trotter integrator, see for details [23] and [28]. For the beginning, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3}.$$

where

$$H_1 = x,\ H_2 = y,\ H_3 = z.$$

Following [23], we obtain the Lie-Trotter integrator:

$$\begin{cases} x^{n+1} = e^{z(0)-y(0)}x^n \\ \\ y^{n+1} = e^{x(0)-z(0)}y^n \\ \\ z^{n+1} = e^{y(0)-x(0)}z^n \end{cases} \quad (9)$$

Now, a direct computation or using MATHEMATICA 7 leads us to:

Proposition 15 *Lie-Trotter integrator (7) has the following properties:*

(i) *It preserves the Poisson structure* Π.

(ii) *It preserves the Casimir C of our Poisson configuration* $(\mathbb{R}^3, \Pi)$.

(iii) *It doesn't preserve the Hamiltonian H of our system (2).*

(iv) *Its restriction to the coadjoint orbit* $(\mathcal{O}_k, \omega_k)$, *where*

$$\mathcal{O}_k = \{(x, y, z) \in \mathbb{R}^3 | xyz = const.\}$$

and ω_k is the Kirilov-Kostant-Souriau symplectic structure on $\mathcal{O}_k$ gives rise to a symplectic integrator.

Proof. The items (i),(ii), (iv) hold because $\Phi_i, i = 1, 2, 3$ are flows of some Hamiltonian vector fields, hence they are Poisson one. Item (iii) is essentially due to the fact that:

$$\{H_i, H_j\} \neq 0,\ i \neq j.$$

Remark 16 *If we make a comparison with the 4th-step Runge-Kutta method we can see that Lie-Trotter integrator and Kahan's integrator give us a good approximation of our dynamics. In fact, Kahan's integrator provides the same results as Runge-Kutta 4th-step. However, Kahan's integrator and the Lie-Trotter integrator have the advantage to be easier implemented, see Figures 2.3, 2.4 and 2.5.*

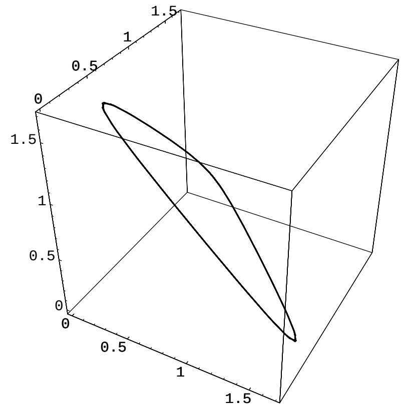

Figure 2.3: The 4th-step Runge-Kutta

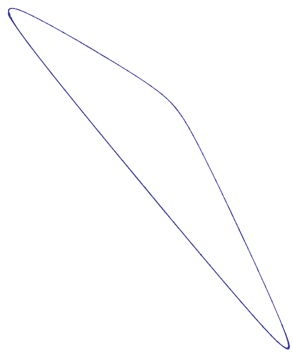

Figure 2.4: Kahan integrator

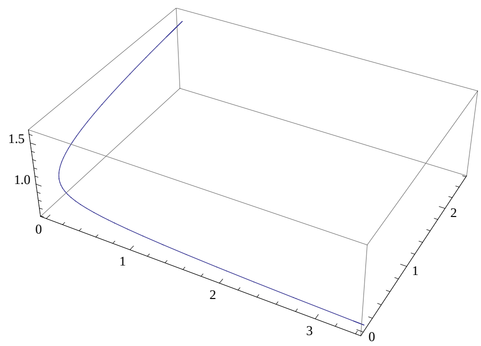

Figure 2.5: Lie-Trotter integrator

3 The Poisson geometry of Lotka-Volterra system for the general case

Let us recall that the three dimensional Lotka-Volterra system of competing species has the following form, see [6]:

$$\begin{cases} \dot{x} = x\,(Cy + z + \lambda) \\ \dot{y} = y\,(Az + x + \mu) \\ \dot{z} = z\,(Bx + y + \nu) \end{cases} \qquad (10)$$

It can be readily verified that, if conditions (2) and (3) hold, the equations (10) have the following Hamilton-Poisson realization (see [20]):

$$\left(\mathbb{R}^3, \Pi, H\right)$$

where

$$\Pi = \begin{bmatrix} 0 & Cxy & BCxz \\ -Cxy & 0 & -yz \\ -BCxz & yz & 0 \end{bmatrix} \qquad (11)$$

and

$$H\,(x,y,z) = ABx + y - Az + \nu ln\; y - \mu ln\; z,$$

$x, y, z > 0.$

Remark 17 *It is not hard to see that the function*

$$C\,(x,y,z) = ABln\; x - Bln\; y + ln\; z,$$

$x, y, z > 0$, *is a Casimir of our configuration (see [7]).*

Remark 18 *The phase curves of the dynamics (10) are the intersection of the surfaces:*

$$ABx + y - Az + \nu ln\; y - \mu ln\; z = constant$$

and

$$ABln\; x - Bln\; y + ln\; z = constant,$$

$x, y, z > 0$, *see the Figures 3.1 and 3.2.*

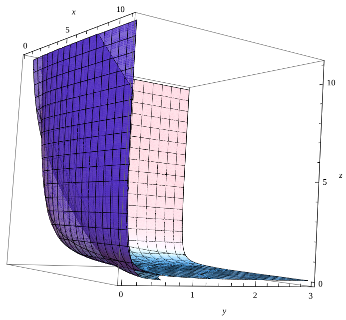

Figure 3.1: The phase curves of the system (10)

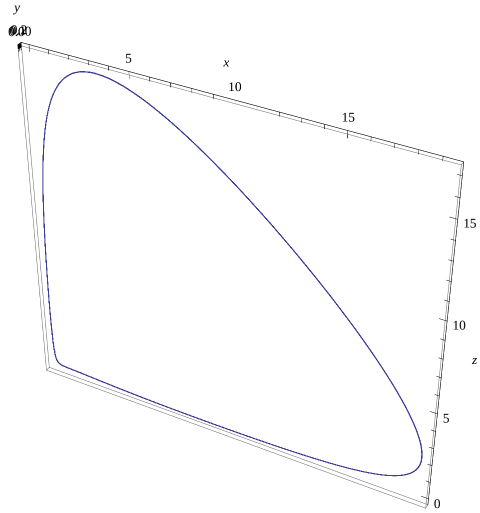

Figure 3.2: The phase curves of the system (10)

4 Stability problems for the system (10)

In this section we analyze the stability properties of the equilibrium states of the Lotka-Volterra system (10), with the assumption that the conditions (2) and (3) hold. The equilibrium states of the Lotka-Volterra system are given by the following family:

$$e^M = (-\mu - AM, AB\lambda + ABM, M), M \in \mathbb{R}_+.$$

Let A be the matrix of the linear part of our system (10) i.e.

$$A = \begin{bmatrix} Cy + z + \lambda & Cx & x \\ y & Az + x + \mu & Ay \\ Bz & z & Bx + y + \nu \end{bmatrix}$$

Then the characteristic polynom of $A(e^M)$ is:

$$q(AM^2 - ABM^2 + A^2BM + \mu(M - BM) +$$

$$+\lambda(M + A(1 + AB)M) - q^2).$$

Its roots are given by:

$$q_1 = 0,$$

$$q_{2,3} = \pm\sqrt{\Omega},$$

where

$$\Omega = M(\mu - B\mu + A(1 + (-1 + A)B)M) +$$

$$+\lambda(M + A(1 + AB)M)$$

so we can conclude that the equilibrium states e^M are spectral stable if one of the following conditions hold:

1. $\mu < 0, \lambda \leq \mu, A < 0,\ M > -\lambda,\ B < 0;$
2. $\mu < 0, \lambda \leq \mu,\ A < -\sqrt{\frac{\mu}{\lambda}},\ 0 < M < \frac{\mu - A^2\lambda}{A^2 - A},$
$$B > -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
3. $\mu < 0, \lambda \leq \mu,\ 0 < A < \frac{\mu}{\lambda},\ 0 < M < -\lambda,$
$$B < -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
4. $\mu < 0, \lambda \leq \mu,\ 0 < A < \frac{\mu}{\lambda},\ -\lambda < M < -\frac{\mu}{A},$
$$0 < B < -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
5. $\mu < 0,\ \ \lambda < \mu,\ \ \frac{\mu}{\lambda} \leq A < \sqrt{\frac{\mu}{\lambda}},$
$$0 < M < \frac{\mu - A^2\lambda}{A^2 - A},$$
$$B < -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
6. $\mu < 0,\ \ \mu < \lambda < 0,\ \ A < -\sqrt{\frac{\mu}{\lambda}},$
$$0 < M < \frac{\mu - A^2\lambda}{A^2 - A},$$
$$B > -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
7. $\mu < 0,\ \mu < \lambda < 0,\ A < 0,\ M > -\lambda,\ B < 0;$
8. $\mu < 0,\ \mu < \lambda < 0,\ 0 < A < \sqrt{\frac{\mu}{\lambda}},$
$$0 < M < -\lambda,$$
$$B < -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
9. $\mu < 0, \mu < \lambda < 0,\ 0 < A < \sqrt{\frac{\mu}{\lambda}},\ -\lambda < M < -\frac{\mu}{A},$
$$0 < B < -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
10. $\mu < 0,\ \mu < \lambda < 0,\ \sqrt{\frac{\mu}{\lambda}} < A < \frac{\mu}{\lambda},$
$$-\frac{\mu + A^2\lambda}{A^2 - A} < M < -\lambda,$$
$$B < -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
11. $\mu < 0,\ \mu < \lambda < 0,\ \sqrt{\frac{\mu}{\lambda}} < A < \frac{\mu}{\lambda},$
$$-\lambda < M < -\frac{\mu}{A},$$
$$0 < B < -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
12. $\mu < 0, \lambda \geq 0,\ A < 0,\ M > 0,\ B < 0;$
13. $\mu < 0, \lambda \geq 0,\ A > 0,\ 0 < M < -\frac{\mu}{A},$
$$0 < B < -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$
14. $\mu = 0, \lambda < 0,\ A < 0,\ 0 < M < -\frac{A^2\lambda}{A^2 - A},$
$$B > -\frac{A\lambda M + AM^2}{A^2\lambda M - AM^2 + A^2M^2};$$
15. $\mu = 0, \lambda < 0,\ A < 0,\ M > -\lambda,\ B < 0;$
16. $\mu = 0, \lambda \geq 0,\ A < 0,\ M > 0,\ B < 0;$
17. $\mu > 0, \lambda < 0,\ A < \frac{\mu}{\lambda},\ M > -\lambda,\ B < 0;$
18. $\mu > 0, \lambda < 0,\ A = \frac{\mu}{\lambda},\ M > -\frac{\mu - A^2\lambda}{A^2 - A},\ B < 0;$
19. $\mu > 0, \lambda < 0,\ \frac{\mu}{\lambda} < A < 0,\ M > -\frac{\mu}{A},\ B < 0;$
20. $\mu > 0, \lambda \geq 0,\ A < 0,\ M > -\frac{\mu}{A},\ B < 0;$
21. $\mu > 0, \lambda < 0,\ A < \frac{\mu}{\lambda},$
$$-\frac{\mu}{A} < M < -\frac{\mu - A^2\lambda}{A^2 - A},$$
$$B > -\frac{\lambda\mu + A\lambda M + \mu M + AM^2}{A^2\lambda M - \mu M - AM^2 + A^2M^2};$$

Let us begin the nonlinear stability analysis using Arnold stability test for the case (i).

Proposition 19 *If:*
$$\mu < 0, \lambda \leq \mu, A < -\sqrt{\frac{\mu}{\lambda}}, B < 0, M > -\lambda \quad (12)$$
then the equilibrium states e^M *are nonlinearly stables.*

Proof. To study the nonlinear stability of the equilibria e^M we are using the Arnold stability test (see [2], [3]). To do that, let $F_p \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be defined by:
$$F_p = H + pC$$
where $p \in \mathbb{R}$ is a real parameter.

Then, we succesively have the following:

(i) $dF_p(e^M) = 0$ if and only if

$$p = \mu + AM.$$

(ii) Let $W = kerdC(e^M)$ i.e.

$$W = span\left(\begin{bmatrix} -\frac{\mu+AM}{A^2B(\lambda+M)} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\mu+AM}{ABM} \\ 0 \\ 1 \end{bmatrix}\right).$$

(iii) $d^2F_{\mu+AM}(e^M) \setminus W \times W$ is positively defined if the following condition holds:

$$\mu < 0, \lambda \le \mu, A < \sqrt{-\frac{\mu}{\lambda}}, B < 0, M > -\lambda.$$

Hence, from the Arnold stability test we conclude that the equilibrium states e^M are nonlinearlly stable if the conditions (12) hold.

5 Numerical integration of the equations (10)

In this section we discuss the numerical integration of the equations (10) via Kahan integrator, Lie-Trotter integrator and Runge-Kutta 4th steps intgrator. Numerical simulations via MATHEMATICA 7 are presented for each case, too.

Kahan integrator of the Lotka-Volterra system (10) is given by:

$$\begin{cases} x^{n+1} - x^n = \frac{h}{2}(Cx^{n+1}y^n + Cy^{n+1}x^n + z^{n+1}x^n \\ \qquad +x^{n+1}z^n + \lambda x^n + \lambda x^{n+1}) \\ y^{n+1} - y^n = \frac{h}{2}(x^{n+1}y^n + y^{n+1}x^n + Az^{n+1}y^n \\ \qquad +Ay^{n+1}z^n + \mu y^n + \mu y^{n+1}) \\ z^{n+1} - z^n = \frac{h}{2}(Bx^{n+1}Z^n + Bz^{n+1}x^n + z^{n+1}y^n \\ \qquad +y^{n+1}z^n + \nu z^n + \nu z^{n+1}) \end{cases} \tag{13}$$

After some long but straightforward computations, we get the following proposition which shows the incompatibility of the Kahan integrator with the Poisson geometric structure of the Lotka-Volterra system.

Proposition 20 *Kahan integrator (13) has the following properties:*

(i) It is not Poisson preserving.

(ii) It does not preserve the Casimir C of our Poisson configuration $(\mathbb{R}^3, \Pi)$.

(iii) It does not preserve the Hamiltonian H of our system (10).

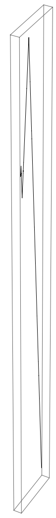

Figure 3.3: Kahan integrator for the system (10)

We shall discuss now the numerical integration of the dynamics (10) via the Lie-Trotter integrator, see for details [23] and [28].

For the beginning, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3} + X_{H_4} + X_{H_5},$$

where

$$H_1 = ABx,\ H_2 = y,\ H_3 = -Az,$$
$$H_4 = \nu\ lny,\ H_5 = -\mu\ lnz.$$

Their corresponding integral curves are respectively given by:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = A_i \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix},\ i = \overline{1,5},$$

where:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{x(0)t} & 0 \\ 0 & 0 & e^{Bx(0)t} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} e^{Cy(0)t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{y(0)t} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} e^{z(0)t} & 0 & 0 \\ 0 & e^{Az(0)t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} e^{C\nu t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\nu t} \end{pmatrix}$$

$$A_5 = \begin{pmatrix} e^{-BC\mu t} & 0 & 0 \\ 0 & e^{\mu t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Following [23], we obtain the Lie-Trotter integrator:

$$\begin{cases} x^{n+1} = e^{(Cy(0)+z(0)+C\nu-BC\mu)t}x^n \\ y^{n+1} = e^{(x(0)+Az(0)+\mu)t}y^n \\ z^{n+1} = e^{(Bx(0)+y(0)+\nu)t}z^n \end{cases} \tag{14}$$

Now, a direct computation or using MATHEMATICA 7 leads us to:

Proposition 21 *Lie-Trotter integrator (14) has the following properties:*

(i) It preserves the Poisson structure Π.

(ii) It preserves the Casimir C *of our Poisson configuration* $(\mathbb{R}^3, \Pi)$.

(iii) It doesn't preserve the Hamiltonian H *of our system (10).*

(iv) Its restriction to the coadjoint orbit $(\mathcal{O}_k, \omega_k)$, *where*

$$\mathcal{O}_k = \{(x, y, z) \in \mathbb{R}^3_+ | \ ABlnx- $$
$$-Blny + lnz = const., ABC = -1\}$$

and ω_k *is the Kirilov-Kostant-Souriau symplectic structure on* $\mathcal{O}_k$ *gives rise to a symplectic integrator.*

Proof. The items (i),(ii), (iv) hold because $\Phi_i, i = 1, \ldots 5$ are flows of some Hamiltonian vector fields, hence they are Poisson one.

Item (iii) is essentially due to the fact that:

$$\{H_i, H_j\} \neq 0, \ i \neq j.$$

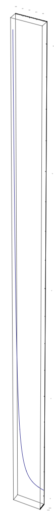

Figure 3.4: Lie-Trotter integrator for the system (10)

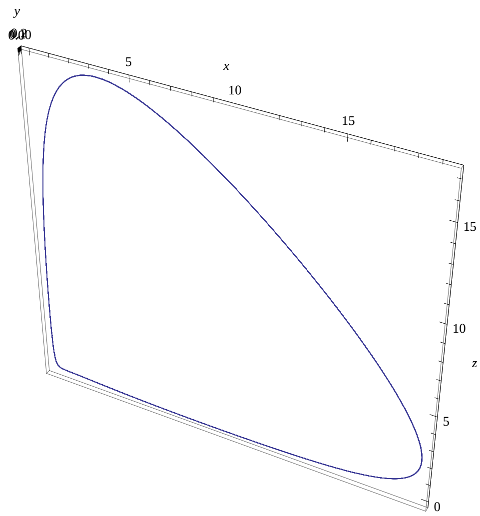

Figure 3.5: Runge-Kutta 4 steps integrator for the system (10)

Remark 22 *If we compare with the 4th-step Runge-Kutta method we can see that Lie-Trotter integrator and Kahan's integrator give us a weak approximation of our dynamics, unlike Runge-Kutta 4th steps integrator which gives us very good results. In fact, Kahan's integrator provides the almost the same results like Lie-Trotter do. However, Kahan's integrator and the Lie-Trotter integrator have the advantage to be easier implemented, see Figures 3.3-3.5. Looking back to the second paragraph, where we have studied the specific case* $A = B = C = -1$ *and* $\lambda = \nu = \mu = 0$, *we remember that Kahan integrator and Runge-Kutta 4th steps integrator have looked the same. Only Lie-Trotter integrator has a different picture, still it stays close to the first two of them.*

6 Conclusion

The paper presents Lotka-Volterra system from the mechanical geometry point of view in the general case and in a specific case. This requests a Hamilton Poisson realization, presented for each of the two cases.

The specific case, when $A = B = C = -1$ and $\lambda = \nu = \mu = 0$, is the subject of the second paragraph. In this case we have studied the stability problems and the existence of the periodic orbits arround the nonlinear equilibrium states we have found. In addition, we have presented a comparison between three numerical integration methods: Runge-Kutta 4th steps, Lie-Trotter algorithm and Kahan algorithm. This is a good example of a system for which all the three methods provided almost the same results unlike other simplier systems for which some of the three methods have failed or have given us a weak approximation of the movement trajectory.

The general case of Lotka-Volterra system and its Hamilton-Poisson realization are disscused in the third paragraph.

The fourth section presents stability problems for the studied system. We have found necessary conditions for spectral and nonlinear stability of the system's equilibria. Due to the complexity of this conditions, we have decided to analyze just one case.

Numerical integration and numerics simulation are presented in the last paragraph. We have used the same three methods like for the specific case $A = B = C = -1$ and $\lambda = \nu = \mu = 0$, but the results are very different.

In this case, both Lie-Trotter and Kahan integrators give us different results, close to each other but far away from Runge-Kutta 4th steps result. We accredit this results to the big number of the parameters of the studied system.

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