High Order Alternating Group Explicit Finite Difference Method For Parabolic Equations

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Abstract: Based on the concept of domain decomposition we construct a class of alternating group explicit method for fourth order parabolic equations. Furthermore, an exponential type alternating group explicit method for 2D convection-diffusion equations is derived, which is effective in convection dominant cases. Both of the two methods have the property of unconditional stability and intrinsic parallelism. Domain decomposition and group computing can be obtained in both of the two methods.

Key–Words: parabolic equations, alternating group, parallel computation, unconditionally stable, finite difference

1 Introduction

Parabolic equations are important partial differential equations. Finite difference method is one of the most frequently used numerical methods in solving differential equations [1-3]. As we all know, Most of explicit methods are short in stability and accuracy, while implicit methods usually have good stability, but are complex in computing, and need to solve large system of equations in the cost of large memory spaces and CPU cycles. Thus it is necessary to construct methods with the advantages of explicit methods and implicit methods, that is, simple for computation and good stability. We notice that a socalled AGE (alternating group explicit) method based on the concept of domain decomposition is widely cared for its intrinsic parallelism and absolute stability, which was originally presented for solving diffusion equations in [4] by Evans. The AGE method is used in computing by applying the special combination of several asymmetry schemes to a group of grid points, and the computation in the whole domain can be divided into many sub-domains, Then the numerical solutions at each group can be obtained independently, which highly cuts down the running computing time, and is suitable for parallel computing. Furthermore, by alternating use of asymmetry schemes at adjacent grid points and different time levels, the AGE method can lead to counteraction of truncation error, and then increase the accurate of numerical solution. The AGE method was soon applied to convection-diffusion equations and hyperbolic equations in [5-6]. In [7-8], AGE method was applied to solve two point boundary value problems. Based on the concept of AGE method, a class of domain splitting method was presented in [9-10]. The developed methods have the same advantages of parallelism and absolute stability as the AGE method in [4], but we notice that almost all the methods have no more than four order accuracy for spatial step. To our knowledge researches on alternating group explicit method for fourth order parabolic equations and 2D convection-diffusion equations have been scarcely presented.

Results about existence and uniqueness of theoretic solution for parabolic equations can be found in [11-13].

We will organize this paper as follows: In section 2, we present an $O(\tau^2 + h^6)$ order unconditionally stable symmetry implicit scheme, and construct an AGE method based on an the scheme for fourth order parabolic equatuins. Stability analysis and convergence analysis are given in section 3. In section 4, we construct a new exponential type alternating group explicit method (EXPAGE) for 2D convection-diffusion equations. Results of numerical experiment are presented at the end of the paper.

2 The AGE Method For Fourth Order Parabolic Equations

In this section we consider the following periodic boundary value problem of fourth order parabolic equations:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0, \quad -\infty < x < \infty, \quad 0 \le t \le T \\ u(x,0) = f(x), \\ u(x+l,t) = u(x,t). \end{cases} \tag{1}$$

The domain $\Omega : [0,l]\times[0,T]$ will be divided into $(m\times \xi)$ meshes with spatial step size h=$\frac{l}{m}$ in x direction and the time step size τ=$\frac{T}{\xi}$. Grid points are denoted by (x_i,t_n), $x_i = ih(i = 0,\ 1,\cdots,m)$, $t_n = n\tau(n = 0,\ 1,\cdots,\frac{T}{\tau})$. The numerical solution of (1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$.

Let

$$\delta_t u_i^n = \frac{u_i^{n+1} - u_i^n}{\tau}$$

$$\delta_x^4 u_i^n = \frac{u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n}{h^4}.$$

We present an implicit finite difference scheme with parameters for solving (1) as below:

$$\lambda_{-2}\delta_t u_{i-2}^n+\lambda_{-1}\delta_t u_{i-1}^n+\lambda_0\delta_t u_i^n+\lambda_1\delta_t u_{i+1}^n+\lambda_2\delta_t u_{i+2}^n = \frac{1}{2}(\delta_x^4 u_i^{n+1} + \delta_x^4 u_i^n) \tag{2}$$

It follows that

$$\lambda_{-2}(\frac{\partial u}{\partial t})_{i-2}^n+\lambda_{-1}(\frac{\partial u}{\partial t})_{i-1}^n+\lambda_0(\frac{\partial u}{\partial t})_i^n+\lambda_1(\frac{\partial u}{\partial t})_{i+1}^n +\lambda_2(\frac{\partial u}{\partial t})_{i+2}^n+\frac{\tau}{2}[\lambda_{-2}(\frac{\partial^2 u}{\partial t^2})_{i-2}^n+ \lambda_{-1}(\frac{\partial^2 u}{\partial t^2})_{i-1}^n+\lambda_0(\frac{\partial^2 u}{\partial t^2})_i^n+\lambda_1(\frac{\partial^2 u}{\partial t^2})_{i+1}^n+\lambda_2(\frac{\partial^2 u}{\partial t^2})_{i+2}^n]$$

$$=(\frac{\partial^4 u}{\partial x^4})_i^n+\frac{h^2}{6}(\frac{\partial^6 u}{\partial x^6})_i^n+\frac{504h^4}{8!}(\frac{\partial^6 u}{\partial x^8})_i^n+\frac{\tau}{2}(\frac{\partial^5 u}{\partial x^4\partial t})_i^n +\frac{60\tau h^2}{720}(\frac{\partial^7 u}{\partial x^6\partial t})_i^n+\frac{252\tau h^4}{8!}(\frac{\partial^9 u}{\partial x^8\partial t})_i^n = O(\tau^2+h^6)$$

Considering $\frac{\partial^k u}{\partial t^k} = (-1)^k\frac{\partial^{4k}u}{\partial x^{4k}}$, we have

$$[-\lambda_{-2}-\lambda_{-1}-\lambda_0-\lambda_1-\lambda_2+1](\frac{\partial^4 u}{\partial x^4})_i^n$$

$$+[2\lambda_{-2}+\lambda_{-1}-\lambda_1-2\lambda_2]h(\frac{\partial^5 u}{\partial x^5})_i^n$$

$$+[-2\lambda_{-2}-\frac{1}{2}\lambda_{-1}-\frac{1}{2}\lambda_1-2\lambda_2+\frac{1}{6}]h^2(\frac{\partial^6 u}{\partial x^6})_i^n$$

$$+[\frac{8}{3!}\lambda_{-2}+\frac{1}{3!}\lambda_{-1}-\frac{1}{3!}\lambda_1-\frac{8}{3!}\lambda_2]h^3(\frac{\partial^7 u}{\partial x^7})_i^n$$

$$+[-\frac{2^4}{4!}\lambda_{-2}-\frac{1}{4!}\lambda_{-1}-\frac{1}{4!}\lambda_1-\frac{2^4}{4!}\lambda_2+\frac{9}{6!}]h^4(\frac{\partial^8 u}{\partial x^8})_i^n$$

$$+[\frac{1}{2}\lambda_{-2}+\frac{1}{2}\lambda_{-1}+\frac{1}{2}\lambda_0+\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_2-\frac{1}{2}]\tau(\frac{\partial^8 u}{\partial x^8})_i^n$$

$$+[-\lambda_{-2}-\frac{1}{2}\lambda_{-1}+\frac{1}{2}\lambda_1+\lambda_2]\tau h(\frac{\partial^9 u}{\partial x^9})_i^n$$

$$+[\lambda_{-2}+\frac{1}{4}\lambda_{-1}+\frac{1}{4}\lambda_1+\lambda_2-\frac{60}{720}]\tau h^2(\frac{\partial^{10} u}{\partial x^{10}})_i^n$$

$$+[-\frac{4}{3!}\lambda_{-2}-\frac{1}{3!}\lambda_{-1}+\frac{1}{3!}\lambda_1+\frac{4}{3!}\lambda_2]\tau h^3(\frac{\partial^{11} u}{\partial x^{11}})_i^n$$

$$+[\frac{8}{4!}\lambda_{-2}+\frac{1}{2\times 4!}\lambda_{-1}+\frac{1}{2\times 4!}\lambda_1+\frac{8}{4!}\lambda_2-\frac{252}{8!}]\tau h^4(\frac{\partial^{12} u}{\partial x^{12}})_i^n$$

$$= O(\tau^2+h^6)$$

Let

$$\begin{cases} -\lambda_{-2}-\lambda_{-1}-\lambda_0-\lambda_1-\lambda_2+1=0 \\ 2\lambda_{-2}+\lambda_{-1}-\lambda_1-2\lambda_2=0 \\ -2\lambda_{-2}-\frac{1}{2}\lambda_{-1}-\frac{1}{2}\lambda_1-2\lambda_2+\frac{1}{6}=0 \\ \frac{8}{3!}\lambda_{-2}+\frac{1}{3!}\lambda_{-1}-\frac{1}{3!}\lambda_1-\frac{8}{3!}\lambda_2=0 \\ -\frac{2^4}{4!}\lambda_{-2}-\frac{1}{4!}\lambda_{-1}-\frac{1}{4!}\lambda_1-\frac{2^4}{4!}\lambda_2+\frac{504}{8!}=0 \\ \frac{1}{2}\lambda_{-2}+\frac{1}{2}\lambda_{-1}+\frac{1}{2}\lambda_0+\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_2-\frac{1}{2}=0 \\ -\lambda_{-2}-\frac{1}{2}\lambda_{-1}+\frac{1}{2}\lambda_1+\lambda_2=0 \\ \lambda_{-2}+\frac{1}{4}\lambda_{-1}+\frac{1}{4}\lambda_1+\lambda_2-\frac{60}{720}=0 \\ -\frac{4}{3!}\lambda_{-2}-\frac{1}{3!}\lambda_{-1}+\frac{1}{3!}\lambda_1+\frac{4}{3!}\lambda_2=0 \\ \frac{8}{4!}\lambda_{-2}+\frac{1}{2\times 4!}\lambda_{-1}+\frac{1}{2\times 4!}\lambda_1+\frac{8}{4!}\lambda_2-\frac{252}{8!}=0 \end{cases}$$

Then $\lambda_{-2} = \lambda_2 = -\frac{1}{720}$, $\lambda_{-1} = \lambda_1 = \frac{124}{720}$, $\lambda_0 = \frac{474}{720}$

We denote (2) as

$$-\frac{1}{720}\delta_t u_{i-2}^n+\frac{124}{720}\delta_t u_{i-1}^n+\frac{474}{720}\delta_t u_i^n+\frac{124}{720}\delta_t u_{i+1}^n -\frac{1}{720}\delta_t u_{i+2}^n = \frac{1}{2}(\delta_x^4 u_i^{n+1}+\delta_x^4 u_i^n) \tag{3}$$

and obviously the truncation error of (3) is $O(\tau^2+h^6)$.

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_m^n)^T$, $r = \frac{360\tau}{h^4}$, then from (3) we have

$$(-1+r)u_{i-2}^{n+1}+(124-4r)u_{i-1}^{n+1}+(474+6r)u_i^{n+1}+ (124-4r)u_{i+1}^{n+1}+(-1+r)u_{i+2}^{n+1} = (-1-r)u_{i-2}^n+ (124+4r)u_{i-1}^n+(474-6r)u_i^n+(124+4r)u_{i+1}^n+(-1-r)u_{i+2}^n \tag{4}$$

Based on (4), we present eight saul'yev asymmetry schemes to solve the solution at the $n+1$ time level with the solution at the n time level known.

$$(474+3r)u_i^{n+1}+(248-4r)u_{i+1}^{n+1}+(-2+r)u_{i+2}^{n+1}=$$
$$-2ru_{i-2}^n+8ru_{i-1}^n+(474-9r)u_i^n+(248+4r)u_{i+1}^n+(-2-r)u_{i+2}^n \quad (5)$$

$$(124-r)u_{i-1}^{n+1}+(474+3r)u_i^{n+1}+(124-3r)u_{i+1}^{n+1}$$
$$+(-2+r)u_{i+2}^{n+1}=-2ru_{i-2}^n+(124+7r)u_{i-1}^n$$
$$+(474-9r)u_i^n+(124+5r)u_{i+1}^n+(-2-r)u_{i+2}^n \quad (6)$$

$$(-2+r)u_{i-2}^{n+1}+(124-5r)u_{i-1}^{n+1}+(474+9r)u_i^{n+1}$$
$$+(124-7r)u_{i+1}^{n+1}+2ru_{i+2}^{n+1}=(-2-r)u_{i-2}^n$$
$$+(124+3r)u_{i-1}^n+(474-3r)u_i^n+(124+r)u_{i+1}^n \quad (7)$$

$$(-2+r)u_{i-2}^{n+1}+(248-4r)u_{i-1}^{n+1}+(474+9r)u_i^{n+1}$$
$$-8ru_{i+1}^{n+1}+2ru_{i+2}^{n+1}=(-2-r)u_{i-2}^n$$
$$+(248+4r)u_{i-1}^n+(474-3r)u_i^n \quad (8)$$

$$2ru_{i-2}^{n+1}-8ru_{i-1}^{n+1}+(474+9r)u_i^{n+1}$$
$$+(248-4r)u_{i+1}^{n+1}+(-2+r)u_{i+2}^{n+1}=(474-3r)u_i^n$$
$$+(248+4r)u_{i+1}^n+(-2-r)u_{i+2}^n \quad (9)$$

$$2ru_{i-2}^{n+1}+(124-7r)u_{i-1}^{n+1}+(474+9r)u_i^{n+1}$$
$$+(124-5r)u_{i+1}^{n+1}+(-2+r)u_{i+2}^{n+1}=(124+r)u_{i-1}^n$$
$$+(474-3r)u_i^n+(124+3r)u_{i+1}^n+(-2-r)u_{i+2}^n \quad (10)$$

$$(-2+r)u_{i+2}^{n+1}+(124-3r)u_{i-1}^{n+1}+(474+3r)u_i^{n+1}$$
$$+(124-r)u_{i+1}^{n+1}=(-2-r)u_{i-2}^n+(124+5r)u_{i-1}^n$$
$$+(474-9r)u_i^n+(124+7r)u_{i+1}^n-2ru_{i+2}^n \quad (11)$$

$$(-2+r)u_{i-2}^{n+1}+(248-4r)u_{i-1}^{n+1}+(474+3r)u_i^{n+1}=$$
$$(-2-r)u_{i-2}^n+(248+4r)u_{i-1}^n+(474-9r)u_i^n+8ru_{i+1}^n-2ru_{i+2}^n \quad (12)$$

If we apply (5)-(12) to $(i+k,n),\ k=0,1,\cdots,7$, and let $U_i^{n+1}=(u_i^{n+1},\ u_{i+1}^{n+1},\ \cdots,u_{i+7}^{n+1})^T$, then it follows

$$A_1U_i^{n+1}=B_1U_i^n+F_i^n \quad (13)$$

Here $F_i^n=(-2ru_{i-2}^n+8ru_{i-1}^n,-2ru_{i-1}^n,$ $0,\cdots,0,-2ru_{i+8}^n,-2ru_{i+9}^n+8ru_{i+8}^n)^T$

$$A_1=\begin{pmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{pmatrix}$$

$$A_{11}=\begin{pmatrix} 474+3r & 248-4r & -2+r & \\ 124-r & 474+3r & 124-3r & -2+r \\ -2+r & 124-5r & 474+9r & 124-7r \\ & -2+r & 248-4r & 474+9r \end{pmatrix}$$

$$A_{12}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2r & 0 & 0 & 0 \\ -8r & 2r & 0 & 0 \end{pmatrix} A_{13}=\begin{pmatrix} 0 & 0 & 2r & -8r \\ 0 & 0 & 0 & 2r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{14}=\begin{pmatrix} 474+9r & 248-4r & -2+r & \\ 124-7r & 474+9r & 124-5r & -2+r \\ -2+r & 124-3r & 474+3r & 124-r \\ & -2+r & 248-4r & 474+3r \end{pmatrix}$$

$$B_1=\begin{pmatrix} B_{11} & O \\ O & B_{14} \end{pmatrix}$$

$$B_{12}=\begin{pmatrix} 474-9r & 248+4r & -2-r & \\ 124+7r & 474-9r & 124+5r & -2-r \\ -2-r & 124+3r & 474-3r & 124+r \\ & -2-r & 248+4r & 474-3r \end{pmatrix}$$

$$B_{14}=\begin{pmatrix} 474-3r & 248+4r & -2-r & \\ 124+r & 474-3r & 124+3r & -2-r \\ -2-r & 124+5r & 474-9r & 124+7r \\ & -2-r & 248+4r & 474-9r \end{pmatrix}$$

Based on (5)-(12), we construct three basic explicit computing point groups:

"$\nu 1$"group: eight grid points are involved, and (5)-(12) are used respectively. From (13) we have

$$U_i^{n+1}=A_1^{-1}(B_1U_i^n+F_i^n) \quad (14)$$

Then the numerical solution at the eight grid nodes can be obtained independently.

"$\nu 2$"group: four inner points are involved. Let $\overline{U}_i^{n+1} = (u_i^{n+1},\ u_{i+1}^{n+1},\ \cdots, u_{i+3}^{n+1})^T$, then it follows

$$A_2 \overline{U}_i^{n+1} = B_2 \overline{U}_i^n + \overline{F}_i^n \qquad (15)$$

Here $\overline{F}_i^n = (-2ru_{i-2}^n + 8ru_{i-1}^n, -2ru_{i-1}^n,$
$-2ru_{i+4}^n, -2ru_{i+5}^{n+1} + 8ru_{i+4}^{n+1})^T$,

$$A_2 = \begin{pmatrix} 474+3r & 248-4r & -2+r & 1 \\ 124-r & 474+3r & 124-3r & -2+r1 \\ -2+r & 124-5r & 474+9r & 124-7r1 \\ & -2+r & 248-4r & 474+9r1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 474-9r & 248+4r & -2-r & \\ 124+7r & 474-9r & 124+5r & -2-r \\ -2-r & 124+3r & 474-3r & 124+r \\ & -2-r & 248+4r & 474-3r \end{pmatrix}$$

And we have

$$\overline{U}_i^{n+1} = A_2^{-1}(B_2 \overline{U}_i^n + \overline{F}_i^n)$$

"$\nu 3$"group: four inner points are involved. Let $\widetilde{U}_i^{n+1} = (u_i^{n+1},\ u_{i+1}^{n+1},\ \cdots, u_{i+3}^{n+1})^T$, then it follows

$$A_3 \widetilde{U}_i^{n+1} = B_3 \widetilde{U}_i^n + \widetilde{F}_i^n \qquad (16)$$

Here $\widetilde{F}_i^n = (-2ru_{i-2}^{n+1} + 8ru_{i-1}^{n+1},$
$-2ru_{i-1}^{n+1}, -2ru_{i+4}^n, -2ru_{i+5}^n + 8ru_{i+4}^n)^T$,

$$A_3 = \begin{pmatrix} 474+9r & 248-4r & -2+r & \\ 124-7r & 474+9r & 124-5r & -2+r \\ -2+r & 124-3r & 474+3r & 124-r \\ & -2+r & 248-4r & 474+3r \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 474-3r & 248+4r & -2-r & \\ 124+r & 474-3r & 124+3r & -2-r \\ -2-r & 124+5r & 474-9r & 124+7r \\ & -2-r & 248+4r & 474-9r \end{pmatrix}$$

Thus we have:

$$\widetilde{U}_i^{n+1} = A_3^{-1}(B_3 \widetilde{U}_i^n + \widetilde{F}_i^n)$$

Applying the basic point groups above we construct the alternating group method in two cases as follows:

Case 1: Let $m-1 = 8s$, here s is an integer. First at the $(n+1)$-th time level, we divide all of the $m-1$ inner grid points into s "$\nu 1$" groups, and (14) is used in each group. Second at the $(n+2)$-th time level, we will have $(s+1)$ point groups. "$\nu 3$" group are applied to get the solution of the left four grid points $(1, n+2), (2, n+2), (3, n+2), (4, n+2)$. (14) are used in the following $(s-1)$ "$\nu 1$" groups, while "$\nu 2$" group are used in the right four grid points $(m-4, n+2), (m-3, n+2), (m-2, n+2), (m-1, n+2)$.

By alternating use of the asymmetry schemes (5)-(12), the computing in the whole domain can be divided into many sub-domains, and grouping explicit computation can be obtained obviously. So the method has the obvious property of parallelism.

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_m^n)^T$. Considering under periodic boundary conditions it follows $u_i^n = u_{i+m}^n$, we can denote the alternating group explicit method I (AGEI) as follows:

$$\begin{cases} AU^{n+1} = BU^n \\ \widehat{A}U^{n+2} = \widehat{B}U^{n+1} \end{cases} \qquad (17)$$

here $A,\ B,\ \widehat{A},\ \widehat{B}$ are all $m \times m$ matrices.

$$A = diag(A_1,\ A_1,\ \cdots,\ A_1,\ A_1)$$

$$\widehat{B} = diag(\overline{B}_1,\ \overline{B}_1,\ \cdots,\ \overline{B}_1,\ \overline{B}_1)$$

$$\widehat{A} = \begin{pmatrix} A_3 & & & & -B_4 \\ & A_1 & & & \\ & & \cdots & & \\ & & & A_1 & \\ -B_4^T & & & & A_2 \end{pmatrix}$$

$$B = \begin{pmatrix} B_2 & & & & B_4 \\ & \overline{B}_1 & & & \\ & & \cdots & & \\ & & & \overline{B}_1 & \\ B_4^T & & & & B_3 \end{pmatrix}$$

$$B_4 = \begin{pmatrix} 0 & 0 & -2r & 8r \\ 0 & 0 & 0 & -2r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{B}_1 = \begin{pmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{13} & \overline{B}_{14} \end{pmatrix}$$

$$\overline{B}_{11} = \begin{pmatrix} 474-3r & 248+4r & -2-r & \\ 124+r & 474-3r & 124+3r & -2-r \\ -2-r & 124+5r & 474-9r & 124+7r \\ & -2-r & 248+4r & 474-9r \end{pmatrix}$$

$$\overline{B}_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2r & 0 & 0 & 0 \\ 8r & -2r & 0 & 0 \end{pmatrix}$$

$$\overline{B}_{13} = \begin{pmatrix} 0 & 0 & -2r & 8r \\ 0 & 0 & 0 & -2r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{B}_{14}=\begin{pmatrix} 474-9r & 248+4r & -2-r & \\ 124+7r & 474-9r & 124+5r & -2-r \\ -2-r & 124+3r & 474-3r & 124+r \\ & -2-r & 248+4r & 474-3r \end{pmatrix}$$

Case 2: Let $m-1=8s+4$, here s is an integer. First at the $(n+1)$-th time level, we divide all of the $m-1$ inner grid points into $s+1$ groups, "$\nu 2$" group are used in the right four grid points $(m-4,n+1),(m-3,n+1),(m-2,n+1),(m-1,n+1)$, while "$\nu 1$" group is used in each of the rest s groups. Second at the $(n+2)$-th time level, we will still have $(s+1)$ point groups. "$\nu 3$" group are applied to get the solution of the left four grid points $(1,n+2),(2,n+2),(3,n+2),(4,n+2)$, while "$\nu 1$" group is used in each of the rest s groups.

We can denote the alternating group explicit method II (AGEII) as follows:

$$\begin{cases} \widetilde{A}U^{n+1}=\widetilde{B}U^{n} \\ \widetilde{\widetilde{A}}U^{n+2}=\widetilde{\widetilde{B}}U^{n+1} \end{cases} \quad (18)$$

here $\widetilde{A},\ \widetilde{B},\ \widetilde{\widetilde{A}},\ \widetilde{\widetilde{B}}$ are all $m\times m$ matrices.

$$\widetilde{A}=\begin{pmatrix} A_1 & & & & \\ & A_1 & & & \\ & & \cdots & & \\ & & & A_1 & \\ -\widetilde{B}_4^T & & & & A_2 \end{pmatrix}$$

$$\widetilde{\widetilde{B}}=\begin{pmatrix} B_3 & & & & \widetilde{B}_4 \\ & \overline{B}_1 & & & \\ & & \cdots & & \\ & & & \overline{B}_1 & \\ & & & & B_1 \end{pmatrix},$$

$$\widetilde{\widetilde{A}}=\begin{pmatrix} A_3 & & & & \widetilde{B}_4 \\ & A_1 & & & \\ & & \cdots & & \\ & & & A_1 & \\ & & & & A_1 \end{pmatrix}$$

$$\widetilde{\widetilde{B}}=\begin{pmatrix} \overline{B}_1 & & & & \\ & \overline{B}_1 & & & \\ & & \cdots & & \\ & & & \overline{B}_1 & \\ \widetilde{B}_4^T & & & & B_2 \end{pmatrix},$$

$$\widetilde{B}_4=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2r & 8r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We point out that computation in each group can also be finished independently. So the parallelism of the AGEII method is obvious.

3 Stability and Convergence Analysis

In order to verify the stability of (17), we present the following lemma [14]:

Lemma1 If M=(m_{ij}) is a $n\times n$ diagonal dominant L-matrix, while N=(n_{ij}) is a $n\times n$ nonnegative definite matrix, then it follows:

$$\min_i(\sum_j n_{ij}/\sum_j m_{ij})\le\rho(M^{-1}N)\le\|M^{-1}N\|_\infty$$

$$\le\max_i(\sum_j n_{ij}/\sum_j m_{ij}) \quad (19)$$

Theorem 1 The AGEI method defined by (17) is unconditionally stable.

Proof: From (17) we have $U^{n+2}=GU^n$, here $G=\widehat{A}^{-1}\widehat{B}A^{-1}B$ is the growth matrix. From the construction of the matrixes above we can see A and $\widehat{A}$ are both strictly diagonally dominant L-matrixes, while B and $\widehat{B}$ are both nonnegative definite real matrixes. Then from lemma 1 we have:

$$\rho(\widehat{A}^{-1}\widehat{B})\le 1,\ \le\rho(A^{-1}B)\le 1$$

Then we have $\rho(G)=\rho(\widehat{A}^{-1}\widehat{B}A^{-1}B)\le\rho(\widehat{A}^{-1}\widehat{B})\rho(A^{-1}B)\le 1$, which shows the AGE method (17) is of unconditional stability.

Considering $(\frac{\partial^k u}{\partial t^k})_i^n=(-1)^k(\frac{\partial^{4k}u}{\partial x^{4k}})_i^n$, applying Taylor's formula to (5)-(12) we can easily obtain that the truncation error is $O(h^2+h^4+\tau+\tau h^2+\tau h^4+\tau^2+h^6)$ respectively. Furthermore alternating use of (5)-(12) can lead to counteraction of the truncation error for the items containing $h^2,\ h^4,\ \tau,\ \tau h,\ \tau h^2,\ \tau h^4$. Then it follows the truncation error of (17) is $O(\tau^2+h^6)$, which shows (17) is compatible with (1).

According to Lax theorem, (17) is convergent under the fact of unconditional stability. So we have:

Theorem 2 The AGEI method defined by (17) is convergent.

Similarly we have:

Theorem 3 The AGEII method defined by (18) is also unconditionally stable and convergent.

4 Construction Of EXPAGE Method For 2D Convection-Diffusion Equations

In section 2, we present a class of alternating group explicit method with intrinsic parallelism. The method is based on an $O(\tau^2+h^6)$ order implicit scheme, which is of absolute stability. We notice the construction of the method is universal, and of course we can apply the concept to other problems.

In this section, we will consider the 2D convection-diffusion equation:

$$\frac{\partial u}{\partial t}+k_1\frac{\partial u}{\partial x}+k_2\frac{\partial u}{\partial y}=\varepsilon_1\frac{\partial^2 u}{\partial x^2}+\varepsilon_2\frac{\partial^2 u}{\partial y^2}$$
$$0\le x\le 1, 0\le y\le 1, 0\le t\le T, \varepsilon_1>0, \varepsilon_2>0 \tag{20}$$

with initial and boundary conditions:

$$\begin{cases} u(x,y,0)=f(x), \\ u(0,y,t)=g_1(y,t), u(1,y,t)=g_2(y,t), \\ u(x,0,t)=h_1(x,t), u(x,1,t)=h_2(x,t). \end{cases} \tag{21}$$

The domain Ω : $[0,1]\times[0,1]\times[0,T]$ will be divided into $(m\times m\times N)$ meshes with spatial step size $h=\frac{1}{m}$ in x, y direction and the time step size τ=$\frac{T}{N}$. Grid points are denoted by (x_i, y_j, t_n) or (i,j,n), $x_i=ih, y_j=jh(i,j=0,\ 1,\cdots,m)$, $t_n=n\tau(n=0,\ 1,\cdots,\frac{T}{\tau})$. The numerical solution of (20)-(21) is denoted by $u^n_{i,j}$, while the exact solution $u(x_i,\ y_j,\ t_n)$. Let $\overline{r}=\frac{\tau}{h^2}$.

Let

$$\delta_x u^n_{i,j}=\frac{u^n_{i+1,j}-u^n_{i,j}}{h},\ \delta_{\overline{x}} u^n_{i,j}=\frac{u^n_{i,j}-u^n_{i-1,j}}{h},$$
$$\delta_{\widehat{x}} u^n_{i,j}=\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2h},\ \delta_y u^n_{i,j}=\frac{u^n_{i,j+1}-u^n_{i,j}}{h},$$
$$\delta_{\overline{y}} u^n_{i,j}=\frac{u^n_{i,j}-u^n_{i,j-1}}{h},\ \delta_{\widehat{y}} u^n_{i,j}=\frac{u^n_{i,j+1}-u^n_{i,j-1}}{2h}$$
$$\delta_t u^n_{i,j}=\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau},\ \delta^2_x u^n_{i,j}=\frac{u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j}}{h^2},$$
$$\delta^2_y u^n_{i,j}=\frac{u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1}}{h^2}.$$

Let $\widehat{U}^n=(u^n_1,u^n_2,\cdots,u^n_{m-1})^T$, $u^n_j=(u^n_{1,j}, u^n_{2,j},\cdots,u^n_{m-1,j})^T$. We present the following exponential type implicit scheme for approaching (20)-(21):

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}+\frac{k_1}{2}(\delta_{\widehat{x}}u^{n+1}_{i,j}+\delta_{\widehat{x}}u^n_{i,j})$$
$$+\frac{k_2}{2}(\delta_{\widehat{y}}u^{n+1}_{i,j}+\delta_{\widehat{y}}u^n_{i,j})=\frac{k_1h}{2}coth(\frac{k_1h}{2\varepsilon})(\delta^2_x u^{n+1}_{i,j}+\delta^2_x u^n_{i,j})$$
$$+\frac{k_2h}{2}coth(\frac{k_2h}{2\varepsilon})(\delta^2_y u^{n+1}_{i,j}+\delta^2_y u^n_{i,j}) \tag{22}$$

In order to fulfil the parallel computation, based on the scheme, we give sixteen asymmetry schemes as follows. Let $\frac{k_1h}{2}coth(\frac{k_1h}{2\varepsilon})=\kappa_1,\ \frac{k_2h}{2}coth(\frac{k_2h}{2\varepsilon})=\kappa_2$.

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}+\frac{k_1}{2}(\frac{u^{n+1}_{i+1,j}-u^n_{i-1,j}}{2h}+\delta_{\widehat{x}}u^n_{i,j})$$
$$+\frac{k_2}{2}(\frac{u^{n+1}_{i,j+1}-u^n_{i,j-1}}{2h}+\delta_{\widehat{y}}u^n_{i,j})=$$
$$\kappa_1(\frac{u^{n+1}_{i+1,j}-u^{n+1}_{i,j}-u^n_{i,j}+u^n_{i-1,j}}{h^2}+\delta^2_x u^n_{i,j})$$
$$+\kappa_2(\frac{u^{n+1}_{i,j+1}-u^{n+1}_{i,j}-u^n_{i,j}+u^n_{i,j-1}}{h^2}+\delta^2_y u^n_{i,j}) \tag{23}$$

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}+\frac{k_1}{2}(\frac{u^{n+1}_{i+1,j}-u^n_{i-1,j}}{2h}+\delta_{\widehat{x}}u^{n+1}_{i,j})$$
$$+\frac{k_2}{2}(\frac{u^{n+1}_{i,j+1}-u^n_{i,j-1}}{2h}+\delta_{\widehat{y}}u^n_{i,j})=$$
$$\kappa_1(\frac{u^{n+1}_{i+1,j}-u^{n+1}_{i,j}-u^n_{i,j}+u^n_{i-1,j}}{h^2}+\delta^2_x u^{n+1}_{i,j})$$
$$+\kappa_2(\frac{u^{n+1}_{i,j+1}-u^{n+1}_{i,j}-u^n_{i,j}+u^n_{i,j-1}}{h^2}+\delta^2_y u^n_{i,j}) \tag{24}$$

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}+\frac{k_1}{2}(\frac{u^n_{i+1,j}-u^{n+1}_{i-1,j}}{2h}+\delta_{\widehat{x}}u^{n+1}_{i,j})$$
$$+\frac{k_2}{2}(\frac{u^{n+1}_{i,j+1}-u^n_{i,j-1}}{2h}+\delta_{\widehat{y}}u^n_{i,j})=$$
$$\kappa_1(\frac{u^n_{i+1,j}-u^n_{i,j}-u^{n+1}_{i,j}+u^{n+1}_{i-1,j}}{h^2}+\delta^2_x u^{n+1}_{i,j})$$
$$+\kappa_2(\frac{u^{n+1}_{i,j+1}-u^{n+1}_{i,j}-u^n_{i,j}+u^n_{i,j-1}}{h^2}+\delta^2_y u^n_{i,j}) \tag{25}$$

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}+\frac{k_1}{2}(\frac{u^n_{i+1,j}-u^{n+1}_{i-1,j}}{2h}+\delta_{\widehat{x}}u^n_{i,j})$$
$$+\frac{k_2}{2}(\frac{u^{n+1}_{i,j+1}-u^n_{i,j-1}}{2h}+\delta_{\widehat{y}}u^n_{i,j})=$$
$$\kappa_1(\frac{u^n_{i+1,j}-u^n_{i,j}-u^{n+1}_{i,j}+u^{n+1}_{i-1,j}}{h^2}+\delta^2_x u^n_{i,j})$$
$$+\kappa_2(\frac{u^{n+1}_{i,j+1}-u^{n+1}_{i,j}-u^n_{i,j}+u^n_{i,j-1}}{h^2}+\delta^2_y u^n_{i,j}) \tag{26}$$

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}+\frac{k_1}{2}(\frac{u^{n+1}_{i+1,j}-u^n_{i-1,j}}{2h}+\delta_{\widehat{x}}u^n_{i,j})$$

$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (27)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n+1})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (28)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n+1})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (29)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (30)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (31)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n+1})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (32)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n+1})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (33)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n+1}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n+1})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2}+\delta_x^2u_{i,j}^{n})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n+1}) \quad (34)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n})$$
$$+\frac{k_2}{2}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n+1}}{2h}+\delta_{\widehat{y}}u_{i,j}^{n})=$$
$$\kappa_1(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\delta_x^2u_{i,j}^{n})$$
$$+\kappa_2(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2}+\delta_y^2u_{i,j}^{n}) \quad (35)$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\frac{k_1}{2}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n}}{2h}+\delta_{\widehat{x}}u_{i,j}^{n+1})$$

$$+\frac{k_2}{2}(\frac{u^n_{i,j+1}-u^{n+1}_{i,j-1}}{2h}+\delta_{\widehat{y}}u^n_{i,j})=$$
$$\kappa_1(\frac{u^{n+1}_{i+1,j}-u^{n+1}_{i,j}-u^n_{i,j}+u^n_{i-1,j}}{h^2}+\delta^2_x u^{n+1}_{i,j})$$
$$+\kappa_2(\frac{u^n_{i,j+1}-u^n_{i,j}-u^{n+1}_{i,j}+u^{n+1}_{i,j-1}}{h^2}+\delta^2_y u^n_{i,j}) \quad (36)$$

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}+\frac{k_1}{2}(\frac{u^n_{i+1,j}-u^{n+1}_{i-1,j}}{2h}+\delta_{\widehat{x}}u^{n+1}_{i,j})$$
$$+\frac{k_2}{2}(\frac{u^n_{i,j+1}-u^{n+1}_{i,j-1}}{2h}+\delta_{\widehat{y}}u^n_{i,j})=$$
$$\kappa_1(\frac{u^n_{i+1,j}-u^n_{i,j}-u^{n+1}_{i,j}+u^{n+1}_{i-1,j}}{h^2}+\delta^2_x u^{n+1}_{i,j})$$
$$+\kappa_2(\frac{u^n_{i,j+1}-u^n_{i,j}-u^{n+1}_{i,j}+u^{n+1}_{i,j-1}}{h^2}+\delta^2_y u^n_{i,j}) \quad (37)$$

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau}+\frac{k_1}{2}(\frac{u^n_{i+1,j}-u^{n+1}_{i-1,j}}{2h}+\delta_{\widehat{x}}u^n_{i,j})$$
$$+\frac{k_2}{2}(\frac{u^n_{i,j+1}-u^{n+1}_{i,j-1}}{2h}+\delta_{\widehat{y}}u^n_{i,j})=$$
$$\kappa_1(\frac{u^n_{i+1,j}-u^n_{i,j}-u^{n+1}_{i,j}+u^{n+1}_{i-1,j}}{h^2}+\delta^2_x u^n_{i,j})$$
$$+\kappa_2(\frac{u^n_{i,j+1}-u^n_{i,j}-u^{n+1}_{i,j}+u^{n+1}_{i,j-1}}{h^2}+\delta^2_y u^n_{i,j}) \quad (38)$$

The schemes (23)-(38) compose the "16-point" group, which will be applied to get the solution on 16 grids points $(i,j,n+1),(i+1,j,n+1),(i+2,j,n+1),(i+3,j,n+1),(i,j+1,n+1),(i+1,j+1,n+1),(i+2,j+1,n+1),(i+3,j+1,n+1),(i,j+2,n+1),(i+1,j+2,n+1),(i+2,j+2,n+1),(i+3,j+2,n+1),(i,j+3,n+1),(i+1,j+3,n+1),(i+2,j+3,n+1),(i+3,j+3,n+1)$. Similar to the "$\omega_1$" group, computation in "16-point" group can also be done independently.

Let $(m-1)=4s$, s is an integer. We describe the exponential type alternating group method (EXPAGE) as follows:

First at the $(n+1)$-th time level, we will have s^2 point groups. "16-point" group is applied in each group. Let $\overline{u}^n_{i,j}=(u^n_j,u^n_{j+1},u^n_{j+2},u^n_{j+3})^T$, $u^n_{j+k}=(u^n_{i,j+k},u^n_{i+1,j+k},u^n_{i+2,j+k},u^n_{i+3,j+k})^T$, $k=0,1,2,3$, then the solution of $\overline{u}^{n+1}_{i,j}$ can be solved in each "16-point" group independently.

Second at the $(n+2)$-th time level, we will have $(s+1)^2$ point groups:

(33),(34),(37),(38) are applied to solve $(u^{n+2}_{1,1},u^{n+2}_{2,1},u^{n+2}_{1,2},u^{n+2}_{2,2})$, which marks "H1" group.

(31),(32),(35),(36) are applied to solve $(u^{n+2}_{m-2,1},u^{n+2}_{m-1,1},u^{n+2}_{m-2,2},u^{n+2}_{m-1,2})$, which marks "H2" group.

(25),(26),(29),(30) are applied to solve $(u^{n+2}_{1,m-2},u^{n+2}_{2,m-2},u^{n+2}_{1,m-1},u^{n+2}_{2,m-1})$, which marks "H3" group.

(23),(24),(27),(28) are applied to solve $(u^{n+2}_{m-2,m-2},u^{n+2}_{m-1,m-2},u^{n+2}_{m-2,m-1},u^{n+2}_{m-1,m-1})$, which marks "H4" group.

(25),(26),(29),(30),(33),(34),(37),(38) are applied to solve $(u^{n+2}_{1,j},u^{n+2}_{2,j},u^{n+2}_{1,j+1},u^{n+2}_{2,j+1},u^{n+2}_{1,j+2},u^{n+2}_{2,j+2},u^{n+2}_{1,j+3},u^{n+2}_{2,j+3}),j=3,7,\cdots,m-6$, which marks "Lx"group.

(23),(24),(27),(28),(31),(32),(35),(36) are applied to solve $(u^{n+2}_{m-2,j},u^{n+2}_{m-1,j},u^{n+2}_{m-2,j+1},u^{n+2}_{m-1,j+1},u^{n+2}_{m-2,j+2},u^{n+2}_{m-1,j+2},u^{n+2}_{m-2,j+3},u^{n+2}_{m-1,j+3})$, $j=3,7,\cdots,m-6$, which marks "Rx"group.

(31),(32),(33),(34),(35),(36),(37),(38) are applied to solve $(u^{n+2}_{i,1},u^{n+2}_{i+1,1},u^{n+2}_{i+2,1},u^{n+2}_{i+3,1},u^{n+2}_{i,2},u^{n+2}_{i+1,2},u^{n+2}_{i+2,2},u^{n+2}_{i+3,2}),i=3,7,\cdots,m-6$, which marks "Ly"group.

(23),(24),(25),(26),(27),(28),(29),(30) are applied to solve $(u^{n+2}_{i,m-2},u^{n+2}_{i+1,m-2},u^{n+2}_{i+2,m-2},u^{n+2}_{i+3,m-2},u^{n+2}_{i,m-1},u^{n+2}_{i+1,m-1},u^{n+2}_{i+2,m-1},u^{n+2}_{i+3,m-1})$, $i=3,7,\cdots,m-6$, which marks "Ry"group.

"16 point" group are applied to solve $(\overline{u}^{n+2}_{i,j},i,j=3,7,\cdots,m-6)$ respectively.

Thus the EXPAGE method is established by alternating use of the schemes (23)-(38) in the two time levels, and computation in each group can be done independently, which shows the method is suitable for parallel computation.

We denote the method as following:

$$\begin{cases}(I+\overline{r}\widehat{G}_1)\widehat{U}^{n+1}=(I-\overline{r}\widehat{G}_2)\widehat{U}^n+\overline{F}^n_1\\(I+\overline{r}\widehat{G}_2)\widehat{U}^{n+2}=(I-\overline{r}\widehat{G}_1)\widehat{U}^{n+1}+\overline{F}^n_2\end{cases} \quad (40)$$

Here $\overline{F}^n_1$ and $\overline{F}^n_2$ are known vectors related to boundary.

Let $\overline{a}=(m-1)^2$, $\overline{b}=4(m-1)$, then

$$\widehat{G}_1=\begin{pmatrix}\widehat{G}_{11}&&\\&...&\\&&\widehat{G}_{11}\end{pmatrix}_{\overline{a}\times\overline{a}}$$

$$\widehat{G}_{11}=\begin{pmatrix}\widehat{A}_1&&\\&...&\\&&\widehat{A}_1\end{pmatrix}_{\overline{b}\times\overline{b}}$$

$$\widehat{G}_2 = \begin{pmatrix} \widehat{G}_{21} & \widehat{E} & & & \\ \widehat{F} & \widehat{G}_{21} & \widehat{E} & & \\ & ... & ... & ... & \\ & & \widehat{F} & G_{21} & \widehat{E} \\ & & & \widehat{F} & G_{21} \end{pmatrix}_{\bar{b}\times\bar{b}}$$

$$G_{21} = \begin{pmatrix} \widehat{A}_2 & \widehat{B} & & & \\ \widehat{C} & \widehat{A}_2 & \widehat{B} & & \\ & ... & ... & ... & \\ & & \widehat{C} & \widehat{A}_2 & \widehat{B} \\ & & & \widehat{C} & \widehat{A}_2 \end{pmatrix}_{\bar{b}\times\bar{b}}$$

$$\widehat{B} = \begin{pmatrix} \widehat{B}_1 & & & \\ & \widehat{B}_1 & & \\ & & \widehat{B}_1 & \\ & & & \widehat{B}_1 \end{pmatrix}, \widehat{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 \end{pmatrix}$$

$$\widehat{C} = \begin{pmatrix} \widehat{C}_1 & & & \\ & \widehat{C}_1 & & \\ & & \widehat{C}_1 & \\ & & & \widehat{C}_1 \end{pmatrix}, \widehat{C}_1 = \begin{pmatrix} 0 & 0 & 0 & -q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\widehat{E} = \begin{pmatrix} O & O \\ \widehat{E}_1 & O \end{pmatrix}, \widehat{E}_1 = \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

$$\widehat{F} = \begin{pmatrix} O & \widehat{F}_1 \\ O & O \end{pmatrix}, \widehat{F}_1 = \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & -q & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}$$

Applying the analysis in section 3, we also have

Theorem 4 The EXPAGE method defined by (40) is also unconditionally stable and convergent.

5 Numerical Experiments

Example 1: We consider the following example:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} = 0, \ 0 \le x \le 2\pi, \ 0 \le t \le T \\ u(x,0) = sinx, \\ u(0,t) = u(2\pi,t) = 0. \end{cases} \quad (41)$$

The exact solution for the problem is $u(x,t) = e^{-t}sinx$. Let $||E_1||_\infty$ denote maximum absolute error, while $||E_2||_\infty$ denote maximum relevant error. $||E_1||_\infty$=$|u_i^n - u(x_i,t_n)|$, $||E_2||_\infty$=100 $\times$ $|u_i^n - u(x_i,t_n)/u(x_i,t_n)|$. In order to verify the presented AGE method, we present the numerical results of comparisons with implicit Crank-Nicolson scheme (C-N) in the following tables:

Table 1: Numerical results at $m = 16, \ t = 100\tau$

	$\tau = 10^{-3}$	$\tau = 10^{-4}$
$\|\|E_1\|\|_\infty$	1.570 $\times10^{-4}$	2.086 $\times10^{-5}$
$\|\|E_1\|\|_\infty(C-N)$	1.214 $\times10^{-4}$	1.673 $\times10^{-5}$
$\|\|E_2\|\|_\infty$	4.106 $\times10^{-2}$	5.452 $\times10^{-3}$
$\|\|E_2\|\|_\infty(C-N)$	3.717 $\times10^{-2}$	4.561 $\times10^{-3}$

Table 2: Numerical results at $m = 24, \ t = 100\tau$

	$\tau = 10^{-3}$	$\tau = 10^{-4}$
$\|\|E_1\|\|_\infty$	5.839 $\times10^{-5}$	1.295 $\times10^{-5}$
$\|\|E_1\|\|_\infty(C-N)$	3.479 $\times10^{-5}$	0.976 $\times10^{-5}$
$\|\|E_2\|\|_\infty$	1.279 $\times10^{-2}$	5.004 $\times10^{-3}$
$\|\|E_2\|\|_\infty(C-N)$	0.931 $\times10^{-2}$	3.547 $\times10^{-3}$

Table 3: Numerical results at $m = 40, \ t = 100\tau$

	$\tau = 10^{-3}$	$m = 24, \tau = 10^{-4}$
$\|\|E_1\|\|_\infty$	9.157 $\times10^{-6}$	3.883 $\times10^{-6}$
$\|\|E_1\|\|_\infty(C-N)$	8.204 $\times10^{-6}$	2.917 $\times10^{-6}$
$\|\|E_2\|\|_\infty$	5.860 $\times10^{-3}$	1.933 $\times10^{-3}$
$\|\|E_2\|\|_\infty(C-N)$	4.658 $\times10^{-3}$	1.041$\times10^{-3}$

From Table 1,2,3 we can see that the present AGE method has nearly the same accurate as the implicit C-N scheme. Furthermore, we notice the method is suitable for parallel computing.

Example 2: Consider the following problem:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$0 \le x \le 2, 0 \le y \le 2, 0 \le t \le T \quad (42)$$

with initial and boundary conditions:

$$\begin{cases} u(x,y,0) = exp(-(x-0.5)^2 - (y-0.5)^2), \\ u(0,y,t) = \frac{1}{4t+1}exp(-\frac{(t+0.5)^2}{(4t+1)} - \frac{(y-t-0.5)^2}{(4t+1)}), \\ u(2,y,t) = \frac{1}{4t+1}exp(-\frac{(1.5-t)^2}{(4t+1)} - \frac{(y-t-0.5)^2}{(4t+1)}), \\ u(x,0,t) = \frac{1}{4t+1}exp(-\frac{(x-t-0.5)^2}{(4t+1)} - \frac{(t+0.5)^2}{(4t+1)}), \\ u(x,2,t) = \frac{1}{4t+1}exp(-\frac{(x-t-0.5)^2}{(4t+1)} - \frac{(1.5-t)^2}{(4t+1)}). \end{cases} \quad (43)$$

The exact solution of the problem above is denoted as below:

$$u(x,y,t) = \frac{1}{4t+1}exp(-\frac{(x-t-0.5)^2}{(4t+1)} - \frac{(y-t-0.5)^2}{(4t+1)})$$

Let $A.E. = |u_i^n - u(x_i, t_n)|$ and $P.E. = 100 \times \frac{|u_i^n - u(x_i, t_n)|}{u(x_i, t_n)}$ denote maximum absolute error and relevant error of the presented method respectively. We compare the numerical results of the EXPAGE method (40) with Crank-Nicolson (C-N) scheme and the methods in [15, 16].

Table 4: Results at $m = 13, l = 3, \tau = 10^{-2}$

	$t = 100\tau$	$t = 1000\tau$
A.E.(EXPAGE)	5.671×10^{-5}	1.664×10^{-8}
P.E.(EXPAGE)	2.146×10^{-2}	1.253×10^{-2}
A.E.[15]	4.216×10^{-4}	5.938×10^{-7}
P.E.[15]	3.124×10^{-1}	6.627×10^{-1}
A.E.[16]	2.305×10^{-4}	2.014×10^{-7}
P.E.[16]	1.876×10^{-1}	3.172×10^{-1}
A.E.(C-N)	3.241×10^{-5}	0.937×10^{-8}
P.E.(C-N)	1.014×10^{-2}	0.894×10^{-2}

Example 3: We will consider a convection dominant problem.

Let $k_1 = k_2 = 1,\ \varepsilon_1 = \varepsilon_2 = 0.1$, then the exact solution of the problem above is denoted as below:

$$u(x, y, t) = \frac{1}{4t+1} exp(-10\frac{(x-t-0.5)^2}{(4t+1)} - 10\frac{(y-t-0.5)^2}{(4t+1)})$$

Under the condition of $m = 81$, the implicit C-N scheme is difficult to implement for computation. But the present methods can be fulfilled effectively because of its intrinsic parallelism. The numerical results of comparisons with the methods [15, 16] are listed in Table 2.

Table 5: Results at $m = 81, l = 5, \tau = 10^{-3}$,

	$t = 100\tau$	$t = 1000\tau$
A.E.(EXPAGE)	8.963×10^{-5}	2.346×10^{-5}
P.E.(EXPAGE)	4.637×10^{-2}	6.917×10^{-2}
A.E.[15]	4.426×10^{-3}	1.869×10^{-4}
P.E.[15]	6.871×10^{-1}	8.723×10^{-1}
A.E.[16]	1.078×10^{-3}	0.685×10^{-4}
P.E.[16]	3.261×10^{-1}	1.325×10^{-1}

The results in Table 4-5 show that the EXPAGE method presented are of higher accurate than the methods in [15, 16], and have nearly the same accurate as the implicit C-N scheme. Results of Table 5 show the EXPAGE method is also effective even in convection dominant cases.

6 Conclusions

In this paper, we present an unconditionally stable alternating group method with intrinsic parallelism for fourth order parabolic equations by use of saul'yev asymmetry schemes. Furthermore, we apply the concept to 2D convection-diffusion equations and construct an EXPAGE method. The results of Table 1-5 show that the two methods are superior to the methods in [15, 16].

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