Genetic Algorithms with Nelder-Mead Optimization in the variational methods of Boundary Value Problems

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Abstract: - The p-Laplacian equation is a generalization of the PDE of Laplace Equation and in this paper, we present a way of its solution using Finite Elements. Our method of Finite Elements leads to an Optimization Problem that can be solved by an appropriate combination of Genetic Algorithms and Nelder-Mead . Our method is illustrated by a numerical example. Other methods for the solution of other equations that contain the p-Laplacian operator are also discussed.

Keywords: - Boundary Value Problems, Finite Elements, Genetic Algorithms with Nelder-Mead. p-Laplacian, non-Newtonian fluid flow.

1 Introduction

The combined method of Genetic Algorithms and Nelder-Mead was proposed by the author in 2005, [2]÷[9] , while the author proposed the solution of ODEs and PDES since July 1996 (See[1])).

Many nonlinear problems in physics and mechanics are formulated in equations that contain the p-Laplacian, (i.e. the p-Laplace operator), where the p-Laplacian operator is defined as follows

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$$

G. Bognar in [17, presented a very interesting numerical and analytic investigation of problems of fluid mechanics that are described with PDEs containing the p-Laplacian operator. Previous publications (also reported in [17]) include reaction-diffusion problems, non-Newtonian fluid flows [18], fluid flows through certain types of porous media ([19], [20], the Lane-Emden equations for equilibrium configurations of spherically symmetric gaseous stellar objects [21], singular solutions for the Emden-Fowler equation [22] and the Einstein-Yang-Mills equations [23], the existence and nonexistence of black hole solutions, nonlinear elasticity [24], glaciology [25] and petroleum extraction [26]. It is clear that for $p = 2$: $\Delta_p = \Delta$. The study of the p-Laplacian equation started more than thirty years ago. In recent years, rapid development has been achieved for the study of equation involving operator Δ_p and a vast literature has appeared on the theory of quasilinear differential equations.). In [27] Strikwerda summarized many Finite Difference Schemes for PDEs. Other relevant studies can be found in [28], [29] and [30].

In [17], Bognar had studied the equation of turbulent filtration in porous media

$$\theta \frac{\partial \rho}{\partial t} = c^{\alpha} \lambda \operatorname{div}\left(\left| \nabla \rho^{n} \right|^{p-2} \nabla \rho^{n} \right), \tag{1}$$

where $\theta > 0$ and the constants $n > 0$ and $p > 1$ satisfy $np > 1$. If we scale out the constants in (1), we derive

$$\frac{\partial u}{\partial t} = \Delta_p \left(u^n \right) \tag{2}$$

where a particular case of (2) is the non-Newtonian filtration equation

$$\frac{\partial u}{\partial t} = \Delta_p u \tag{3}$$

which is also called evolution p-Laplacian equation. The case $p > 1 + \frac{1}{n}$ is called the slow diffusion and the case $p < 1 + \frac{1}{n}$, the fast diffusion.

Also in the paper [17], Bognar studied the equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\left| \nabla u \right|^{p-2} \nabla u \right) + \lambda u^{q}, \tag{4}$$

where $p > 1$, $q > 0$ and λ are some constants, in which the nonlinear term λu^q describes the nonlinear source in the diffusion process, called "heat source" if $\lambda > 0$ and "cold source" if $\lambda < 0$. Just as the Newtonian equation ($p = 2$), the appearance of nonlinear sources will exert a great influence to the properties of solutions and the influence of "heat source" and "cold source" is completely different.

In [31], an attempt is made by the author to solve the equations (2), (3) and (4) using various numerical schemes.

In this paper we will solve the boundary value problem

$$\operatorname{div}\left(\left| \nabla u \right|^{p-2} \nabla u \right) = 0$$

where u is known on the boundary of our domain using Variational Techniques (Finite elements).

The Problem is reduced to an Optimization problem that can be solved by Genetic Algorithms with Nelder-Mead. An early paper of the author with the title “Solving Differential Equations via Genetic Algorithms” was presented in [1]. Actually, the author presented in 1996 the solution of ODE and PDE using Genetic Algorithms optimization, while the author use the same method to solve various problems in [2]÷[9].

The main Results are given in Section 2 and a numerical example illustrates the method in Section 3.

A discussion for the numerical solution of (2), (3) and (4) by finite elements is also included in Section 4.

2 Main Results

We start solving the boundary value problem

$$\operatorname{div}\left(\left| \nabla u \right|^{p-2} \nabla u \right) = 0 \tag{4}$$

where u is a known function on the boundary of our domain.

As one can see in [32] and[33], the solution of this p-Laplacian equation with Dirichlet boundary conditions in a domain Ω is the minimizer of the energy functional

$$J(u) = \int |\nabla u|^p \, dv \qquad (5)$$

We consider that u is written as

$$u = \sum_n \lambda_n f_n$$

$$u = \sum_n f_n \qquad (6)$$

where λ_n have been incorporated in f_n

So, we have the minimization problem

$$\min \int \left| \nabla (\sum_n f_n) \right|^p dv$$

One can select a triangular mesh and appropriate functions f_n that have non-zero value only in the *n-th* triangle ("finite elements"). So, in a triangular mesh, for example of $\mathbb{R}^2$, we can have $f_n = a_n x + b_n y + c_n$ for the *n-th* triangle. Without loss of generality we consider the case $\mathbb{R}^2$ here u in (4).

To avoid to write **continuity conditions** on the common vertices of the triangles of the mesh, one can find that in the *n-th* triangle of the points *s,q,r* (see Figure 1)

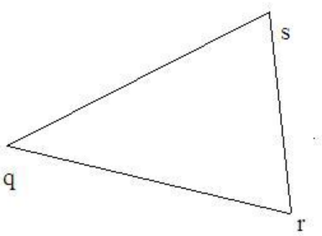

Fig.1 A triangle in a 2-D mesh

$$u_p = a_n x_s + b_n y_s + c_n \qquad (7.1)$$

$$u_q = a_n x_q + b_n y_q + c_n \qquad (7.2)$$

$$u_r = a_n x_r + b_n y_r + c_n \qquad (7.3)$$

There three equations can be solved with respect to a_n, b_n, c_n and give

$$a_n = \frac{\begin{vmatrix} u_s & y_s & 1 \\ u_q & y_q & 1 \\ u_r & y_r & 1 \end{vmatrix}}{D} \qquad (8.1)$$

$$b_n = \frac{\begin{vmatrix} x_s & u_s & 1 \\ x_q & u_q & 1 \\ x_r & u_r & 1 \end{vmatrix}}{D} \qquad (8.2)$$

$$c_n = \frac{\begin{vmatrix} x_s & y_s & u_s \\ x_q & y_q & u_q \\ x_r & y_r & u_r \end{vmatrix}}{D} \qquad (8.3)$$

$$D = \begin{vmatrix} x_s & y_s & 1 \\ x_q & y_q & 1 \\ x_r & y_r & 1 \end{vmatrix}$$ (which is by the way 2*E where E is the algebraic area of the triangle)

So, from the minimization problem

$$\min \int \left| \nabla (\sum_n f_n) \right|^p dv$$

we find the equivalent minimization problem

$$\boxed{\min \int |\phi(u_n)|^p \, dv \qquad (9)}$$

where $\phi(u_n)$ is the function that we find after replacing $f_n = a_n x + b_n y + c_n$ in $\nabla(\sum_n f_n)$

and a_n, b_n, c_n are evaluated using (8.1), (8.2), (8.3) for each triangle of the mesh.

Equation (9) can be solved now by a variety of techniques. The author uses Genetic Algorithms with Nelder-Meade for Non-linear Problems as in [2], [3], [4], [5], [6], [7], [8].

The same optimization scheme: Genetic Algorithms with Nelder-Meade will be also applied for (9).

Before proceeding in the solution of the problem, some background on GA (Genetic Algorithms) and Nelder-Mead is necessary. In [4], the author has also proposed a hybrid method that includes a) Genetic Algorithm for finding rather the neiborhood of the global minimum than the global minimu itself and b) Nelder-Mead algorithm to find the exact point of the global minimum itself.

So, with this Hybrid method of Genetic Algorithm + Nelder-Mead we combine the advantages of both methods, that are a) the convergence to the global minimum (genetic algorithm) plus b) the high accuracy of the Nelder-Mead method.

If we use only a Genetic Algorithm then we have the problem of low accuracy.

If we use only Nelder-Mead, then we have the problem of the possible convergence to a local (not to the global) minimum.

These disadvantages are removed in the case of our Hybrid method that combines Genetic Algorithm with Nelder-Mead method. We recall the following definitions from the Genetic Algorithms literature:

Fitness function is the objective function we want to minimize.
Population size specifies how many individuals there are in each generation. We can use various Fitness Scaling Options (rank, proportional, top, shift linear, etc), as well as various Selection Options (like Stochastic uniform, Remainder, Uniform, Roulette, Tournament). Fitness Scaling Options: We can use scaling functions. A Scaling function specifies the function that performs the scaling. A scaling function converts raw fitness scores returned by the fitness function to values in a range that is suitable for the selection function.

We have the following options:
Rank Scaling Option: scales the raw scores based on the rank of each individual, rather than its score. The rank of an individual is its position in the sorted scores. The rank of the fittest individual is 1, the next fittest is 2 and so on. Rank fitness scaling removes the effect of the spread of the raw scores.
Proportional Scaling Option: The Proportional Scaling makes the expectation proportional to the raw fitness score. This strategy has weaknesses when raw scores are not in a "good" range.
Top Scaling Option: The Top Scaling scales the individuals with the highest fitness values equally.

Shift linear Scaling Option: The shift linear scaling option scales the raw scores so that the expectation of the fittest individual is equal to a constant, which you can specify as Maximum survival rate, multiplied by the average score.
We can have also option in our Reproduction in order to determine how the genetic algorithm creates children at each new generation.
For example,
Elite Counter specifies the number of individuals that are guaranteed to survive to the next generation.
Crossover combines two individuals, or parents, to form a new individual, or child, for the next generation.
Crossover fraction specifies the fraction of the next generation, other than elite individuals, that are produced by crossover.
Scattered Crossover: Scattered Crossover creates a random binary vector. It then selects the genes where the vector is a 1 from the first parent, and the genes where the vector is a 0 from the second parent, and combines the genes to form the child.
Mutation: Mutation makes small random changes in the individuals in the population, which provide genetic diversity and enable the GA to search a broader space. Gaussian Mutation: We call that the Mutation is Gaussian if the Mutation adds a random number to each vector entry of an individual. This random number is taken from a Gaussian distribution centered on zero. The variance of this distribution can be controlled with two parameters. The Scale parameter determines the variance at the first generation. The Shrink parameter controls how variance shrinks as

generations go by. If the Shrink parameter is 0, the variance is constant. If the Shrink parameter is 1, the variance shrinks to 0 linearly as the last generation is reached.

Migration is the movement of individuals between subpopulations (the best individuals from one subpopulation replace the worst individuals in another subpopulation). We can control how migration occurs by the following three parameters.

Direction of Migration: Migration can take place in one direction or two. In the so-called "Forward migration" the nth subpopulation migrates into the (n+1)'th subpopulation. while in the so-called "Both directions Migration", the nth subpopulation migrates into both the (n-1)th and the (n+1)th subpopulation. Migration wraps at the ends of the subpopulations. That is, the last subpopulation migrates into the first, and the first may migrate into the last. To prevent wrapping, specify a subpopulation of size zero.

Fraction of Migration is the number of the individuals that we move between the subpopulations. So, Fraction of Migration is the fraction of the smaller of the two subpopulations that moves. If individuals migrate from a subpopulation of 50 individuals into a population of 100 individuals and Fraction is 0.1, 5 individuals (0.1 * 50) migrate. Individuals that migrate from one subpopulation to another are copied. They are not removed from the source subpopulation.

Interval of Migration counts how many generations pass between migrations.

The Nelder-Mead simplex algorithm appeared in 1965 and is now one of the most widely used methods for nonlinear unconstrained optimization [33]÷[35]. The Nelder-Mead method attempts to minimize a scalar-valued nonlinear function of n real variables using only function values, without any derivative information (explicit or implicit).

The Nelder-Mead method thus falls in the general class of direct search methods. The method is described as follows: Let $f(x)$ be the function for minimization.

x is a vector in n real variables. We select $n+1$ initial points for x and we follow the steps:

Step 1. *Order*. Order the $n+1$ vertices to satisfy $f(x_1) \le f(x_2) \le \ldots \le f(x_{n+1})$, using the tie-breaking rules given below.

Step 2. *Reflect*. Compute the *reflection point* x_r from $x_r = \bar{x} + \rho(\bar{x} - x_{n+1}) = (1+\rho)\bar{x} - \rho x_{n+1}$, where $\bar{x} = \sum_{i=1}^{n} x_i / n$ is the centroid of the n best points (all vertices except for x_{n+1}). Evaluate $f_r = f(x_r)$. If $f_1 \le f_r < f_n$, accept the reflected point x_r and terminate the iteration.

Step 3. *Expand*. If $f_r < f_1$, calculate the *expansion point* x_e,

$$x_e = \bar{x} + \chi(x_r - \bar{x}) = \bar{x} + \rho\chi(\bar{x} - x_{n+1}) = (1+\rho\chi)\bar{x} - \rho\chi x_{n+1}$$

and evaluate $f_e = f(x_e)$. If $f_e < f_r$, accept x_e and terminate the iteration; otherwise (if $f_e \ge f_r$), accept x_r and terminate the iteration.

Step 4. *Contract*. If $f_r \ge f_n$, perform a *contraction* between $\bar{x}$ and the better of x_{n+1} and x_r.

Outside. If $f_n \le f_r < f_{n+1}$ (i.e. x_r is strictly better than x_{n+1}), perform an *outside contraction*: calculate

$$x_c = \bar{x} + \gamma(x_r - \bar{x}) = \bar{x} + \gamma\rho(\bar{x} - x_{n+1}) = (1+\rho\gamma)\bar{x} - \rho\gamma x_{n+1}$$

and evaluate $f_c = f(x_c)$. If $f_c \le f_r$, accept x_c and terminate the iteration; otherwise, go to step 5 (perform a shrink).

b. *Inside*. If $f_r \ge f_{n+1}$, perform an *inside contraction*: calculate

$x_{cc} = \bar{x} - \gamma(\bar{x} - x_{n+1}) = (1-\gamma)\bar{x} + \gamma x_{n+1}$, and evaluate $f_{cc} = f(x_{cc})$. If $f_{cc} < f_{n+1}$, accept x_{cc} and terminate the iteration; otherwise, go to step 5 (perform a shrink).

Step 5. *Perform a shrink step*. Evaluate f at the n points $v_i = x_1 + \sigma (x_i - x_1)$, $i = 2, \ldots, n+1$. The (unordered) vertices of the simplex at the next iteration consist of $x_1, v_2, \ldots, v_{n+1}$.

After this preparation, we are ready to solve the $\min \int |\phi(u_n)|^p \, dv$ of (9) as minimization problem.

The minimization is achieved by using Genetic Algorithms (GA) and the method of Nelder-Mead exactly as we described previously. We can use the MATLB software package (MATLAB, Version 7.0.0, by Math Works).

In the next numerical example (Section 3) our GA has the following Parameters

Population type:
Double Vector Population size: 30

Creation function: Uniform

Fitness scaling: Rank

Selection function: roulette

Reproduction: 6 – Crossover fraction 0.8

Mutation: Gaussian – Scale 1.0,
Shrink 1.0

Crossover: Scattered

Migration: Both – fraction 0.2, interval: 20

Stopping criteria: 50 generations

3 Numerical Example

Consider now the following problem (Fig.2)

$$\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \tag{4}$$

in the domain $u \in [0,2]\times[0,2]-[0,1]\times[0,1]$

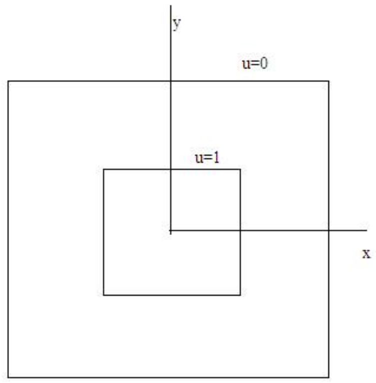

Fig.2

with $u = 0$ in the external boundary:
$x = \pm 2, -2 \le y \le 2$
$y = \pm 2, -2 \le x \le 2$

and $u = 1$ in the internal boundary

$x = \pm 1, -1 \le y \le 1$
$y = \pm 1, -1 \le x \le 1$

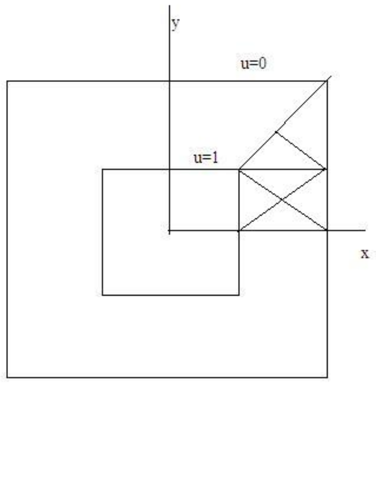

Fig.3

Due to symmetry, we can split the domain in 8 same trapezoids (trapezia). It is sufficient to solve the problem $\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)=0$ in one of them with the boundary conditions $u=0$ in the external boundary and $u=1$ in the internal boundary.

Taking one of these trapezoids and splitting it into 6 triangles like in Fig.3, we have in some enlargement the following Figure: Fig.4

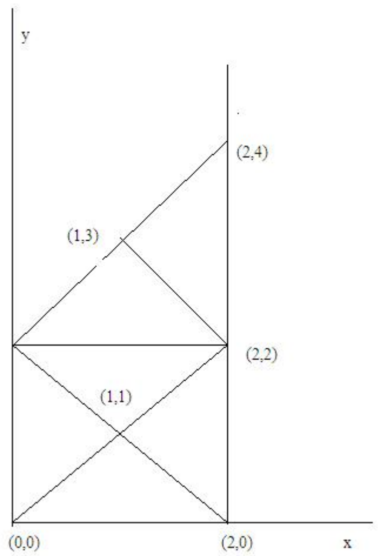

Fig.4

We consider as $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ the value of the u at the points
$(0,0),(2,0),(2,2),(2,4),(1,3),(0,2),(1,1)$
i.e.

$u_1 = u(0,0),$

$u_2 = u(2,0),$

$u_3 = u(2,2),$

$u_4 = u(2,4),$

$u_5 = u(1,3),$

$u_6 = u(0,2),$

$u_7 = u(1,1)$

Then by considering

$$u_p = a_n x_s + b_n y_s + c_n \qquad (7.1)$$

$$u_q = a_n x_q + b_n y_q + c_n \qquad (7.2)$$

$$u_r = a_n x_r + b_n y_r + c_n \qquad (7.3)$$

in every one of the 6 triangles, we solve as in (8.1), (8.2), (8.3) and finally we introduce to

$$J(u) = \int |\nabla u|^p \, dv \qquad (5)$$

We have, considering also that $u_1 = u_6 = 1$ and $u_2 = u_3 = u_4 = 0$

So, after some algebraic manipulation we find that we have to minimize the quantity I

where

$$I = |2u_5|^p + \left|\sqrt{1^2+(1-2u_5)^2}\right|^p + \left|\sqrt{1^2+(1-2u_7)^2}\right|^p + \\ + |(2-2u_7)|^p + \left|\sqrt{1^2+(1-2u_7)^2}\right|^p + |(2u_7)|^p$$

with respect to u_5, u_7

Now, in order to find the global minimum of I we use **GA** *(Population type: Double Vector Population size: 30 / Creation function: Uniform /Fitness scaling: Rank / Selection function: roulette / Reproduction: 6 – Crossover fraction 0.8 / / Mutation: Gaussian – Scale 1.0, Shrink 1.0 / / Crossover: Scattered / Migration: Both – fraction 0.2, interval: 20 /Stopping criteria: 50 generations)*
and continue with **Nelder-Mead**

So we find the following results for different values of *p*.

p	u_5	u_7	I
2	0.2500	0.5000	5.5000
3	0.3145	0.5000	5.4623
4	0.3471	0.5000	5.4280
5	0.3678	0.5000	5.3994
6	0.3824	0.5000	5.3754
7	0.3935	0.5000	5.3550
8	0.4024	0.5000	5.3373
10	0.4155	0.5000	5.3082
20	0.4468	0.5000	5.2246
50	0.4721	0.5000	5.1375
200	0.4903	0.5000	5.0582

4 Solution of the equations (2), (3) and (4)

We remind the problems:

$$\theta \frac{\partial \rho}{\partial t} = c^{\alpha} \lambda \operatorname{div}\left(\left| \nabla \rho^{n} \right|^{p-2} \nabla \rho^{n} \right), \qquad (1)$$

If we scale out the constants in (1), we derive

$$\frac{\partial u}{\partial t} = \Delta_{p}\left(u^{n}\right) \qquad (2)$$

where a particular case of (2) is the non-Newtonian filtration equation

$$\frac{\partial u}{\partial t} = \Delta_{p} u \qquad (3)$$

and

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\left| \nabla u \right|^{p-2} \nabla u \right) + \lambda u^{q}, \qquad (4)$$

Consider that u can be written as $u = \sum_{n} \lambda_n(t) f_n$ or $u = \sum_{n} f_n(t)$ where λ_n have been incorporated to $f_n(t)$

In this "dynamic" case, in a triangular mesh of $\mathbb{R}^2$ we can have $f_n = a_n(t)x + b_n(t)y + c_n(t)$ for the *n-th* triangle.

$$u_s = a_n(t)x_s + b_n(t)y_s + c_n(t) \qquad (7.1)$$

$$u_q = a_n(t)x_q + b_n(t)y_q + c_n(t) \qquad (7.2)$$

$$u_r = a_n(t)x_r + b_n(t)y_r + c_n(t) \qquad (7.3)$$

Of course, we can use higher degree polynomials like quadratic or cubic.

For quadratic:

$$u_s = a_n(t)x_s + b_n(t)y_s + c_n(t) + d_n(t)x_s^2 + e_n(t)d_s^2 + h_n(t)x_s y_s$$

$$u_q = a_n(t)x_q + b_n(t)y_q + c_n(t) + d_n(t)x_q^2 + e_n(t)d_q^2 + h_n(t)x_q y_q$$

$$u_r = a_n(t)x_r + b_n(t)y_r + c_n(t) + d_n(t)x_r^2 + e_n(t)d_r^2 + h_n(t)x_r y_r$$

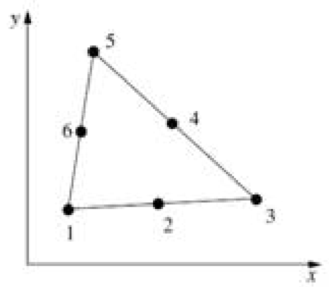

Fig.5

We express $a_n(t), b_n(t), c_n(t), d_n(t), e_n(t), h_n(t)$ with respect not only u in vertices, but also in a node along the midside of each edge. See Fig.5.

Finally using the so-called **collocation method** or **least square method** or **the method of moments** ([35]÷[40]) we can obtain a system of non-linear Ordinary Differential Equations that can be solved in a variety of methods (Runge – Kutta etc…).

5 Conclusion

In this paper, we have examined the boundary value problem $\operatorname{div}\left(\left| \nabla u \right|^{p-2} \nabla u \right) = 0$ where u is a known function on the boundary of our domain using Variational Principle (Finite elements). The Problem is reduced to an Optimization problem that can be solved by Genetic Algorithms plus Nelder-Mead search. An early paper of the author with the title "Solving Differential Equations via Genetic Algorithms" was presented in [1] while the author use the same method to solve various problems in [2]÷[9].

With the Hybrid method of Genetic Algorithm + Nelder-Mead we have combined the advantages of both methods, that are **a)** the convergence to the global minimum (genetic algorithm) plus **b)** the high accuracy of the Nelder-Mead method.

Also, we have discussed briefly the solution of $\frac{\partial u}{\partial t} = \Delta_{p}\left(u^{n}\right)$

$$\frac{\partial u}{\partial t} = \Delta_p u$$

and

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\left|\nabla u\right|^{p-2} \nabla u\right) + \lambda u^q ,$$

using the so-called collocation method or least square method or the method of moments.

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