Finite Difference Methods with intrinsic parallelism For parabolic Equations

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Abstract: Based on eight saul'yev asymmetry schemes and the concept of domain decomposition, a class of finite difference method (AGE) with intrinsic parallelism for 1D diffusion equations is constructed. Stability analysis for the method is done. We also pay attention to the implementation of the parallel algorithms for 2D convection-diffusion equations. Based on another group of saul'yev asymmetry schemes and the Crank-Nicolson scheme we construct a class of alternating group explicit Crank-Nicolson method(AGEC-N). Both of the present methods are suitable for parallel computation. Stability analysis are also given. In order to verify the methods, we present several numerical examples at the end of the paper. Results of numerical examples show all the methods are of high accuracy.

Key–Words: parallel computing, domain decomposition, alternating group, parabolic equations, finite difference

1 Introduction

In scientific and engineering computing, we usually need to solve large systems of equation. Many parallel finite difference methods for parabolic equations have been presented [1-4], which are sorted by explicit methods and implicit methods in general. Considering the stability and accuracy of explicit schemes and the computation difficulty of implicit schemes, it is necessary to construct methods with the advantages of explicit methods and implicit methods, that is, simple for computation and good stability. Recently many scientists payed much attention to the finite difference methods with the property of intrinsic parallelism. Evans [5] presented an unconditional stable AGE method based on the concept of domain decomposition originally. The AGE method is used in computing by applying the special combination of several asymmetry schemes to a group of grid points, and the computation in the whole domain can be divided into many sub-domains, Then the numerical solutions at each group can be obtained independently, which highly cuts down the running computing time. So the AGE method is of obvious parallelism. Furthermore, by alternating use of asymmetry schemes at adjacent grid points and different time levels, the AGE method can lead to counteraction of truncation error partly, and then increase the accurate of numerical solution. The AGE method was soon developed to solve other problems such as two-point linear and non-linear boundary value problems, hyperbolic equations, and poisson equations and so on [6-10]. Evans [11] applied the AGE method to 1D convection-diffusion equations. In [12, 13], AGE methods for solving two-dimension convection-diffusion equations were presented, But the accurate of the two methods need to be increased. We will organize this paper as follows: In section 2, we present a group of asymmetric schemes. Based on the schemes a class of unconditionally stable alternating group explicit finite difference method (AGE) with accurate of order four in spatial step size for 1D diffusion equations is derived. In section 3, stability analysis for the AGE method is given. In section 4, we construct a new alternating group explicit Crank-Nicolson method (AGEC-N) for 2D convection-diffusion equations. Stability analysis for AGEC-N method will be given in section 5. Results of comparison with the methods in [5, 12, 13] are presented in section 6. Some conclusions are given at the end of the paper.

2 The AGE Method

In this section, we will consider the periodic boundary value problem of 1D diffusion equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2}, \quad 0 \le t \le T \\ u(x,0) = f(x), \\ u(x,t) = u(x+1,t). \end{cases} \tag{1}$$

The domain $\Omega : (0,1) \times (0,T)$ will be divided into $(m \times N)$ meshes with spatial step size $h = \frac{1}{m}$ in x direction and the time step size $\tau=\frac{T}{N}$. Grid points are denoted by (x_i, t_n) or (i, n), $x_i = ih(i = 0,\ 1, \cdots, m)$, $t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. The numerical solution of (1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$. Let $r = \frac{a\tau}{24h^2}$.

We first present eight asymmetry schemes to approach (1) at $(i, n + \frac{1}{2})$ as follows:

$$(1+15r)u_i^{n+1}-16ru_{i+1}^{n+1}+ru_{i+2}^{n+1} = -2ru_{i-2}^{n}+32ru_{i-1}^{n} +(1-45r)u_i^n + 16ru_{i+1}^n - ru_{i+2}^n \quad (2)$$

$$-14ru_{i-1}^{n+1}+(1+29r)u_i^{n+1}-16ru_{i+1}^{n+1}+ru_{i+2}^{n+1} = -2ru_{i-2}^{n} +18ru_{i-1}^n + (1-31r)u_i^n + 16ru_{i+1}^n - ru_{i+2}^n \quad (3)$$

$$ru_{i-2}^{n+1}-16ru_{i-1}^{n+1}+(1+31r)u_i^{n+1}-18ru_{i+1}^{n+1}+2ru_{i+2}^{n+1} = -ru_{i-2}^n + 16ru_{i-1}^n + (1-29r)u_i^n + 14ru_{i+1}^n \quad (4)$$

$$ru_{i-2}^{n+1}-16ru_{i-1}^{n+1}+(1+45r)u_i^{n+1}-32ru_{i+1}^{n+1}+2ru_{i+2}^{n+1} = -ru_{i-2}^n + 16u_{i-1}^n + (1-15r)u_i^n \quad (5)$$

$$2ru_{i-2}^{n+1}-32ru_{i-1}^{n+1}+(1+45r)u_i^{n+1}-16ru_{i+1}^{n+1}+ru_{i+2}^{n+1} = (1-15r)u_i^n + 16ru_{i+1}^n - ru_{i+2}^n \quad (6)$$

$$2ru_{i-2}^{n+1}-18ru_{i-1}^{n+1}+(1+31r)u_i^{n+1}-16ru_{i+1}^{n+1}+ru_{i+2}^{n+1} = 14ru_{i-1}^n + (1-29r)u_i^n + 16ru_{i+1}^n - ru_{i+2}^n \quad (7)$$

$$ru_{i-2}^{n+1} - 16ru_{i-1}^{n+1} + (1+29r)u_i^{n+1} - 14ru_{i+1}^{n+1} = -ru_{i-2}^n+16ru_{i-1}^n+(1-31r)u_i^n+18ru_{i+1}^n-2ru_{i+2}^n \quad (8)$$

$$ru_{i-2}^{n+1}-16ru_{i-1}^{n+1}+(1+15r)u_i^{n+1} = -ru_{i-2}^n+16ru_{i-1}^n +(1-45r)u_i^n + 32ru_{i+1}^n - 2ru_{i+2}^n \quad (9)$$

Using the schemes mentioned above, we will have three basic point groups:

"G1"group: eight inner points are involved, and $(2)-(9)$ are used at each grid point respectively.

"G2"group: four inner points are involved, and $(2)-(5)$ are used respectively.

(3)"G3"group: two inner points are involved, and $(6)-(9)$ are used respectively.

Let $m = 4s$, here s is an integer. Based on the basic point groups above, the alternating group method will be presented as following:

First at the $(n+1)$-th time level, we will have s point groups. "G1" are used in each group.

Second at the $(n+2)$-th time level, we will have $(s+1)$ point groups. "G3" are used in the left four grid points. "G1" are used in the following $(s-1)$ point groups, while "G2" are used in the right four grid points.

The alternating use of the asymmetry schemes (2)-(9) can lead to partly counteracting of truncation error, and then can increase the numerical accuracy. On the other hand, grouping computation can be obviously obtained. Thus computing in the whole domain can be divided into many sub-domains, and can be worked out with several parallel computers. So the method has the obvious property of parallelism.

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_m^n)^T$, then we can denote the alternating group method as follows:

$$\begin{cases} (I+rA)U^{n+1} = (I-rB)U^n \\ (I+rB)U^{n+2} = (I-rA)U^{n+1} \end{cases} \quad (10)$$

$$A = \begin{pmatrix} A_1 & & & & \\ & A_1 & & & \\ & & \cdots & & \\ & & & A_1 & \\ & & & & A_1 \end{pmatrix}_{m\times m}$$

$$B = \begin{pmatrix} A_3 & & & & D \\ & A_1 & & & \\ & & \cdots & & \\ & & & A_1 & \\ E & & & & A_2 \end{pmatrix}_{m\times m},$$

$$A_1 = \begin{pmatrix} 15 & -16 & & & & & & \\ -14 & 29 & -16 & & & & & \\ & -16 & 31 & 18 & 2 & & & \\ & & -16 & 45 & 32 & 2 & & \\ & & 2 & -32 & 45 & -16 & & \\ & & & 2 & -18 & 31 & -16 & \\ & & & & & -16 & 29 & 14 \\ & & & & & & -16 & 15 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 15 & -16 & & \\ -14 & 29 & -16 & \\ & -16 & 31 & 18 \\ & & -16 & 45 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 32 & 2 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 45 & -16 & & \\ -18 & 31 & -16 & \\ & -16 & 29 & 14 \\ & & -16 & 15 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 2 & -32 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying Taylor's formula to (2)-(9) at (x_i, t_n), we can easily obtain that the truncation error is $\mathcal{O}(\tau^2 + \tau h + \tau h^2 + \tau h^3 + h^4)$ respectively, and alternating use of (2)-(9) can lead to counteraction of the truncation error for the items containing τh, τh^2 and τh^3. Then we can denote the truncation error of (10) as $\mathcal{O}(\tau^2 + h^4)$.

3 Stability Analysis

Kellogg Lemma[14] Let $r > 0$, and G is nonnegative definite real matrix, then:

$$\begin{cases} \|(I + rG)^{-1}\|_2 \le 1 \\ \|(I - rG)(I + rG)^{-1}\|_2 \le 1 \end{cases} \tag{11}$$

Theorem 1 The alternating group method denoted by (10) is unconditionally stable.

Proof: From the construction of the matrices above we can see A and B are both diagonally dominant matrices, which shows A and B are both nonnegative definite real matrices. Then we have:

$$\|(I + rA)^{-1}\|_2 \le 1, \|(I - rA)(I + rA)^{-1}\|_2 \le 1$$

$$\|(I + rB)^{-1}\|_2 \le 1, \|(I - rB)(I + rB)^{-1}\|_2 \le 1.$$

Let n be an even integer, from (10) we have

$$U^n = GU^{n-2} = G^{\frac{n}{2}} U^0,$$

here

$$G = (I + rB)^{-1}(I - rA)(I + rA)^{-1}(I - rB).$$

Let $\overline{G} = (I + rB)G(I + rB)^{-1} = (I - rA)(I + rA)^{-1}(I - rB)(I + rB)^{-1}$, then we have $\rho(G) = \rho(\overline{G}) \le \|\overline{G}\|_2 \le 1$, which shows the AGE method (10) is unconditionally stable.

4 The Construction Of AGEC-N Method

In this section, we will consider the 2D convection-diffusion equation:

$$\frac{\partial u}{\partial t} + k_1 \frac{\partial u}{\partial x} + k_2 \frac{\partial u}{\partial y} = \varepsilon_1 \frac{\partial^2 u}{\partial x^2} + \varepsilon_2 \frac{\partial^2 u}{\partial y^2}$$
$$0 \le x \le 1, 0 \le y \le 1, 0 \le t \le T, \varepsilon_1 > 0, \varepsilon_2 > 0 \tag{12}$$

with initial and boundary conditions:

$$\begin{cases} u(x, y, 0) = f(x), \\ u(0, y, t) = g_1(y, t), u(1, y, t) = g_2(y, t), \\ u(x, 0, t) = h_1(x, t), u(x, 1, t) = h_2(x, t). \end{cases} \tag{13}$$

The domain $\Omega : [0,1] \times [0,1] \times [0,T]$ will be divided into $(m \times m \times N)$ meshes with spatial step size $h = \frac{1}{m}$ in x, y direction and the time step size $\tau = \frac{T}{N}$. Grid points are denoted by (x_i, y_j, t_n) or (i, j, n), $x_i = ih, y_j = jh (i, j = 0, 1, \cdots, m)$, $t_n = n\tau (n = 0, 1, \cdots, \frac{T}{\tau})$. The numerical solution of (12)-(13) is denoted by $u^n_{i,j}$, while the exact solution $u(x_i, y_j, t_n)$. Let $\overline{r} = \frac{\tau}{h^2}$.

Let

$$\delta_x u^n_{i,j} = \frac{u^n_{i+1,j} - u^n_{i,j}}{h}, \ \delta_{\overline{x}} u^n_{i,j} = \frac{u^n_{i,j} - u^n_{i-1,j}}{h},$$

$$\delta_{\widehat{x}} u^n_{i,j} = \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2h}, \ \delta_y u^n_{i,j} = \frac{u^n_{i,j+1} - u^n_{i,j}}{h},$$

$$\delta_{\overline{y}} u^n_{i,j} = \frac{u^n_{i,j} - u^n_{i,j-1}}{h}, \ \delta_{\widehat{y}} u^n_{i,j} = \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2h}$$

$$\delta_t u^n_{i,j} = \frac{u^{n+1}_{i,j} - u^n_{i,j}}{\tau}, \ \delta^2_x u^n_{i,j} = \frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{h^2},$$

$$\delta^2_y u^n_{i,j} = \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{h^2}.$$

We first present sixteen saul'yev asymmetry schemes to approach (12)-(13) at $(i, j, n + \frac{1}{2})$ as follows:

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\tau} + \frac{k_1}{2}\left(\frac{u^{n+1}_{i+1,j} - u^n_{i-1,j}}{2h} + \delta_{\widehat{x}} u^n_{i,j}\right)$$
$$+ \frac{k_2}{2}\left(\frac{u^{n+1}_{i,j+1} - u^n_{i,j-1}}{2h} + \delta_{\widehat{y}} u^n_{i,j}\right) =$$
$$\varepsilon_1\left(\frac{u^{n+1}_{i+1,j} - u^{n+1}_{i,j} - u^n_{i,j} + u^n_{i-1,j}}{h^2} + \delta^2_x u^n_{i,j}\right)$$
$$+ \varepsilon_2\left(\frac{u^{n+1}_{i,j+1} - u^{n+1}_{i,j} - u^n_{i,j} + u^n_{i,j-1}}{h^2} + \delta^2_y u^n_{i,j}\right) \tag{14}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n+1}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n}\right)\end{gathered}\tag{15}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n+1}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n}\right)\end{gathered}\tag{16}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n}\right)\end{gathered}\tag{17}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n+1}\right)\end{gathered}\tag{18}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n+1}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n+1}\right)\end{gathered}\tag{19}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n+1}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n+1}\right)\end{gathered}\tag{20}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n+1}\right)\end{gathered}\tag{21}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n+1}\right)\end{gathered}\tag{22}$$

$$\begin{gathered}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1}\right)\\ +\frac{k_2}{2}\left(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1}\right)=\\ \varepsilon_1\left(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n+1}\right)\\ +\varepsilon_2\left(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n+1}\right)\end{gathered}\tag{23}$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\varepsilon_1(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n+1})$$
$$+\varepsilon_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (24)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\varepsilon_1(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n})$$
$$+\varepsilon_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (25)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n})=$$
$$\varepsilon_1(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n})$$
$$+\varepsilon_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n}) \quad (26)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n})=$$
$$\varepsilon_1(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n+1})$$
$$+\varepsilon_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n}) \quad (27)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n})=$$
$$\varepsilon_1(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n+1})$$
$$+\varepsilon_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n}) \quad (28)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n})=$$
$$\varepsilon_1(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n})$$
$$+\varepsilon_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n}) \quad (29)$$

In the construction of AGEC-N method, we will use the Crank-Nicolson scheme as follows:

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\delta_{\widehat{x}}u_{i,j}^{n+1}+\delta_{\widehat{x}}u_{i,j}^{n})+\frac{k_2}{2}(\delta_{\widehat{y}}u_{i,j}^{n+1}+\delta_{\widehat{y}}u_{i,j}^{n})$$
$$=\varepsilon_1(\delta_x^2u_{i,j}^{n+1}+\delta_x^2u_{i,j}^{n})+\varepsilon_2(\delta_y^2u_{i,j}^{n+1}+\delta_y^2u_{i,j}^{n}) \quad (30)$$

Using the schemes mentioned above, we will have nine basic computing point groups:

"$\omega 1$"group: $(2l)^2$ grid points $(i+p, j+q, n+1), p, q = 1, 2, \cdots, 2l$ are involved. Here $l \geq 3$. $(14), (15), (16), (17)$ are used to get the solution of $(u_{i+1,j+1}^{n+1}, u_{i+l,j+1}^{n+1}, u_{i+l+1,j+1}^{n+1}, u_{i+2l,j+1}^{n+1})$. $(18), (19), (20), (21)$ are used to get the solution of $(u_{i+1,j+l}^{n+1}, u_{i+l,j+l}^{n+1}, u_{i+l+1,j+l}^{n+1}, u_{i+2l,j+l}^{n+1})$. $(22), (23), (24), (25)$ are used to get the solution of $(u_{i+1,j+l+1}^{n+1}, u_{i+l,j+l+1}^{n+1}, u_{i+l+1,j+l+1}^{n+1}, u_{i+2l,j+l+1}^{n+1})$. $(26), (27), (28), (29)$ are used to get the solution of $(u_{i+1,j+2l}^{n+1}, u_{i+l,j+2l}^{n+1}, u_{i+l+1,j+2l}^{n+1}, u_{i+2l,j+2l}^{n+1})$. (30) are used to get the solution of other grid points.

From (14)-(29) we notice that the solution at the sixteen grid points can be obtained independently in the group.

"$\omega 2$"group: l^2 inner points are involved, named $(i+p, j+q, n+1), i, j = 1, 2, \cdots, l$. (24) is used to solve $u_{i+1,j+1}^{n+1}$. (25) is used to solve $u_{i+l,j+1}^{n+1}$. (28) is used to solve $u_{i+1,j+l}^{n+1}$. (29) is used to solve $u_{i+l,j+l}^{n+1}$. (30) are used for the rest grid points.

"$\omega 3$"group: l^2 inner points are involved, named $(i+p, j+q, n+1), i, j = 1, 2, \cdots, l$. (22) is used to solve $u_{i+1,j+1}^{n+1}$. (23) is used to solve $u_{i+l,j+1}^{n+1}$. (26) is used to solve $u_{i+1,j+l}^{n+1}$. (27) is used to solve $u_{i+l,j+l}^{n+1}$. (30) are used for the rest grid points.

"$\omega 4$"group: l^2 inner points are involved, named $(i+p, j+q, n+1), i, j = 1, 2, \cdots, l$. (16) is used to solve $u^{n+1}_{i+1,j+1}$. (17) is used to solve $u^{n+1}_{i+l,j+1}$. (20) is used to solve $u^{n+1}_{i+1,j+l}$. (21) is used to solve $u^{n+1}_{i+l,j+l}$. (30) are used for the rest grid points.

"ω5"group: l^2 inner points are involved, named $(i+p, j+q, n+1), i, j = 1, 2, \cdots, l$. (14) is used to solve $u^{n+1}_{i+1,j+1}$. (15) is used to solve $u^{n+1}_{i+l,j+1}$. (18) is used to solve $u^{n+1}_{i+1,j+l}$. (19) is used to solve $u^{n+1}_{i+l,j+l}$. (30) are used for the rest grid points.

"ω6"group: $2l^2$ inner points are involved, named $(i+p, j+q, n+1), i = 1, 2, \cdots, 2l, j = 1, 2, \cdots, l$. (22) is used to solve $u^{n+1}_{i+1,j+1}$. (23) is used to solve $u^{n+1}_{i+l,j+1}$. (24) is used to solve $u^{n+1}_{i+l+1,j+1}$. (25) is used to solve $u^{n+1}_{i+2l,j+1}$. (26) is used to solve $u^{n+1}_{i+1,j+l}$. (27) is used to solve $u^{n+1}_{i+l,j+l}$. (28) is used to solve $u^{n+1}_{i+l+1,j+l}$. (29) is used to solve $u^{n+1}_{i+2l,j+l}$. (30) are used for the rest grid points.

"ω7"group: $2l^2$ inner points are involved, named $(i+p, j+q, n+1), i = 1, 2, \cdots, 2l, j = 1, 2, \cdots, l$. (14) is used to solve $u^{n+1}_{i+1,j+1}$. (15) is used to solve $u^{n+1}_{i+l,j+1}$. (16) is used to solve $u^{n+1}_{i+l+1,j+1}$. (17) is used to solve $u^{n+1}_{i+2l,j+1}$. (18) is used to solve $u^{n+1}_{i+1,j+l}$. (19) is used to solve $u^{n+1}_{i+l,j+l}$. (20) is used to solve $u^{n+1}_{i+l+1,j+l}$. (21) is used to solve $u^{n+1}_{i+2l,j+l}$. (30) are used for the rest grid points.

"ω8"group: $2l^2$ inner points are involved, named $(i+p, j+q, n+1), i = 1, 2, \cdots, l, j = 1, 2, \cdots, 2l$. (16) is used to solve $u^{n+1}_{i+1,j+1}$. (17) is used to solve $u^{n+1}_{i+l,j+1}$. (20) is used to solve $u^{n+1}_{i+1,j+l}$. (21) is used to solve $u^{n+1}_{i+l,j+l}$. (24) is used to solve $u^{n+1}_{i+1,j+l+1}$. (25) is used to solve $u^{n+1}_{i+l,j+l+1}$. (28) is used to solve $u^{n+1}_{i+1,j+2l}$. (29) is used to solve $u^{n+1}_{i+l,j+2l}$. (30) are used for the rest grid points.

"ω9"group: $2l^2$ inner points are involved, named $(i+p, j+q, n+1), i = 1, 2, \cdots, l, j = 1, 2, \cdots, 2l$. (14) is used to solve $u^{n+1}_{i+1,j+1}$. (15) is used to solve $u^{n+1}_{i+l,j+1}$. (18) is used to solve $u^{n+1}_{i+1,j+l}$. (19) is used to solve $u^{n+1}_{i+l,j+l}$. (22) is used to solve $u^{n+1}_{i+1,j+l+1}$. (23) is used to solve $u^{n+1}_{i+l,j+l+1}$. (26) is used to solve $u^{n+1}_{i+1,j+2l}$. (27) is used to solve $u^{n+1}_{i+l,j+2l}$. (30) are used for the rest grid points.

Let $\widehat{U}^n = (u^n_1, u^n_2, \cdots, u^n_{m-1})^T, u^n_j = (u^n_{1,j}, u^n_{2,j}, \cdots, u^n_{m-1,j})^T$. Based on the basic point groups above, the AGEC-N method will be presented as following:

Let $m-1 = 2lk$, here k is an positive integer, $k \geq 2$. First in order to get the solution of U^{n+1} with U^n known, we divide all the $(m-1)^2$ grid points into k^2 "ω1" groups, and computation in each group can be finished independently.

Second in order to get the solution of U^{n+2} with U^{n+1} known, we divide all the grid points into $(k+1)^2$ groups.

"ω2" group are used to solve $(u^{n+2}_{1,1}, u^{n+2}_{2,1}, u^{n+2}_{1,2}, u^{n+2}_{2,2})$.

"ω3" group is used to solve $(u^{n+2}_{m-2,1}, u^{n+2}_{m-1,1}, u^{n+2}_{m-2,2}, u^{n+2}_{m-1,2})$.

"ω4" group is used to solve $(u^{n+2}_{1,m-2}, u^{n+2}_{2,m-2}, u^{n+2}_{1,m-1}, u^{n+2}_{2,m-1})$.

"ω5" group is used to solve $(u^{n+2}_{m-2,m-2}, u^{n+2}_{m-1,m-2}, u^{n+2}_{m-2,m-1}, u^{n+2}_{m-1,m-1})$.

"ω6" group is used to solve $(u^{n+2}_{2la+l+p,q}, p = 1, 2, \cdots, 2l, q = 1, 2, \cdots, l)$, $a = 0, 1, \cdots, \frac{m-1-4l}{2l}$.

"ω7" group is used to solve $(u^{n+2}_{2la+l+p,q}, p = 1, 2, \cdots, 2l, q = m-l, m-(l-1), \cdots, m-1)$, $a = 0, 1, \cdots, \frac{m-1-4l}{2l}$.

"ω8" group is used to solve $(u^{n+2}_{p,2la+l+q}, p = 1, 2, \cdots, l, q = 1, 2, \cdots, 2l)$, $a = 0, 1, \cdots, \frac{m-1-4l}{2l}$.

"ω9" group is used to solve $(u^{n+2}_{p,2la+l+q}, p = m-l, m-(l-1), \cdots, m-1), q = 1, 2, \cdots, 2l, a = 0, 1, \cdots, \frac{m-1-4l}{2l}$.

We point out that computation in each group can also be finished independently. So the parallelism of the AGEC-N method is obvious.

The AGEC-N method can be denoted as follows:

$$\begin{cases} (I + rG_1)\widehat{U}^{n+1} = (I - rG_2)\widehat{U}^n + \widetilde{F}^n_1 \\ (I + rG_2)\widehat{U}^{n+2} = (I - rG_1)\widehat{U}^{n+1} + \widetilde{F}^n_2 \end{cases} \quad (31)$$

Here $\widetilde{F}^n_1$ and $\widetilde{F}^n_2$ are known vectors relevant to the boundary value conditions.

Let $(m-1)^2 = s_1,\ 2l(m-1) = s_2,\ b = -\frac{\varepsilon}{2} + \frac{kh}{4},\ c = -\frac{\varepsilon}{2} - \frac{kh}{4},$ then

$$G_1 = diag(G_{11}, G_{11}, \cdots, G_{11}, G_{11})_{s_1 \times s_1}$$

$$G_{11} = diag(\overline{A}_1, \overline{A}_1, \cdots, \overline{A}_1, \overline{A}_1)_{s_2 \times s_2}$$

$$\overline{A}_1 = \begin{pmatrix} \widehat{s_{11}} & \widehat{s_{12}} \\ \widehat{s_{13}} & \widehat{s_{14}} \end{pmatrix}$$

$$\widehat{s_{11}} = \begin{pmatrix} A_{11} & A_{12} & & & \\ \overline{A}_{12} & A_{22} & A_{12} & & \\ & \cdots & \cdots & \cdots & \\ & & \overline{A}_{12} & A_{22} & A_{12} \\ & & & \overline{A}_{12} & A_{33} \end{pmatrix}_{\frac{(2l)^2}{2} \times \frac{(2l)^2}{2}}$$

$$\widehat{s_{12}} = \begin{pmatrix} O & O & O & O & O \\ O & O & O & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & O & O & O \\ A_{34} & O & O & O & O \end{pmatrix}_{\frac{(2l)^2}{2} \times \frac{(2l)^2}{2}}$$

$$\widehat{s_{13}} = \begin{pmatrix} O & O & O & O & \overline{A}_{34} \\ O & O & O & O & O \\ \cdots & \cdots & \cdots & \cdots & \\ O & O & O & O & O \\ O & O & O & O & O \end{pmatrix}_{\frac{(2l)^2}{2} \times \frac{(2l)^2}{2}}$$

$$\widehat{s_{14}} = \begin{pmatrix} A_{33} & A_{12} & & & \\ \overline{A}_{12} & A_{22} & A_{12} & & \\ & \cdots & \cdots & \cdots & \\ & & \overline{A}_{12} & A_{22} & A_{12} \\ & & & \overline{A}_{12} & A_{11} \end{pmatrix}_{\frac{(2l)^2}{2} \times \frac{(2l)^2}{2}}$$

$$A_{11} = \begin{pmatrix} \varepsilon & b & & & & & & & & \\ c & \frac{3\varepsilon}{2} & b & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & c & \frac{3\varepsilon}{2} & b & & & & & \\ & & & c & 2\varepsilon & 2b & & & & \\ & & & & 2c & 2\varepsilon & b & & & \\ & & & & & c & \frac{3\varepsilon}{2} & b & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & c & \frac{3\varepsilon}{2} & b \\ & & & & & & & & c & \varepsilon \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} \frac{3\varepsilon}{2} & b & & & & & & & & \\ c & 2\varepsilon & b & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & c & 2\varepsilon & b & & & & & \\ & & & c & \frac{5\varepsilon}{2} & 2b & & & & \\ & & & & 2c & \frac{5\varepsilon}{2} & b & & & \\ & & & & & c & 2\varepsilon & b & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & c & 2\varepsilon & b \\ & & & & & & & & c & \frac{3\varepsilon}{2} \end{pmatrix}$$

$$A_{33} = \begin{pmatrix} 2\varepsilon & b & & & & & & & & \\ c & \frac{5\varepsilon}{2} & b & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & c & \frac{5\varepsilon}{2} & b & & & & & \\ & & & c & 3\varepsilon & 2b & & & & \\ & & & & 2c & 3\varepsilon & b & & & \\ & & & & & c & \frac{5\varepsilon}{2} & b & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & c & \frac{5\varepsilon}{2} & b \\ & & & & & & & & c & 2\varepsilon \end{pmatrix}$$

$A_{12} = diag(b, b, \cdots, b, b)$, $\overline{A}_{12} = diag(c, c, \cdots, c, c)$,

$$A_{34} = 2A_{12},\ \overline{A}_{34} = 2\overline{A}_{12}.$$

$A_{11}, A_{22}, A_{33}, A_{12}, \overline{A}_{12}, A_{34}, \overline{A}_{34}$ are all $(2l) \times (2l)$ matrices.

$$G_2 = \begin{pmatrix} P & M^T & & & \\ \overline{M} & P & M^T & & \\ & \cdots & \cdots & \cdots & \\ & & \overline{M} & P & M^T \\ & & & \overline{M} & P \end{pmatrix}_{s_1 \times s_1}$$

$$P = \begin{pmatrix} B_1 & E_1^T & & & \\ \overline{E}_1 & B_1 & E_1^T & & \\ & \cdots & \cdots & \cdots & \\ & & \overline{E}_1 & B_1 & E_1^T \\ & & & \overline{E}_1 & B_1 \end{pmatrix}_{s_2 \times s_2}$$

$$B_1 = \begin{pmatrix} S_1 & \\ & S_2 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} B_{11} & A_{12} & & & \\ \overline{A}_{12} & B_{22} & A_{12} & & \\ & \cdots & \cdots & \cdots & \\ & & \overline{A}_{12} & B_{22} & A_{12} \\ & & & \overline{A}_{12} & B_{33} \end{pmatrix}_{\frac{(2l)^2}{2} \times \frac{(2l)^2}{2}}$$

$$S_2 = \begin{pmatrix} B_{33} & A_{12} & & & \\ \overline{A}_{12} & B_{22} & A_{12} & & \\ & \cdots & \cdots & \cdots & \\ & & \overline{A}_{12} & B_{22} & A_{12} \\ & & & \overline{A}_{12} & B_{11} \end{pmatrix}_{\frac{(2l)^2}{2} \times \frac{(2l)^2}{2}}$$

$$B_{11} = \begin{pmatrix} 3\varepsilon & b & & & & & & & & \\ c & \frac{5\varepsilon}{2} & b & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & c & \frac{5\varepsilon}{2} & b & & & & & \\ & & & c & 2\varepsilon & & & & & \\ & & & & & 2\varepsilon & b & & & \\ & & & & & c & \frac{5\varepsilon}{2} & b & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & c & \frac{5\varepsilon}{2} & b \\ & & & & & & & & c & 3\varepsilon \end{pmatrix}$$

$$B_{22} = \begin{pmatrix} \frac{5\varepsilon}{2} & b & & & & & & & & \\ c & 2\varepsilon & b & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & c & 2\varepsilon & b & & & & & \\ & & & c & \frac{3\varepsilon}{2} & & & & & \\ & & & & & \frac{3\varepsilon}{2} & b & & & \\ & & & & & c & 2\varepsilon & b & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & c & 2\varepsilon & b \\ & & & & & & & & c & \frac{5\varepsilon}{2} \end{pmatrix}$$

$$B_{33} = \begin{pmatrix} 2\varepsilon & b & & & & & & & & \\ c & \frac{3\varepsilon}{2} & b & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & c & \frac{3\varepsilon}{2} & b & & & & & \\ & & & c & \varepsilon & & & & & \\ & & & & & \varepsilon & b & & & \\ & & & & & c & \frac{3\varepsilon}{2} & b & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & c & \frac{3\varepsilon}{2} & b \\ & & & & & & & & c & 2\varepsilon \end{pmatrix}$$

B_{11}, B_{22}, B_{33} are all $(2l) \times (2l)$ matrices.

$$E_1 = \begin{pmatrix} E_{11} & & & \\ & E_{11} & & \\ & & \cdots & \\ & & & E_{11} \end{pmatrix}_{(2l)^2 \times (2l)^2},$$

$$\overline{E}_1 = \begin{pmatrix} \overline{E}_{11} & & & \\ & \overline{E}_{11} & & \\ & & \cdots & \\ & & & \overline{E}_{11} \end{pmatrix}_{(2l)^2 \times (2l)^2}$$

$$E_{11} = \begin{pmatrix} O & 2b \\ O & O \end{pmatrix}_{2l \times 2l}, \ \overline{E}_{11} = \begin{pmatrix} O & 2c \\ O & O \end{pmatrix}_{2l \times 2l}$$

$$M = \begin{pmatrix} M_1 & & & \\ & M_1 & & \\ & & \cdots & \\ & & & M_1 \end{pmatrix}$$

$$\overline{M} = \begin{pmatrix} \overline{M}_1 & & & \\ & \overline{M}_1 & & \\ & & \cdots & \\ & & & \overline{M}_1 \end{pmatrix}_{s_2 \times s_2}$$

$$M_1 = \begin{pmatrix} O & M_{11} \\ O & O \end{pmatrix}_{(2l)^2 \times (2l)^2}$$

$$\overline{M}_1 = \begin{pmatrix} O & \overline{M}_{11} \\ O & O \end{pmatrix}_{(2l)^2 \times (2l)^2}$$

$$M_{11} = diag(2b, 2b, \cdots, 2b)_{2l \times 2l}$$

$$\overline{M}_{11} = diag(2c, 2c, \cdots, 2c)_{2l \times 2l}.$$

5 Stability Analysis

Theorem 2 The alternating group method (AGEC-N) defined by (31) is unconditionally stable.

Proof: From the construction of the matrices above we can see G_1 and G_2 are both diagonally dominant matrices, which shows G_1 and G_2 are both nonnegative definite real matrixes. Then from Kellogg lemma we have:

$$\|(I+\overline{r}G_1)^{-1}\|_2 \le 1, \|(I-\overline{r}G_1)(I+\overline{r}G_1)^{-1}\|_2 \le 1$$

$$\|(I+\overline{r}G_2)^{-1}\|_2 \le 1, \|(I-\overline{r}G_2)(I+\overline{r}G_2)^{-1}\|_2 \le 1.$$

Let n be an even integer, from (31) it follows

$$U^n = \widehat{G}U^{n-2} + (I+\overline{r}G_2)^{-1}\widetilde{F}_2^n$$

$$+(I+\overline{r}G_2)^{-1}(I-\overline{r}G_1)(I+\overline{r}G_1)^{-1}\widetilde{F}_1^n.$$

Here

$$\widehat{G} = (I+\overline{r}G_2)^{-1}(I-\overline{r}G_1)(I+\overline{r}G_1)^{-1}(I-\overline{r}G_2)$$

is the growth matrix.

Let

$$\widehat{\widehat{G}} = (I+\overline{r}G_2)\widehat{G}(I+\overline{r}G_2)^{-1} =$$

$$(I-\overline{r}G_1)(I+\overline{r}G_1)^{-1}(I-\overline{r}G_2)(I+\overline{r}G_2)^{-1},$$

then $\rho(\widehat{G}) = \rho(\widehat{\widehat{G}}) \le \|\widehat{\widehat{G}}\|_2 \le 1$. So the method defined by (31) is unconditionally stable.

6 Numerical Experiments

Example 1: Consider (1) with the initial conditions:

$$u(x,0) = sin(2\pi x)$$

The exact solution of the problem above is denoted as:$u(x,t) = e^{-4a\pi^2 t} sin(2\pi x)$.

Let $A.E = |u_i^n - u(x_i,t_n)|$, $P.E = \dfrac{|u_i^n - u(x_i,t_n)|}{u(x_i,t_n)}$ denote maximum absolute error and relevant error respectively. we compare the numerical results of (10) with the results in [5] as follows:

Table 1: Results of comparisons $m = 16, \ a = 1$

	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 1000\tau$
$A.E.$	6.448 $\times 10^{-5}$	1.869 $\times 10^{-5}$
$A.E.$[1]	3.384 $\times 10^{-3}$	9.913 $\times 10^{-4}$
$P.E.$	9.755 $\times 10^{-3}$	9.769 $\times 10^{-2}$
$P.E.$[1]	5.068 $\times 10^{-1}$	5.139

Table 2: Results of comparisons $m = 24, \ a = 0.1$

	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 1000\tau$
$A.E.$	1.965 $\times 10^{-6}$	1.371 $\times 10^{-5}$
$A.E.$[1]	2.162 $\times 10^{-4}$	1.517 $\times 10^{-3}$
$P.E.$	2.264 $\times 10^{-4}$	2.062 $\times 10^{-3}$
$P.E.$[1]	2.251 $\times 10^{-2}$	2.252 $\times 10^{-1}$

Table 3: Results of comparisons $m = 32, \ a = 0.01$

	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 1000\tau$
$A.E.$	6.474 $\times 10^{-8}$	6.247 $\times 10^{-7}$
$A.E.$[1]	2.241 $\times 10^{-5}$	1.218 $\times 10^{-4}$
$P.E.$	6.670 $\times 10^{-6}$	6.536 $\times 10^{-5}$
$P.E.$[1]	2.250 $\times 10^{-3}$	1.267 $\times 10^{-2}$

The results in Table 1-3 show that the method (10) is of higher accurate than the original AGE method in [5].

Example 2: Consider the following problem:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$0 \le x \le 2, 0 \le y \le 2, 0 \le t \le T \qquad (32)$$

with initial and boundary conditions:

$$\begin{cases} u(x,y,0) = exp(-(x-0.5)^2 - (y-0.5)^2), \\ u(0,y,t) = \frac{1}{4t+1} exp(-\frac{(t+0.5)^2}{(4t+1)} - \frac{(y-t-0.5)^2}{(4t+1)}), \\ u(2,y,t) = \frac{1}{4t+1} exp(-\frac{(1.5-t)^2}{(4t+1)} - \frac{(y-t-0.5)^2}{(4t+1)}), \\ u(x,0,t) = \frac{1}{4t+1} exp(-\frac{(x-t-0.5)^2}{(4t+1)} - \frac{(t+0.5)^2}{(4t+1)}), \\ u(x,2,t) = \frac{1}{4t+1} exp(-\frac{(x-t-0.5)^2}{(4t+1)} - \frac{(1.5-t)^2}{(4t+1)}). \end{cases} \qquad (33)$$

The exact solution of the problem above is denoted as below:

$$u(x,y,t) = \frac{1}{4t+1} exp(-\frac{(x-t-0.5)^2}{(4t+1)} - \frac{(y-t-0.5)^2}{(4t+1)})$$

We compare the numerical results of (31) with Crank-Nicolson (C-N) scheme denoted by (30) and the methods in [12, 13] in Table 4.

Table 4: Results at $m = 13, l = 3, \tau = 10^{-2}$

	$t = 100\tau$	$t = 1000\tau$
A.E.(AGEC-N)	7.064×10^{-5}	1.946×10^{-8}
P.E.(AGEC-N)	3.094×10^{-2}	1.598×10^{-2}
A.E.[12]	4.216×10^{-4}	5.938×10^{-7}
P.E.[12]	3.124×10^{-1}	6.627×10^{-1}
A.E.[13]	2.305×10^{-4}	2.014×10^{-7}
P.E.[13]	1.876×10^{-1}	3.172×10^{-1}
A.E.(C-N)	3.241×10^{-5}	0.937×10^{-8}
P.E.(C-N)	1.014×10^{-2}	0.894×10^{-2}

The results in Table 4 show that the AGEC-N method (31) is of higher accurate than the methods in [12, 13].,

Example 3: We will consider a convection dominant problem.

Let $k_1 = k_2 = 1$, $\varepsilon_1 = \varepsilon_2 = 0.1$, then the exact solution of the problem above is denoted as below:

$$u(x,y,t) = \frac{1}{4t+1} exp(-10\frac{(x-t-0.5)^2}{(4t+1)} - 10\frac{(y-t-0.5)^2}{(4t+1)})$$

Under the condition of $m = 81$, the implicit C-N scheme is difficult to implement for computation. But the present methods can be fulfilled effectively because of its intrinsic parallelism. The numerical results of comparisons with the methods [12, 13] are listed in Table 2.

Table 5: Results at $m = 81, l = 5, \tau = 10^{-3}$,

	$t = 100\tau$	$t = 1000\tau$
A.E.(AGEC-N)	1.197×10^{-4}	3.358×10^{-5}
P.E.(AGEC-N)	6.626×10^{-2}	7.684×10^{-5}
A.E.[12]	4.426×10^{-3}	1.869×10^{-4}
P.E.[12]	6.871×10^{-1}	8.723×10^{-1}
A.E.[13]	1.078×10^{-3}	0.685×10^{-4}
P.E.[13]	3.261×10^{-1}	1.325×10^{-1}

The results in Table 5 show that the AGEC-N is still of higher accurate than the methods in [12, 13], even in convection dominant cases.

7 Conclusions

In this paper, we present a class of alternating group explicit method for 1D diffusion equations, which is suitable for parallel computation, and verified to be unconditionally stable. Then we apply the concept to 2D convection-diffusion equations, and construct another AGEC-N method. From Table 1-5 we can see that the numerical results for the two methods are of higher accurate than the original AGE method and the methods in [12,13]. Based on the property of domain decomposition, the two methods are more effective than the transitional implicit methods in solving large system of equations.

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