Embedding Geodesic and Balanced Cycles into Hypercubes

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Abstract: A graph G is said to be *pancyclic* if it contains cycles of all lengths from 4 to $|V(G)|$ in G. For any two vertices $u, v \in V(G)$, a cycle is called a *geodesic cycle* with u and v if a shortest path joining u and v lies on the cycle. Let G be a bipartite graph. For any two vertices u and v in G, a cycle C is called a *balanced cycle* between u and v if $d_C(u,v) = max\{d_C(x,y) \mid d_G(x,u)$ and $d_G(y,v)$ are even, resp. for all $x, y \in V(G)$ $\}$. A bipartite graph G is *geodesic bipancyclic* (respectively, *balanced bipancyclic*) if for each pair of vertices $u, v \in V(G)$, it contains a geodesic cycle (respectively, balanced cycle) of every even length of k satisfying $max\{2d_G(u,v), 4\} \le k \le |V(G)|$ between u and v. In this paper, we prove that Q_n is geodesic bipancyclic and balanced bipancyclic if $n \ge 2$.

Key–Words: Hypercube; Interconnection networks; Edge-bipancyclic; Geodesic bipancyclic; Balanced bipancyclic

1 Introduction

An interconnection network topology is usually represented by a graph where vertices represent processors and edges represent links between processors. There are various kinds of graphs applied to design interconnection networks. For example, a ring structure is often used as a connection structure of local area network and as a control and data flow structure in distributed networks due to its good properties such as low connectivity, simplicity, and their feasible implementation [19]. There are a lot of mutually conflicting requirements in designing the topology of computer networks. It is almost impossible to design a network which is optimum in all aspects. Existence of various cycles (rings) in an interconnection network is essential for parallel algorithms that communicate data in token-ring mode. Probably the most effective measure of a communication network performance is the transmission delay encountered by a message in traveling through the network from its source to its destination. In a store-forward network a message may have to be stored and forwarded by several intermediate processors before reaching its destination. The transmission delay is approximately proportional to the number of edges a message must travel.

The hypercube proposed in [25] is a popular interconnection network with many attractive properties. It has been used in a wide variety of parallel systems such as Intel iPSC, the nCUBE [11], the Connection Machine CM-2 [27], and SGI Origin 2000 [26]. The n-dimensional hypercube network, Q_n, is widely used in interconnection network topology based on its many attractive properties such as regularity, recursive structure, node and edge symmetry, maximum connectivity, and effective routing and broadcasting algorithms [19]. Many variations of hypercube network are also proposed, for example, crossed cube, twisted cube, möbius cube, augmented cube, and folded hypercube. Cycles (rings) are one of the most fundamental networks for parallel and distributed computation. They are suitable for designing simple algorithms with low communication costs. Many efficient algorithms designed on rings for solving various algebraic problems and graph problems can be found in [19]. These applications motivate the embedding of various length of cycles in networks.The ring embedding problem, which deals with all possible lengths of cycles in a given graph, is investigated in a variety of interconnection networks [3, 6, 9, 10, 12, 13, 16, 17, 18, 20, 21, 23, 25, 28, 30, 31, 32, 33, 34, 35]. Indeed, this problem has been studied for n-dimensional hypercube Q_n [15, 18, 20, 22, 24, 25, 28, 29]. Saad and Schultz [25] proved that Q_n is bipancyclic in the sense that an even cycle of length k exists for each even integer between 4 and $|V(Q_n)|$. Latifi et al [18] found that Q_n is hamiltonian with up to $n-2$ edge faults. Li et al. [20] proved that Q_n is still edge-bipancyclic in the sense that every edge of Q_n lies on a cycle of every even length from 4 to $|V(Q_n)|$ even if it is up to $n-2$ edge faults. Recently, Tsai [28] proved that Q_n is bipancyclic with up to $2n-5$ edge faults if these faults satisfy a specified condition.

In this paper, we address the existence of cycles with some properties in Q_n. The rest of this paper is organized as follows. In the next section, we propose notions of geodesic bipancyclic and balanced bipancyclic that are restrictions of the concept of bipancyclic as new measures of cycle embedding capability of a bipartite graph. Section 3 shows that Q_n is geodesic bipancyclic. Secion 4 proves that Q_n is balanced bipancyclic too. The last section contains conclusions and discussions.

2 Preliminaries

Our fundamental graph terminologies refer to [2]. A graph $G=(V,E)$ is bipartite if the node set $V(G)=B\cup W$ is the union of two disjoints node sets B and W (also called the *partite sets*), such that every edge joins B and W. Two vertices, u and v, have the same color if and only if u and v are in the same partite set. We also use $G=(B\cup W,E)$ to denote a bipartite graph. Two vertices a and b are *adjacent* if $(a,b)\in E$. A path is a sequence of adjacent vertices, written as $\langle v_0,v_1,v_2,\ldots,v_m\rangle$, in which all the vertices $v_0,v_1,\ldots,v_m$ are distinct except possibly $v_0=v_m$. We also write the path $\langle v_0,P[v_0,v_m],v_m\rangle$, where $P[v_0,v_m]=\langle v_0,v_1\ldots,v_m\rangle$ as well as v_0 and v_m are two *end-vertices* of $P[v_0,v_m]$. The *length* of a path P denoted by $l(P)$ is the number of edges in P. Two paths are vertex-disjoint (also called disjoint) if and only if they do not have any vertices in common. Two edges (u,v) and (w,z) are disjoint if $u\notin\{w,z\}$ and $v\notin\{w,z\}$. Let u and v be two vertices of G. The *distance* between u and v denoted by $d_G(u,v)$ is the length of a shortest path of G joining u and v.

A *cycle* C is a special path with at least three vertices such that the first vertex is the same as the last one. A cycle C is called k-cycle if $l(C)=k$. A path (respectively, cycle) which traverses each vertex of G exactly once is a *hamiltonian path* (respectively, *hamiltonian cycle*). They are defined as follows.

Definition 1 *Let G be a graph. For any two vertices $u,v\in V(G)$, a cycle C in G is called a geodesic cycle between u and v if the shortest path joining u and v in C is also a shortest path joining u and v in G.*

In definition 1, we define a geodesic k-cycle between two distinct vertices, u and v, such that the distance of u and v in the cycle is the smallest over all k-cycles passing through u and v in G. The transmission delay between u and v in this cycle will be the minimum.

Definition 2 *Let G be a graph. For any two vertices $u,v\in V(G)$, a cycle C is called a balanced cycle between u and v if $d_C(u,v)=max\{d_C(x,y)\mid x,y\in V(C)\}$.*

Consequently, if C is a balanced k-cycle between u and v, $d_C(u,v)=\lfloor\frac{k}{2}\rfloor$. In a bipartite graph, there are only even cycles and vertex set is divided into two partite sets. Hence we modify definition 2 for bipartite graphs.

Definition 3 *Let $G=(B\cup W,E)$ be a bipartite graph. For any two vertices u and v in G, a cycle C is called a balanced cycle between u and v if $d_C(u,v)=max\{d_C(x,y)\mid x$ and u are in the same partite set, and y and v are in the same partite set. $\}$.*

One can observe that a balanced $2l$-cycle C with $l\geq 2$ between u and v in a bipartite graph satisfies one of the following four conditions: (For example see Figure 1.)

(a) u and v are in different partite sets, l is odd, and $d_C(u,v)=l$.

(b) u and v are in different partite sets, l is even, and $d_C(u,v)=l-1$.

(c) u and v are in the same partite set, l is even, and $d_C(u,v)=l$.

(d) u and v are in the same partite set, l is odd, and $d_C(u,v)=l-1$.

A bipartite graph is *vertex-bipancyclic* [23] if every vertex lies on a cycle of every even length from 4 to $|V(G)|$ inclusive. Similarly, a bipartite graph is *edge-bipancyclic* if every edge lies on a cycle of every even length from 4 to $|V(G)|$ inclusive. Obviously, every edge-bipancyclic graph is vertex-bipancyclic. A bipartite graph G is *geodesic bipancyclic* (respectively, *balanced bipancyclic*) if for each pair of vertices $u,v\in V(G)$, it contains a geodesic cycle (respectively, balanced cycle) of every even length of k satisfying $max\{2d_G(u,v),4\}\leq k\leq|V(G)|$ between u and v. It is observed that every geodesic bipancyclic graph is edge-bipancyclic.

Let $u=u_{n-1}u_{n-2}\ldots u_1u_0$ be a n-bit binary strings. The *Hamming weight* of u, denoted by $w(u)$, is the number of u_i such that $u_i=1$. Let $u=u_{n-1}u_{n-2}\ldots u_1u_0$ and $v=v_{n-1}v_{n-2}\ldots v_1v_0$ be two distinct n-bit binary strings. The *Hamming distance* $h(u,v)$ between two vertices u and v is the number of different bits in the corresponding strings

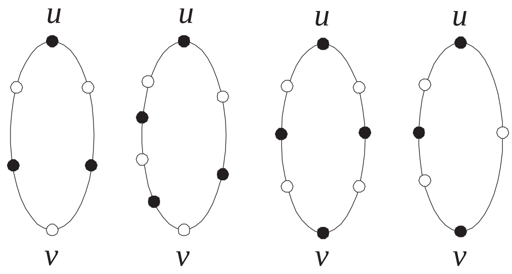

Figure 1: (a) A balanced 6-cycle between u and v that are in different partite sets. (b) A balanced 8-cycle between u and v that are in different partite sets. (c) A balanced 8-cycle between u and v that are in the same partite set. (d) A balanced 6-cycle between u and v that are in the same partite set.

of both vertices. The *n-dimensional hypercube*, denoted by Q_n, consists of all n-bit binary strings as its vertices and two vertices u and v are adjacent if and only if $h(u, v) = 1$. For $0 \leq k < n$, we use v^k to denote the binary string u derived from the binary string $v_{n-1}v_{n-2} \dots v_1v_0$ such that $u_k = 1 - v_k$ and $u_i = v_i$ if $i \neq k$. An edge (u, v) in $E(Q_n)$ is of *dimension* i if $u = v^i$. It is known that $d_{Q_n}(u, v) = h(u, v)$. For a given $0 \leq i < n$, we can partition Q_n along dimension i into two $(n-1)$-cubes such that Q^0_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_i = 0\}$ and Q^1_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_i = 1\}$. We have Q^0_{n-1} and Q^1_{n-1} being isomorphic to Q_{n-1}. The following lemmas are useful in our later proofs.

Lemma 1 *[25] For $n \geq 2$, Q_n is edge-bipancyclic.*

Lemma 2 *[20] Let u and v be two arbitrary distinct vertices with the same partite set in Q_n for $n \geq 2$. Then, for any vertex w such that $h(w, u)$ is odd, there exists a path joining u and v passing all vertices of Q_n except w.*

Lemma 3 *[20] Let u and v be two arbitrary distinct vertices in Q_n and $h(u, v) = d$, where $n \geq 2$. There are paths formed by $\langle u, P[u, v], v\rangle$ in the Q_n with lengths d,$d+2$,$d+4, \dots, c$, where $c = 2^n - 1$ if d is odd, and $c = 2^n - 2$ if d is even.*

3 Geodesic bipancyclicity of Hypercubes

In this section, we address the existence of geodesic cycles between any pair of vertices in Q_n. Given two vertices u and v, the transmission delay from u to v is minimum in a geodesic cycle. For any pair of vertices u and v, a cycle C is called geodesic cycle between u and v in Q_n if $h(u, v)$=$d_C(u, v)$. Q_n is geodesic bipancyclic if Q_n contains a geodesic cycle of every even length of k satisfying $max\{2h(u, v), 4\} \leq k \leq 2^n$. The following lemma about shortest path properties of Q_n will be used in the proof of Theorem 1.

Lemma 4 *Let $u, v \in Q_n$.*

(1) *If $u, v \in Q^0_{n-1}$(respectively, Q^1_{n-1}), then there exists a shortest path joining u and v in Q_n with all its vertices in Q^0_{n-1} (respectively, Q^1_{n-1}).*

(2) *Let $u \in Q^0_{n-1}$ and $v \in Q^1_{n-1}$. Then, (i) There exists a shortest path S joining u and v in Q_n with all its vertices (except u) in Q^1_{n-1}. (ii) There exists a shortest path S joining u and v in Q_n with all its vertices (except v) in Q^0_{n-1}.*

Proof. Let $u = u_{n-1}u_{n-2} \dots u_1u_0$ and $v = v_{n-1}v_{n-2} \dots v_1v_0$ be any two distinct vertices of Q_n and $h(u, v) = d$ where $d \geq 1$.

Case 1: $u, v \in V(Q^0_{n-1})$ or $u, v \in V(Q^1_{n-1})$.

Hence $u_{n-1} = v_{n-1}$ and $h(u, v) \leq n - 1$. Let $\alpha = u_{n-2}u_{n-3} \dots u_0$ and $\beta = v_{n-2}v_{n-3} \dots v_0$ be two $(n-1)$-bit binary strings. Hence α and β are two vertices of Q_{n-1}. Obviously $h(\alpha, \beta) = h(u, v) \leq n - 1$. Let $S = \langle \alpha^0 = \alpha, \alpha^1, \alpha^2, \dots, \alpha^{h(u,v)} = \beta\rangle$ denote a shortest path joining α and β in Q_{n-1} where $\alpha^i = \alpha^i_{n-2}\alpha^i_{n-3} \dots \alpha^i_0$ is an $(n-1)$-bit binary string for $0 \leq i \leq h(u, v)$. Let $u^i = u_{n-1}\alpha^i_{n-2}\alpha^i_{n-3} \dots \alpha^i_0$ be an n-bit binary string with first bit u_{n-1} where $0 \leq i \leq h(u, v)$. Therefore, all vertices lying on the path $R = \langle u^0, u^1, u^2, \dots u^{h(u,v)}\rangle$ are in $Q^{u_{n-1}}_{n-1}$ where $u^0 = u$, $u^{h(u,v)} = v$, and $u_{n-1} = 0, 1$. Obviously, the path R is a shortest path joining u and v in Q_n.

Case 2: $u \in V(Q^i_{n-1})$ and $v \in V(Q^{1-i}_{n-1})$ where $i = 0, 1$.

Without loss of generality, we may assume that $i = 0$, i.e., $u \in V(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Let $x = \overline{u_{n-1}}u_{n-2} \dots u_0$ and $y = \overline{v_{n-1}}v_{n-2} \dots v_0$. Obviously, $x \in V(Q^1_{n-1})$ and $y \in V(Q^0_{n-1})$, respectively. Also $h(x, u) = 1$ and $h(y, v) = 1$.

By case 1, there exists a shortest path, denoted $\langle u, u^1, u^2, \dots, y\rangle$, joining u and y in Q^0_{n-1} and a shortest path, denoted $\langle v, v^1, v^2, \dots, x\rangle$, connecting v and x in Q^1_{n-1}, respectively. Therefore, there exists a shortest path $S = \langle u, u^1, u^2, \dots, y, v\rangle$ joining u and v in Q_n where all vertices (except u) are in Q^1_{n-1}. Meanwhile there exists a shortest path $R = \langle v, v^1, v^2, \dots, x, u\rangle$ joining u and v in Q_n where all vertices (except v) are in Q^0_{n-1}. □

Theorem 1 *Q_n is geodesic bipancyclic if $n \geq 2$.*

Proof. Let $u = u_{n-1}u_{n-2}\dots u_1u_0$ and $v = v_{n-1}v_{n-2}\dots v_1v_0$ be any two distinct vertices of Q_n and $h(u,v) = d$.

Case 1: $h(u,v) = 1$, i.e. u and v are adjacent. Applying Lemma 1, we have that every edge in Q_n lies on a cycle of every even length from 4 to 2^n. The theorem holds for $h(u,v) = 1$.

Case 2: $h(u,v) = d \geq 2$, i.e. u and v are not adjacent. We prove this theorem by induction on n. Obviously, the theorem holds for $n = 2$. Assume that the theorem is true for every integer $2 \leq k < n$. To prove this theorem, we establish every geodesic cycle of even length k between u and v in Q_n where $2d \leq k \leq 2^n$. Partitioning Q_n along dimension 0, we obtain two disjoint $(n-1)$-subcubes Q^0_{n-1} and Q^1_{n-1} such that Q^0_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_0 = 0\}$ and Q^1_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_0 = 1\}$. The proof of case 2 is divided into two cases: u and v are in the same subcube Q^0_{n-1} (or Q^1_{n-1}), and u and v are in different subcubes.

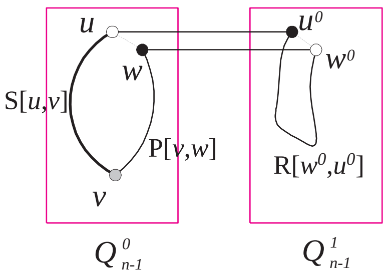

Figure 2: The geodesic cycle between u and v of even length of $2^{n-1} + 1 + l(R)$ where $l(R) = 1, 3, 5, 7, \dots, 2^{n-1} - 1$, $l(S) = h(u,v)$, and $l(S) + l(P) = 2^{n-1} - 1$.

Subcase 2-1: $u, v \in Q^0_{n-1}$ or $u, v \in Q^1_{n-1}$. (See Figure 2.) Without loss of generality, we may assume that u and v are in Q^0_{n-1}., i.e. $u_0 = v_0 = 0$. By induction hypothesis, Q^0_{n-1} is geodesic bipancyclic. Q^0_{n-1} contains every geodesic cycle of even length k satisfying $2d \leq k \leq 2^{n-1}$ between u and v. Applying Lemma 4, we have that $d_{Q_n}(u,v) = d_{Q^0_{n-1}}(u,v) = h(u,v) = d$. It is observed that every geodesic cycle between u and v in Q^0_{n-1} is a geodesic cycle between u and v in Q_n. Thus, geodesic cycle between u and v with even length of k satisfying $2d \leq k \leq 2^{n-1}$ in Q_n can be found in Q^0_{n-1}.

The rest of the proof of this subcase is to find every geodesic cycle of even length from $2^{n-1} + 2$ to 2^n between u and v in Q_n. Let C be a geodesic cycle of length 2^{n-1} between u and v in Q^0_{n-1}. Hence C is a hamiltonian cycle in Q^0_{n-1} passing through u and v, and $d_C(u,v) = h(u,v) = d$. Since $n \geq 3$, $l(C) \geq 4$. One may choice a adjacent vertex w of u in C such that the cycle C can be written as $\langle u, S[u,v], v, P[v,w], w, u \rangle$ where $l(S) = h(u,v)$. Hence $l(S) = d$ and $l(P) \geq 1$. Since (u, u^0) and (w, w^0) are two edges of dimension 0, u^0 and w^0 are two adjacent vertices in Q^1_{n-1}. By Lemma 3, there exist paths formed by $\langle w^0, R[w^0, u^0], u^0 \rangle$ in the Q^1_{n-1} with length $1,3,5,\dots,2^n - 1$. Therefore, we can construct a cycle $C' = \langle u,\ S[u,v],\ v,\ P[v,w],\ w,\ w^0,\ R[w^0,u^0],\ u^0,\ u \rangle$ containing the shortest path $S[u,v]$. It is observed that C' is a geodesic cycle with even length of $l(C') = 2^{n-1} - 1 + 2 + l(R)$ between u and v in Q_n where $l(R) = 1, 3, 5, \dots, 2^{n-1} - 1$. Therefore, $2^{n-1} + 2 \leq l(C') \leq 2^n$. The proof of this subcase is completed.

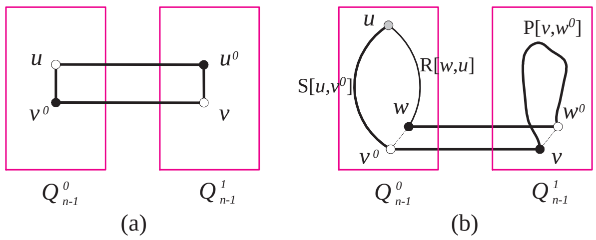

Figure 3: (a) A geodesic 4-cycle between u and v if $h(u,v) = 2$. (b) The geodesic cycle between u and v of even length of $m + k + 1$ where $m = 1, 3, 5, 7, \dots, 2^{n-1} - 1$, k is an even integer satisfying $max\{2h(u,v) - 2, 4\} \leq k \leq 2^{n-1}$, $l(S) = h(u,v) - 1$, $l(P) = m$, and $l(S) + l(R) = k - 1$.

Subcase 2-2: $u \in Q^0_{n-1}$ and $v \in Q^1_{n-1}$ (or $v \in Q^0_{n-1}$ and $u \in Q^1_{n-1}$). (See Figure 3.) Without loss of generality, we may assume that $u \in Q^0_{n-1}$ and $v \in Q^1_{n-1}$. Applying Lemma 4, there exists a shortest path $S[u,v]$ joining u and v in Q_n with all vertices (except v) in Q^0_{n-1}. We write the path $S[u,v]$ as $\langle u, P[u,v^0], v^0, v \rangle$. Hence $l(P[u,v^0]) = d - 1$. Meanwhile, there exists a shortest path $S_1[v,u]$ joining v and u in Q_n with all vertices (except u) in Q^1_{n-1}. We write the path $S_1[v,u]$ as $\langle v, P_1[v,u^0], u^0, u \rangle$. Then, $l(P_1[v,u^0]) = d - 1$. Therefore, $\langle u, P[u,v^0], v^0, v, P_1[v,u^0], u^0, u \rangle$ forms a geodesic cycle of length $2d$ between u and v in Q_n. If $d = 2$, the geodesic cycle of length 4 between u and v exists.

Since u and v^0 are in Q^0_{n-1}, by induction hypothesis, Q^0_{n-1} is geodesic bipancyclic. Let C be a geodesic cycle with even length of k satisfying $max\{2(d-1), 4\} \leq k \leq 2^{n-1}$ between u and v^0 in Q^0_{n-1}. The cycle C can be rewritten as $\langle u, S[u, v^0], v^0, w, R[w, u], u\rangle$ where $l(S[u, v^0]) = d - 1$ and $l(R[w, u]) = k - d$. Hence $\langle u, S[u, v^0], v^0, v\rangle$ is a shortest path of Q_n joining u and v. Since v^0 and w are adjacent, v and w^0 are adjacent in Q^1_{n-1}. By Lemma 3, there are paths formed by $\langle v, P[v, w^0], w^0\rangle$ in the Q^1_{n-1} with length $m =$ 1, 3, 5, ..., $2^{n-1} - 1$. Therefore, we can construct a cycle $C' = \langle u, S[u, v^0], v^0, v, P[v, w^0], w^0, w, R[w, u], u\rangle$ containing the shortest path $S[u, v]$. It is observed that C' is a geodesic cycle with even length of $l(C') = d - 1 + 1 + m + 1 + k - d = m + k + 1$ between u and v in Q_n. Therefore, $max\{2d, 6\} \leq l(C') \leq 2^n$. The theorem is proved. □

4 Balanced bipancyclicity of Hypercubes

To route a packet from u to v in a k-cycle, one may first breaks the packet into two smaller pieces. Then, route the two pieces along two internal vertex-disjoint paths to the two intermediate vertices v_1, v_2. In the second phase, symmetrically, the two pieces are routed from the intermediate vertices v_1, v_2 to their common destination v. The packet is combined in v until all pieces of this packet arrived. Therefore, this kind of transmission delay between u and v in a cycle is determined by the longest path between u and v in this cycle. It is of interest to find a cycle passing through u and v such that lengths of two disjoint paths between u and v in this cycle are as equal as possible.

Q_n is balanced bipancyclic if for each pair of vertices $u, v \in V(Q_n)$, it contains a balanced cycle of every even length of $2l$ satisfying $max\{h(u, v), 2\} \leq l \leq 2^{n-1}$ between u and v. The following lemma is useful in the proof of Theorem 2.

Lemma 5 *For any two disjoint edges (u, v) and (w, z) in Q_n with $n \geq 2$, there exist two disjoint paths, $\langle u, P_1[u, v], v\rangle$ and $\langle w, P_2[w, z], z\rangle$, in Q_n with equal length k where $k =$ 1, 3, 5 , 7, ..., $2^{n-1} - 1$.*

Proof. We prove this lemma by induction on n. Obviously, the lemma holds for $n = 2$. Assume that the lemma is true for every integer $2 \leq m < n$. Suppose that (u, v) is an edge of dimension i and (w, z) is an edge of dimension j where $0 \leq i, j \leq n - 1$. Since $n \geq 3$, there exists an integer r such that $r \neq i$ and $r \neq j$. We may partition Q_n along dimension r into two $(n-1)$-subcubes such that Q^0_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_r = 0\}$ and Q^1_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_r = 1\}$. Since $r \neq i$ and $r \neq j$, (u, v) and (w, z) are in the same subcube or in different subcubes. The proof is divided into two cases: (u, v) and (w, z) lie on the same subcube Q^0_{n-1} (or Q^1_{n-1}), and (u, v) and (w, z) are in different subcubes.

Case 1: (u, v) and (w, z) lie on the same subcube Q^0_{n-1} (or Q^1_{n-1}).

Without loss of generality, we may assume that (u, v) and (w, z) lie on Q^0_{n-1}. By induction hypothesis, there exist two disjoint paths $P_1[u, v]$ and $P_2[w, z]$ in Q^0_{n-1} with equal length of k where $k = 1, 3, 5, 7, \ldots, 2^{n-2} - 1$.

In the rest of Case 1, we construct two disjoint paths $P_1[u, v]$ and $P_2[w, z]$ in Q_n with equal length of k where $k = 2^{n-2} + 1, 2^{n-2} + 3, 2^{n-2} + 5, \ldots, 2^{n-1} - 1$. Let $R_1[u, v]$ and $R_2[w, z]$ be two disjoint paths of length $2^{n-2} - 1$ in Q^0_{n-1}. Since $n \geq 3$, $l(R_1) = l(R_2) \geq 1$. We can rewrite $R_1[u, v]$ (respectively, $R_2[w, z]$) as $\langle u, x, S_1[x, v], v\rangle$ (respectively, $\langle w, y, S_2[y, z], z\rangle$) where $l(S_1) = l(S_2) \geq 0$, and $x = v$ if $l(S_1) = 0$ (respectively, $y = z$ if $l(S_2) = 0$). Let (u, u^r), (x, x^r), (w, w^r), and (y, y^r) be four disjoint edges of dimension r. Hence (u^r, x^r) and (w^r, y^r) are two edges lying Q^1_{n-1}. By the induction hypothesis, there exist two disjoint paths $T_1[u^r, x^r]$ and $T_2[w^r, y^r]$ in Q^1_{n-1} with equal length of m where $m = 1, 3, 5, 7 \ldots, 2^{n-2} - 1$. Therefore, two paths can be constructed as $P_1 = \langle u, u^r, T_1[u^r, x^r], x^r, x, S_1[x, v], v\rangle$ and $P_2 = \langle w, w^r, T_2[w^r, y^r], y^r, y, S_2[y, z], z\rangle$ where $l(P_1) = l(P_2) = 2^{n-2} + m$ and $m =$ 1, 3, 5, 7, ..., $2^{n-2} - 1$. Hence $P_1[u, v]$ and $P_2[w, z]$ are two disjoint paths with equal length of $2^{n-2} + 1, 2^{n-2} + 3, 2^{n-2} + 5, \ldots, 2^{n-1} - 1$.

Case 2: (u, v) and (w, z) are in different subcubes.

Applying Lemma 3, there are paths formed by $\langle u, P_1[u, v], v\rangle$ in the Q^0_{n-1} with length 1, 3, 5, 7, ..., $2^{n-1} - 1$ and there are paths formed by $\langle w, P_2[w, z], z\rangle$ in the Q^1_{n-1} with length 1, 3, 5, 7, ..., $2^{n-1} - 1$. The Lemma is proved. □

Theorem 2 *Q_n is balanced bipancyclic if $n \geq 2$.*

Proof. Let $u = u_{n-1}u_{n-2} \ldots u_1u_0$ and $v = v_{n-1}v_{n-2} \ldots v_1v_0$ be any two distinct vertices of Q_n and $h(u, v) = d$. To prove the theorem, we will find every balanced $2l$-cycle between u and v where $max\{d, 2\} \leq l \leq 2^{n-1}$. The proof is divided into two parts: $d = 1$ and $d \geq 2$.

Case 1: $d = 1$, i.e. u and v are adjacent. (See Figure 4.)

Without loss of generality, we may assume that (u, v) is an edge of dimension 0. We may partition Q_n along dimension 1 into two $(n-1)$-subcubes such that Q^0_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_1 = 0\}$ and Q^1_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_1 = 1\}$. Therefore, u and v are in the same subcube Q^0_{n-1} or

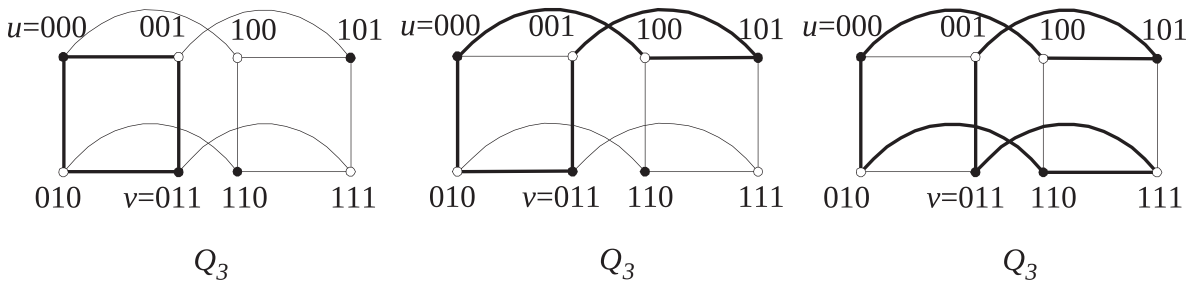

Figure 5: Three balanced cycles between $u = 000$ and $v = 011$ in Q_3.

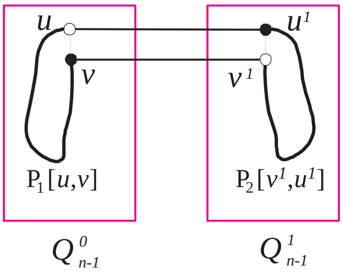

Figure 4: (a) Let $l(P_1) = l(P_2) = k$. Then, a balanced $(2k + 2)$-cycle between u and v is constructed, where $k = 1, 3, 5, \ldots, 2^{n-1} - 1$. (b) Let $l(P_1) = k + 2$ and $l(P_2) = k$. Then, a balanced $(2k + 4)$-cycle between u and v is constructed, where $k = 1, 3, 5, \ldots, 2^{n-1} - 3$.

Q^1_{n-1}. Without loss of generality, we suppose that u and v are in Q^0_{n-1}.

Let (u, u^1) and (v, v^1) be two edges of dimension 1. Hence $h(u^1, v^1) = 1$ and $u^1, v^1 \in V(Q^1_{n-1})$. Applying Lemma 3, there are paths formed by $\langle u, P_1[u, v], v\rangle$ in the Q^0_{n-1} with length $k_1 = 1$, 3, 5, 7, ..., $2^{n-1} - 1$ and there are paths formed by $\langle v^1, P_2[v^1, u^1], u^1\rangle$ in the Q^1_{n-1} whose lengths are $k_2 = 1, 3, 5, 7, \ldots, 2^{n-1} - 1$. We can construct a cycle as $C = \langle u, P_1[u, v], v, v^1, P_2[v^1, u^1], u^1, u\rangle$ of length $l(C) = k_1 + k_2 + 2$ where $k_1 = l(P_1)$ and $k_2 = l(P_2)$. Obviously, the cycle C passes through u and v.

(a). Balanced $(2k + 2)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-1} - 1$. Let $k_1 = k$ and $k_2 = k$. Then, $l(C) = 2k + 2$ where $k = 1, 3, 5, \ldots, 2^{n-1} - 1$. Hence $d_C(u, v) = k = \frac{l(C)}{2} - 1$. Since d is odd, $\frac{l(C)}{2}$ is even, and $d_C(u, v) = \frac{l(C)}{2} - 1$, the cycle C is a balanced $(2k+2)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-1} - 1$.

(b). Balanced $(2k + 4)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-1} - 3$. Let $k_1 = k + 2$ and $k_2 = k$. Then, $l(C) = 2k + 4$ where $k = 1, 3, 5, \ldots, 2^{n-1} - 3$. Hence $d_C(u, v) = k + 2 = \frac{l(C)}{2}$. Since d is odd, $\frac{l(C)}{2}$ is odd, and $d_C(u, v) = \frac{l(C)}{2}$, the cycle C is a balanced $(2k + 4)$-cycle between u and v where $k = 1, 3, 5, \ldots, 2^{n-1} - 3$.

Case 2: $d \geq 2$, i.e. u and v are not adjacent.

We prove this case by induction on n. Obviously, the proof of case 2 holds for $n = 2$. Assume that the proof of case 2 is true for every integer $2 \leq m < n$. Let $u = u_{n-1}u_{n-2} \ldots u_1u_0$ and $v = v_{n-1}v_{n-2} \ldots v_1v_0$ be any two distinct vertices of Q_n and $h(u, v) = d$. Partitioning Q_n along dimension 0, Q_n can be divided into two $(n - 1)$-subcubes where Q^0_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_0 = 0\}$ and Q^1_{n-1} denotes the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_0 = 1\}$.

Subcase 2-1: $u, v \in Q^0_{n-1}$ or $u, v \in Q^1_{n-1}$. (See Figure 5 and Figure 6.)

Without loss of generality, we may assume that $u, v \in Q^0_{n-1}$. For the basis of this proof, we consider Q_3. It is clear that Q_3 is balanced bipancyclic. (See Figure 5 for an illustration).

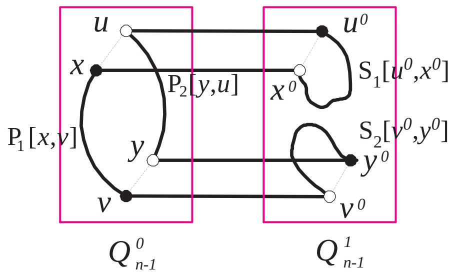

Figure 6: Let $l(S_1) = l(S_2) = k$ where $k = 1, 3, 5, \ldots, 2^{n-1}$. Then, a balanced $(m + 2k + 2)$-cycle between u and v is constructed, where $\langle u, x, P_1[x, v], v, y, P_2[y, u], u\rangle$ is balanced m-cycle between u and v of Q^0_{n-1} where $m \geq 6$.

Suppose that $n \geq 4$. By induction hypothesis, Q^0_{n-1} is balanced bipancyclic. Every balanced $2l$-

cycle between u and v in Q_n can be found in Q^0_{n-1} where $d \le l \le 2^{n-2}$. Let C be a balanced m-cycle with $m \ge 6$ between u and v in Q^0_{n-1}. Hence we rewrite the cycle C as $\langle u,\ x,\ P_1[x,v],\ v,\ y,\ P_2[y,u],\ u\rangle$. Let (u,u^0), (x,x^0), (v,v^0), and (y,y^0) be four edges of dimension 0. It is observed that u^0, x^0, v^0, and y^0 are four distinct vertices in Q^1_{n-1}, and that (u^0,x^0) and (v^0,y^0) are two disjoint edges in Q^1_{n-1}. Applying Lemma 5, there exist two disjoint paths $S_1[u^0,x^0]$ and $S_2[v^0,y^0]$ in Q^1_{n-1} such that $l(S_1) = l(S_2) = k$ where $k = 1,3,5,7,\dots,2^{n-2}-1$. Therefore, we may construct a cycle $C' = \langle u,\ u^0,\ S_1[u^0,x^0],\ x^0,\ x,\ P_1[x,v],\ v,\ v^0,\ S_2[v^0,y^0],\ y^0,\ y,\ P_2[y,u],\ u\rangle$ passing through u and v. Hence $l(C') = m+2k+2$.

Subcase 2-1-1: Balanced $(2^{n-1}+2k)$-cycle between u and v where $k = 1,3,5,\dots,2^{n-2}-1$. Let $m = 2^{n-1}-2$. Therefore, $l(C') = 2^{n-1}+2k$.

(a). Suppose that d is odd. Since C is a balanced $(2^{n-1}-2)$-cycle between u and v, and $\frac{l(C)}{2} = 2^{n-2}-1$ is odd, $d_C(u,v) = 2^{n-2}-1$. It is clearly that $d_{C'}(u,v) = d_C(u,v)+k+1 = 2^{n-2}+k$ and $\frac{l(C')}{2} = 2^{n-2}+k$. Since d is odd, $\frac{l(C')}{2}$ is odd, and $d_{C'}(u,v) = 2^{n-2}+k = \frac{l(C')}{2}$, the cycle C' is a balanced $(2^{n-1}+2k)$-cycle between u and v in Q_n where $k = 1,3,5,\dots,2^{n-2}-1$.

(b). Suppose that d is even. Since C is a balanced $(2^{n-1}-2)$-cycle between u and v, and $\frac{l(C)}{2} = 2^{n-2}-1$ is odd, $d_C(u,v) = 2^{n-2}-2$. It is clearly that $d_{C'}(u,v) = d_C(u,v)+k+1 = 2^{n-2}+k-1$ and $\frac{l(C')}{2} = 2^{n-2}+k$. Since d is even, $\frac{l(C')}{2}$ is odd, and $d_{C'}(u,v) = 2^{n-2}+k-1 = \frac{l(C')}{2}-1$, the cycle C' is a balanced $(2^{n-1}+2k)$-cycle between u and v in Q_n where $k = 1,3,5,\dots,2^{n-2}-1$.

Subcase 2-1-2: Balanced $(2^{n-1}+2k+2)$-cycle between u and v where $k = 1, 3, 5, \dots, 2^{n-2}-1$. Let $m = 2^{n-1}$. Therefore, $l(C') = 2^{n-1}+2k+2$.

(a). Suppose that d is odd. Since C is a balanced 2^{n-1}-cycle between u and v, and $\frac{l(C)}{2} = 2^{n-2}$ is even, $d_C(u,v) = 2^{n-2}-1$. It is clearly that $d_{C'}(u,v) = d_C(u,v)+k+1 = 2^{n-2}+k$ and $\frac{l(C')}{2} = 2^{n-2}+k+1$. Since d is odd, $\frac{l(C')}{2}$ is even, and $d_{C'}(u,v) = 2^{n-2}+k = \frac{l(C')}{2}-1$, the cycle C' is a balanced $(2^{n-1}+2k+2)$-cycle between u and v in Q_n where $k = 1,3,5,\dots,2^{n-2}-1$.

(b). Suppose that d is even. Since C is a balanced 2^{n-1}-cycle between u and v, and $\frac{l(C)}{2} = 2^{n-2}$ is even, $d_C(u,v) = 2^{n-2}$. It is clearly that $d_{C'}(u,v) = d_C(u,v)+k+1 = 2^{n-2}+k+1$ and $\frac{l(C')}{2} = 2^{n-2}+k+1$. Since d is even, $\frac{l(C')}{2}$ is even, and $d_{C'}(u,v) = 2^{n-2}+k+1 = \frac{l(C')}{2}$, the cycle C' is a balanced $(2^{n-1}+2k+2)$-cycle between u and v in Q_n where $k = 1,3,5,\dots,2^{n-2}-1$.

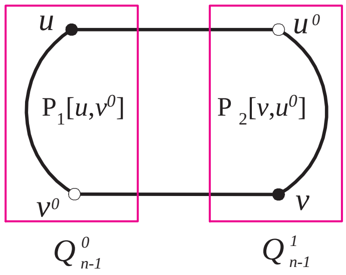

Figure 7: $h(u,v) = d$ is even. (a) Let $l(P_1) = l(P_2) = k$. Then, a balanced $(2k+2)$-cycle between u and v is constructed, where $k = d-1, d+1, d+3,\dots,2^{n-1}-1$. (b) Let $l(P_1) = k+2$ and $l(P_2) = k$. Then, a balanced $(2k+4)$-cycle between u and v is constructed, where $k = d-1, d+1, d+3,\dots,2^{n-1}-3$.

Subcase 2-2: $u \in Q^0_{n-1}$ and $v \in Q^1_{n-1}$ (or $v \in Q^0_{n-1}$ and $u \in Q^1_{n-1}$).

Without loss of generality, we may assume that $u \in Q^0_{n-1}$ and $v \in Q^1_{n-1}$. Let (u,u^0) and (v,v^0) be two edges of dimension 0. Hence $u^0 \in V(Q^1_{n-1})$ and $v^0 \in V(Q^0_{n-1})$, and $h(u,v^0) = h(v,u^0) = d-1$.

Subcase 2-2-1: d is even, i.e. u and v are in the same partite set. (See Figure 7.) Hence u^0 and v are in different partite sets. Similarly, v^0 and u are in different partite sets. By Lemma 3, there exists a path $P_1[u,v^0]$ (respectively, $P_2[v,u^0]$) connecting u and v^0 (respectively, v and u^0) where $l(P_1) = d-1, d+1, d+3,\dots,2^{n-1}-1$ (respectively, $l(P_2) = d-1, d+1, d+3,\dots,2^{n-1}-1$). The cycle C can be constructed as $\langle u, P_1[u,v^0], v^0, v, P_2[v,u^0], u^0, u\rangle$. Therefore, the cycle C passing through u and v, and $l(C) = k_1+k_2+2$ where $k_1 = l(P_1)$ and $k_2 = l(P_2)$.

(a). Balanced $(2k+2)$-cycle between u and v where $k = d-1,\ d+1,\ d+3,\ \dots,\ 2^{n-1}-1$. Let $k_1 = k$ and $k_2 = k$ where $k = d-1, d+1, d+3, \dots, 2^{n-1}-1$. Therefore, $l(C) = 2k+2$. One can observe that $\frac{l(C)}{2} = k+1$ and $d_C(u,v) = k+1$. Since d is even, $\frac{l(C)}{2}$ is even, and $d_C(u,v) = \frac{l(C)}{2}$, the cycle C is a balanced $(2k+2)$-cycle between u and v where $k = d-1, d+1, d+3, \dots, 2^{n-1}-1$.

(b). Balanced $(2k+4)$-cycle between u and v where $k = d-1,\ d+1,\ d+3,\ \dots,\ 2^{n-1}-3$. Let $k_1 = k+2$ and $k_2 = k$ where $k = d-1,\ d+1,\ d+3,\ \dots,\ 2^{n-1}-3$. Therefore, $l(C) = 2k+4$. One can observe that $\frac{l(C)}{2} = k+2$ and $d_C(u,v) = k+1$. Since d is even, $\frac{l(C)}{2}$ is odd, and $d_C(u,v) = \frac{l(C)}{2}-1$, the cycle C is a balanced $(2k+4)$-cycle between u and v where $k = d-1, d+1, d+3, \ldots, 2^{n-1}-3$.

Subcase 2-2-2: d is odd, i.e. u and v are in different partite sets. (See Figure 8.) Hence u^0 and v are in the same partite set. Similarly, v^0 and u are in the same partite set. By Lemma 3, there exists a paths $P_1[u, v^0]$ (respectively, $P_2[v, u^0]$) connecting u and v^0 (respectively, v and u^0) where $l(P_1) = d-1, d+1, d+3, 2^{n-1}-2$ (respectively, $l(P_2) = d-1, d+1, d+3, \ldots, 2^{n-1}-2$). The cycle C can be constructed as $\langle u, P_1[u, v^0], v^0, v, P_2[v, u^0], u^0, u\rangle$. Therefore, the cycle C passing through u and v, and $l(C) = k_1 + k_2 + 2$ where $k_1 = l(P_1)$ and $k_2 = l(P_2)$.

(a). Balanced $(2k+2)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-2$. Let $k_1 = k$ and $k_2 = k$ where $k = d-1, d+1, d+3, \ldots$, $2^{n-1}-2$. Therefore, $l(C) = 2k+2$. One can observe that $\frac{l(C)}{2} = k+1$ and $d_C(u, v) = k+1$. Since d is odd, $\frac{l(C)}{2}$ is odd, and $d_C(u, v) = \frac{l(C)}{2}$, the cycle C is a balanced $(2k+2)$-cycle between u and v where $k = d-1, d+1, d+3, \ldots, 2^{n-1}-2$.

(b). Balanced $(2k+4)$-cycle between u and v where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-4$. Let $k_1 = k+2$ and $k_2 = k$ where $k = d-1$, $d+1$, $d+3$, …, $2^{n-1}-4$. Therefore, $l(C) = 2k+4$. One can observe that $\frac{l(C)}{2} = k+2$ and $d_C(u, v) = k+1$. Since d is odd, $\frac{l(C)}{2}$ is even, and $d_C(u, v) = \frac{l(C)}{2} - 1$, the cycle C is a balanced $(2k+4)$-cycle between u and v where $k = d-1, d+1, d+3, \ldots, 2^{n-1}-4$.

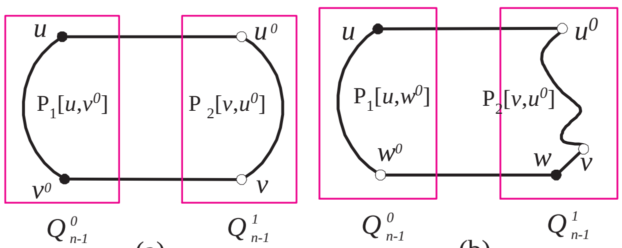

Figure 8: $h(u, v) = d$ is odd. (a.1) Let $l(P_1) = l(P_2) = k$. Then, a balanced $(2k+2)$-cycle between u and v is constructed, where $k = d-1, d+1, d+3, \ldots, 2^{n-1}-2$. (a.2) Let $l(P_1) = k+2$ and $l(P_2) = k$. Then, a balanced $(2k+4)$-cycle between u and v is constructed, where $k = d-1, d+1, d+3, \ldots, 2^{n-1}-4$. (b) A balanced hamiltoian cycle between u and v where $l(P_1) = 2^{n-1}-1$ and $l(P_2) = 2^{n-1}-2$.

(c). Balanced 2^n-cycle between u and v. Let $w \in V(Q^1_{n-1})$ and $h(w, v) = 1$. It is observed that $h(w, u^0)$ is odd. By Lemma 2, there exists a path $P[v, u^0]$ of length $2^{n-1}-2$ joining v and u^0 passing all vertices of Q^1_{n-1} except w. Let (w, w^0) be an edge of dimension 0. Hence w^0 is in Q^0_{n-1}, and w^0 and u are in different partite sets. By Lemma 3, there exists a hamiltonian path $P_1[u, w^0]$ joining u and w^0 in Q^0_{n-1}. Therefore, the longest cycle C between u and v in Q_n can be constructed as $\langle u, P_1[u, w^0], w^0, w, v, P_2[v, u^0], u^0, u\rangle$. Therefore, the cycle C passing through u and v, such that $l(C) = 2^{n-1}-1+1+1+2^{n-1}-2+1 = 2^n$ and $d_C(u, v) = 2^{n-1}-1 = \frac{l(C)}{2} - 1$. Since d is odd, $\frac{l(C)}{2}$ is even, and $d_C(u, v) = \frac{l(C)}{2} - 1$, the cycle C is a balanced cycle between u and v. The theorem is proved. □

5 Conclusions

In this paper, we address the existence of geodesic cycles and balanced cycles between any pair of vertices in Q_n. Given two vertices u and v, the transmission delay from u to v is minimum in a geodesic cycle. We prove that Q_n is a geodesic bipancyclic, i.e. for any two distinct vertices u and v, there exists a geodesic cycle of every even length of k satisfying $max\{2h(u, v), 4\} \leq k \leq 2^n$ in Q_n.

We also deal with the other kind of transmission delay from one vertex to others. To route a packet from u to v in a cycle, one may first breaks the packet into two smaller pieces. Then, route the two pieces along two internal vertex-disjoint paths to destination v. The packet is combined in v until these two pieces arrived. It is of interest to find a cycle passing through u and v such that lengths of two disjoint paths between u and v in this cycle are as equal as possible. Therefore, we define the notion of balanced cycle between u and v. We prove that Q_n is balanced bipancyclic., i.e. for any two distinct vertices u and v, there exists a balanced cycle of every even length of k satisfying $max\{2h(u, v), 4\} \leq k \leq 2^n$ in Q_n.

Numerous variants of hypercube, for example, Augmented cubes [4], Crossed cubes [7], Möbius cubes [5], Twisted cubes [1], and Folded hypercubes [8], have been proposed and proved that they are pancyclic. Geodesic and balanced pancyclicities of Augmented cubes and Crossed cubes are shown in [14] and [17]. However, finding geodesic and banlanced cycles in other variants of hypercube is still open. Our further work tends towards the investigation whether there are more classes of interconnection networks, such as these variations of hypercube, to possess the property of geodesic and balanced pancyclicities.

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