The Ramsey Model with Logistic Population Growth and Benthamite Felicity Function Revisited

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Abstract: In this paper we extend the study done by Ferrara and Guerrini [12], where two different research lines within the Ramsey model were joined together: the one studying the role of a logistic population growth rate (Accinelli and Brida [2]), and the one analyzing the effects of a Benthamite formulation for the utility function. The results obtained in [12] for the special case of a constant intertemporal elasticity of substitution (CIES) utility function and a Cobb-Douglas production function are provided to be still true for a general utility function and a neoclassical production function. We have that the model is described by a three dimensional dynamical system, whose unique non-trivial steady state equilibrium is a saddle point with a two dimensional stable manifold. Consequently, the speed of convergence is determined by two stable roots, rather than only one as in the basic Ramsey model. In addition, in the special case of a CIES utility function and a Cobb-Douglas technology, an explicit solution for the model can be derived, when capital's share is equal to the reciprocal of the intertemporal elasticity of substitution.

Key-Words: Ramsey, Logistic population, Benthamite, Saddle point, Exact solution.

1 Introduction

The starting point for any study of economic growth is the neoclassical growth model, also known as the Solow-Swan model, which originated from the work of Solow [25] and Swan [27]. Swan's paper was published ten months later than Solow, but included a more complete analysis of technical progress, which Solow treated separately in Solow [26]. The neoclassical model extends the Harrod-Domar model, which was independently developed by Harrod [15] and Domar [7], by adding labor as a factor of production, by requiring diminishing returns to labor and capital separately and constant returns to scale for both factors combined, by introducing a time-varying technology variable distinct from capital and labor. As well, the capital-output and capital-labor ratios are not fixed as they are in the Harrod-Domar model. These refinements allow increasing capital intensity to be distinguished from technological progress. Therefore, on the basis of these assumptions, an economy, regardless of its starting point, converges towards a steady state rate of growth, a condition of the economy in which output per worker (productivity of labor) and capital per worker (capital intensity) do not change over time, where long-run growth of output and capital are determined solely by the rate of labor-augmenting technological progress and the rate of population growth (see, for example, Barro and Sala-i-Martin [3] for details).

As simple hypotheses go, in the neoclassical growth model of Solow and Swan, the assumption that the savings rate is constant over time at an exogenous level, and, hence, the ratio of consumption to income, essentially precludes the possibility of doing any significant analysis on optimal policy.

To paint a more complete picture of the process of economic growth, we need to allow for the path of consumption, and, hence, the saving rate, to be determined by optimizing households and firms that interact on competitive markets. The idea is to deal with infinitely lived households that choose consumption and saving to maximize their dynastic utility, subject to an intertemporal budget constraint. This specification of consumer behavior is a key element in the Ramsey growth model, as constructed by Ramsey [20], and refined by Cass [5] and Koopmans [16]. The brilliant idea of Ramsey was to determine the saving rate endogenously, through a dynamic maximization process. As a result, unlike the Solow-Swan model, the saving rate in general is not constant and the convergence of the economy to its steady state is not uniform. In other words, maintaining the same structure of the neoclassical growth model of Solow and Swan, Ramsey, Cass and Koopmans brought explicitly into the model a utility maximizing behavior on the part of consumers. Under this new modelling approach, consumers maximized time aggregate utility, making simultaneous decisions every period on consumption and savings which, in turn, provide resources for capital accumulation on the part of firms. And these decisions are taken under an intertemporal optimality criterion.

In the Ramsey model, population (labor) is assumed to grow at a positive given rate, so leading to exponential growth. Therefore, this population growth model, usually associated with the name of Malthus (Malthusian model [17]), predicts unbounded growth. This constant population growth assumption is manifestly unrealistic. In addition, the Malthusian model considers homogeneous populations, namely it supposes that all the individuals of such a population are physiologically identical, as well as that the population lives isolated in an invariable habitat and with limitless resources, so that the population depends, respectively, on constant fertility and mortality rates. Although one assumes that variations do not take place in the external habitat, the population itself causes changes in life conditions due to competition for the survival resources. Consequently, one could admit that the fertility and mortality rates depend on the total size of the population, replacing the linear model of Malthus by a non-linear model.

In this direction, a major contribution to population growth models came from Verhulst [28], who showed that the population growth not only depends on the population size but also on how far this size is from its upper limit. His model, known as logistic model, improved upon the exponential growth model of Malthus by incorporating a limiting population value, or carrying capacity, that the environment can support. Above this value, lack of food or other resources causes the death rate to rise so that it equals the birth rate. It does not account for oscillations that may occur when food runs out suddenly, but is otherwise quite accurate, and has been shown to give a close match to real populations.

On the other hand, it is a very well known stylized fact that since the 1950s, population growth rate is decreasing, and it is projected to decrease to zero during the next six decades. This decrease, which is particularly relevant in the group of developed countries, but it can also be observed on a global scale, is predominantly due to the aging of the population, causing a dramatic increase in the number of deaths. From 2030 to 2050, the world population would grow more slowly than ever before in its history (see Day [6]). Then, as described by Maynard Smith [18], a more realistic law of population growth would be the logistic model. In economic growth modelling, this approach have been recently analyzed in different directions (see, for example, [1],[2],[8] − [12],[14],[21],[22]).

The main objective of this paper is to generalize the results of [12], where a Benthamite utility function was introduced into the Ramsey model with logistic population growth (Accinelli and Brida [2]). The felicity function is now multiplied by the size of the family, which indicates that at any point in time overall utility is equal to the addition of the felicities of all family members alive at that time. Whereas in the model of [2], the total utility does no longer depend on the size of the population, but only on the per capita felicity discounted by the rate of time preference. As well, this alternative formulation of the planner's problem is investigated working with a general utility function and a neoclassical production function, in contrast to the model of [12], where the special case of a CIES utility function and a Cobb-Douglas production function was considered. These assumptions lead the model to be described by a three dimensional dynamical system, which has a unique non-trivial steady state equilibrium. This is proved to be a saddle point with a two dimensional stable manifold. As a result, the speed of convergence is determined by two stable roots, rather than only one as in the basic Ramsey model. In addition, in case of CIES utility and Cobb-Douglas technology, we derive a closed form solution for our modified Ramsey model, when capital's share is equal to the reciprocal of the intertemporal elasticity of substitution.

2 Setup of the model

We consider a closed economy, which consists of a fixed number of identical infinitely lived households that, for simplicity, is normalized to one. Time is taken to be continuous. Each household has access to a technology described by a neoclassical production function

$$f : \mathbb{R}_+ \to \mathbb{R}, \ \ y_t = f(k_t),$$

where y_t and k_t denote output and capital spent producing goods, all in per capita terms, respectively. Therefore, f is C^2, strictly increasing, strictly concave, linearly homogeneous, satisfying $f(0) = 0$, and the Inada conditions

$$\lim_{k_t\to 0^+} f'(k_t) = +\infty, \ \ \lim_{k_t\to +\infty} f'(k_t) = 0.$$

One simple production function that is often thought to provide a reasonable description of actual economies is the Cobb-Douglas function

$$f(k_t) = k_t^{\alpha}, \quad \alpha \in (0,1), \tag{1}$$

Note that the Cobb-Douglas form satisfies the properties of a neoclassical production function since

$$f'(k_t) = \alpha k_t^{\alpha-1} > 0, \; f''(k_t) = -(1-\alpha)\alpha k_t^{\alpha-2} < 0,$$

$$\lim_{k_t \to 0^+} f'(k_t) = +\infty, \quad \lim_{k_t \to +\infty} f'(k_t) = 0.$$

Each household's preferences are represented by an instantaneous utility function

$$u : \mathbb{R}_+ \to \mathbb{R},$$

which depends on per capita consumption c_t. This utility function is assumed to be C^2, strictly increasing, strictly concave, and satisfying the Inada conditions

$$\lim_{c_t \to 0^+} u'(c_t) = +\infty, \quad \lim_{c_t \to +\infty} u'(c_t) = 0.$$

A special case is that of a constant intertemporal elasticity of substitution (CIES) utility function,

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}, \tag{2}$$

where $\theta > 0$, $\theta \neq 1$, is the reciprocal of σ, the intertemporal elasticity of substitution of consumption. The logarithmic utility, $u(c_t) = \ln c_t$, is the particular case of CIES with $\theta = 1$.

Following Accinelli and Brida [2], L_t is assumed to evolve according to the logistic law

$$\frac{\dot{L}_t}{L_t} = a - bL_t \equiv n(L_t), \tag{3}$$

with $a > b > 0$. For simplicity, the initial population has been normalized to one, $L_0 = 1$.

Given the initial capital stock, $k_0 > 0$, the representative household seeks to maximize the following discounted sum of instantaneous utilities

$$\int_0^{\infty} u(c_t) L_t e^{-\rho t} dt, \tag{4}$$

where $\rho > 0$ is the rate of time preference, subject to the constraint (3), and to the budget constraint

$$\dot{k}_t = f(k_t) - [\delta + n(L_t)] k_t - c_t. \tag{5}$$

The positive constant δ denotes the depreciation rate of capital accumulation. Contrary to Accinelli and Brida [2], the felicity function $u(c_t)$, which appears in (3), is multiplied by the size of the family, namely we have $u(c_t)L_t$ (Benthamite welfare function), so that the number of family members receiving the given utility level is taken into account. At any point in time overall utility is equal to the addition of the felicities of all family members alive at that time.

3 Optimality conditions

In the previous section, our problem has been formulated as one of optimal control, which can easily be dealt with the Pontryagin maximum principle (see Pontryagin et al. [19]). The current-value Hamiltonian for the dynamic optimization planner's problem is

$$H(k_t, c_t, L_t, \lambda_t, \mu_t) = u(c_t)L_t + \lambda_t \{f(k_t) - [\delta + n(L_t)] k_t - c_t\} + \mu_t [L_t n(L_t)], \tag{6}$$

where μ_t and λ_t are the costate variables associated to (3) and (5), respectively. The Pontryagin conditions for optimality are

$$H_{c_t} = 0, \;\; \dot{\lambda}_t = \rho\lambda_t - H_{k_t}, \;\; \dot{\mu}_t = \rho\mu_t - H_{L_t}, \tag{7}$$

$$\dot{k}_t = H_{\lambda_t}, \;\; \dot{L}_t = H_{\mu_t}, \tag{8}$$

together with the transversality conditions

$$\lim_{t \to +\infty} e^{-\rho t} \lambda_t k_t = 0, \quad \lim_{t \to +\infty} e^{-\rho t} \mu_t L_t = 0.$$

Conditions (7) and (8) give

$$u'(c_t) L_t = \lambda_t, \tag{9}$$

$$\dot{\lambda}_t = -\lambda_t [f'(k_t) - \delta - \rho - n(L_t)], \tag{10}$$

$$\dot{\mu}_t = \mu_t [\rho - n(L_t) + bL_t] - b\lambda_t k_t - u(c_t),$$

plus the equations (3) and (5). We want now to eliminate the costate variable λ_t from the above equations. Differentiating (9) with respect to time, we obtain

$$u''(c_t)\dot{c}_t L_t + u'(c_t)\dot{L}_t = \dot{\lambda}_t. \tag{11}$$

Thus, combining (11) with (10), and recalling (9), we get

$$\dot{c}_t = -\frac{u'(c_t)}{u''(c_t)} \left[f'(k_t) - \delta - \rho \right]. \tag{12}$$

Using the inverse of the instantaneous elasticity of substitution of consumption, namely

$$\theta(c_t) = -\frac{u''(c_t)c_t}{u'(c_t)} = \frac{1}{\sigma(c_t)},$$

we can write condition (12) as

$$\dot{c}_t = \sigma(c_t) c_t \left[f'(k_t) - \delta - \rho \right]. \tag{13}$$

Remark 1. *Equation* (13) *shows that optimal consumption increases, decreases or stays constant at each point in time, depending on whether the marginal product of physical capital net of total depreciation is greater, lower, or equal to the social rate of time discount.*

Remark 2. *Contrary to Accinelli and Brida* [2], *population growth has now no effect on the growth rate of per capita consumption.*

In this way, we have reduced the differential equation for the costate variable to a differential equation for the control variable. Thus, the canonical equations of our optimal control problem become

$$\begin{cases} \dot{k}_t = f(k_t) - [\delta + n(L_t)]\, k_t - c_t, \\ \dot{c}_t = \sigma(c_t)c_t[f'(k_t) - \delta - \rho], \\ \dot{L}_t = L_t n(L_t), \\ \dot{\mu}_t = \mu_t\,[\rho - n(L_t) + bL_t] - b\lambda_t k_t - u(c_t), \end{cases}$$

together with the transversality conditions

$$\lim_{t\to+\infty} e^{-\rho t} u'(c_t) L_t k_t = 0, \quad \lim_{t\to+\infty} e^{-\rho t} \mu_t L_t = 0.$$

Lemma 3.

$$\mu_t = -\frac{u'(c_t)\,\{f(k_t) - [\delta + n(L_t)]\, k_t - c_t\}}{n(L_t)}.$$

Proof: Since the current-value Hamiltonian of the system is autonomous, we must have $dH/dt = 0$. Taking partial differentials of the current-value Hamiltonian (6) with respect to time yields

$$H_{k_t}\dot{k}_t + H_{c_t}\dot{c}_t + H_{L_t}\dot{L}_t + H_{\lambda_t}\dot{\lambda}_t + H_{\mu_t}\dot{\mu}_t = 0.$$

Hence, using (7), (8), we have

$$\lambda_t \dot{k}_t + \mu_t \dot{L}_t = 0,$$

which implies

$$\mu_t = -\frac{\dot{k}_t}{\dot{L}_t}\,\lambda_t. \tag{14}$$

Now (3) and (5) can be substituted in (14) to obtain the statement. □

In conclusion, we have derived that it is possible to reduce our system to a system of three dynamical equations only, i.e.

$$\begin{cases} \dot{k}_t = f(k_t) - [\delta + n(L_t)]\, k_t - c_t, \\ \dot{c}_t = \sigma(c_t)c_t[f'(k_t) - \delta - \rho], \\ \dot{L}_t = L_t n(L_t), \end{cases} \tag{15}$$

with the transversality conditions

$$\lim_{t\to+\infty} e^{-\rho t} u'(c_t) L_t k_t = 0, \quad \lim_{t\to+\infty} \frac{e^{-\rho t} u'(c)\dot{k}_t}{n(L_t)} = 0.$$

This reduce-form system captures the complete dynamic of the economy. Given $k_0 > 0, c_0 > 0$ this Cauchy problem has a unique solution (k_t, c_t, L_t) defined on $[0, +\infty)$ (see Birkhoff and Rota [4]).

4 Local dynamics

Before analyzing the dynamics of system (15), we look at the steady state of the optimal allocation. We recall that in steady state both per capita capital stock k_t, the level of consumption per capita c_t, and L_t are constant. Let us denote the steady state values of these variables by k_*, c_*, and L_*, respectively. Note that our analysis is restricted to interior steady states only, i.e. we exclude the economically meaningless solutions such as $k_* = 0, c_* = 0$, or $L_* = 0$.

Lemma 4. *There exists a unique steady state equilibrium* (k_*, c_*, L_*), *where the capital-labor ratio* k_*, *per capita consumption* c_*, *and population* L_* *are given by*

$$f'(k_*) = \delta + \rho, \quad c_* = f(k_*) - \delta k_*, \quad L_* = \frac{a}{b}.$$

Proof: A steady state satisfies

$$\dot{k}_t = \dot{c}_t = \dot{L}_t = 0.$$

From (15), we find that this implies

$$c_t = f(k_t) - \delta k_t, \quad f'(k_t) = \delta + \rho, \quad n(L_t) = 0.$$

It is now immediate that equation $n(L_t) = 0$ yields $L_* = a/b$. Next, let us consider $f'(k_t) = \delta + \rho$, and plot the two curves $f'(k_t)$ and $\delta + \rho$ versus k_t. The curve $f'(k_t)$ is downward sloping since $f'(k_t)$ has negative with respect to k_t, it asymptotes to infinity at $k_t = 0$, and it approaches zero as k_t tends to infinity. The curve $\delta + \rho$ is instead a horizontal line. Since $\delta + \rho > 0$, and $f'(k_t)$ falls monotonically from infinity to zero, these two curves intersect once and only once. Hence, the steady-state capital-labor ratio $k_* > 0$ exists, and it is unique. Consequently, there is a unique c_* satisfying the identity $c_t = f(k_t) - \delta k_t$. □

Proposition 5. *The steady state equilibrium is a saddle point with a two dimensional stable manifold.*

Proof: Setting

$$F^1(k_t, c_t, L_t) = f(k_t) - [\delta + n(L_t)]\, k_t - c_t,$$
$$F^2(k_t, c_t, L_t) = \sigma(c_t)c_t\,[f'(k_t) - \delta - \rho]\,,$$
$$F^3(k_t, c_t, L_t) = L_t n(L_t),$$

the nonlinear system of differential equations (15) rewrites as

$$\begin{cases} \dot{k}_t = F^1(k_t, c_t, L_t), \\ \dot{c}_t = F^2(k_t, c_t, L_t), \\ \dot{L}_t = F^3(k_t, c_t, L_t). \end{cases}$$

A local linear approximation of this system near the equilibrium point (k_*, c_*, L_*) is given by the expansion in a Taylor series of the coordinate functions F^1, F^2, F^3 truncated after the first-order terms. Setting $P_* = (k_*, c_*, L_*)$, this means (in first approximation)

$$\begin{aligned}\frac{d(k_t-k_*)}{dt} &= F^1(P_*) + F^1_{k_t}(P_*)\,(k_t-k_*)\\ &\quad + F^1_{c_t}(P_*)\,(c_t-c_*) + F^1_{L_t}(P_*)\,(L_t-L_*)\\ \frac{d(c_t-c_*)}{dt} &= F^2(P_*) + F^2_{k_t}(P_*)\,(k_t-k_*)\\ &\quad + F^2_{c_t}(P_*)\,(c_t-c_*) + F^2_{L_t}(P_*)\,(L_t-L_*)\\ \frac{d(L_t-L_*)}{dt} &= F^3(P_*) + F^3_{k_t}(P_*)\,(k_t-k_*)\\ &\quad + F^3_{c_t}(P_*)\,(c_t-c_*) + F^3_{L_t}(P_*)\,(L_t-L_*)\end{aligned}$$

Considering that the first term on the right hand side of each equation is equal to zero, i.e. $F^i(P_*) = 0$, $i = 1, 2, 3$, the linearized system can be written compactly as follows

$$\begin{bmatrix} \dot{k}_t \\ \dot{c}_t \\ \dot{L}_t \end{bmatrix} = J^* \begin{bmatrix} k_t - k_* \\ c_t - c_* \\ L_t - L_* \end{bmatrix}, \qquad (16)$$

where J^*, called the Jacobian matrix, denotes the matrix of first partial derivatives evaluated at the equilibrium point P_*, namely

$$J^* = \begin{bmatrix} F^1_{k_t}(P_*) & F^1_{c_t}(P_*) & F^1_{L_t}(P_*) \\ F^2_{k_t}(P_*) & F^2_{c_t}(P_*) & F^2_{L_t}(P_*) \\ F^3_{k_t}(P_*) & F^3_{c_t}(P_*) & F^3_{L_t}(P_*) \end{bmatrix}.$$

Computing these elements yields

$$J^* = \begin{bmatrix} \rho & -1 & bk_* \\ \sigma(c_*)c_* f''(k_*) & 0 & 0 \\ 0 & 0 & -a \end{bmatrix}.$$

In order to characterize the local stability of the system, we need to compute the eigenvalues of the Jacobian matrix J^*. These correspond to the values of ξ that solve the following cubic form

$$\det \begin{bmatrix} \rho-\xi & -1 & bk_* \\ \sigma(c_*)c_* f''(k_*) & -\xi & 0 \\ 0 & 0 & -a-\xi \end{bmatrix} = 0.$$

It is straightforward to verify that one eigenvalue is given by

$$\xi_1 = -a,$$

and the other two eigenvalues are the solutions of the equation

$$\xi^2 - \rho\xi + \sigma(c_*)c_* f''(k_*) = 0. \qquad (17)$$

We have

$$\xi_{2,3} = \frac{\rho \pm \sqrt{\rho^2 - 4\sigma(c_*)c_* f''(k_*)}}{2}.$$

Since $\sigma(c_*)c_* f''(k_*) < 0$, the two roots of (17) are real. Moreover, their signs can be derived looking at the trace and the determinant of J^*. In fact, recalling that the trace of a matrix is equal to the sum of its eigenvalues, and the determinant of a matrix is also equal to the product of its eigenvalues, we can write

$$\xi_1 + \xi_2 + \xi_3 = tr(J^*) = \rho - a \;\Rightarrow\; \xi_2 + \xi_3 > 0$$

$$\begin{aligned}\xi_1 \cdot \xi_2 \cdot \xi_3 \;&=\; det(J^*) \;=\; -a\sigma(c_*)c_* f''(k_*)\\ &\Rightarrow\; \xi_2 \cdot \xi_3 < 0.\end{aligned}$$

Consequently, we derive that one root is negative and one is positive. In conclusion, the matrix J^* has one real positive (unstable) and two real negative (stable) roots. Thus, the system dynamics exhibits saddle point stability with the stable manifold, which is the hyperplane generated by the associated eigenvectors, being two dimensional (see Simon and Blume [23]). □

Remark 6. *An equilibrium point of a system of differential equations is hyperbolic if the Jacobian matrix calculated at that point has no zero or purely imaginary eigenvalues (no eigenvalue has real part equal to zero). There exists a general result in the theory of differential equations, known as the Hartman-Grobman theorem (see Guckenheimer and Holmes* [13]*), which guarantees that, if the equilibrium point is hyperbolic, in a neighbourhood of the equilibrium point the qualitative properties of the nonlinear system* (15) *are preserved by the linearization* (16).

For the linearized model (16) we can then derive a closed-form analytic solution given by

$$\begin{cases} k_t - k_* = \beta_1 v_{11} e^{\xi_1 t} + \beta_2 v_{12} e^{\xi_2 t} + \beta_3 v_{13} e^{\xi_3 t}, \\ c_t - c_* = \beta_1 v_{21} e^{\xi_1 t} + \beta_2 d_{22} e^{\xi_2 t} + \beta_3 v_{23} e^{\xi_3 t}, \\ L_t - L_* = \beta_1 v_{31} e^{\xi_1 t} + \beta_2 v_{32} e^{\xi_2 t} + \beta_3 v_{33} e^{\xi_3 t}, \end{cases}$$

where $\beta_1, \beta_2, \beta_3$ are arbitrary constants to be determined using the initial conditions and the transversality conditions, and where $[v_{11}\ v_{21}\ v_{31}]^T$, $[v_{12}\ v_{22}\ v_{32}]^T$, $[v_{13}\ v_{23}\ v_{33}]^T$ are the eigenvectors associated with each of the three eigenvalues ξ_1, ξ_2, ξ_3. Solving the system

$$\begin{bmatrix} \rho - \xi_i & -1 & bk_* \\ \sigma(c_*)c_* f''(k_*) & -\xi_i & 0 \\ 0 & 0 & -a - \xi_i \end{bmatrix} \begin{bmatrix} u \\ v \\ z \end{bmatrix} = 0,$$

we get the eigenvectors associated with the eigenvalues ξ_i $(i = 1, 2, 3)$. Therefore, substituting these values yields

$$\begin{cases} k_t - k_* = \beta_1 abk_* e^{\xi_1 t} + \beta_2 \xi_3 e^{\xi_2 t} + \beta_3 \xi_4 e^{\xi_3 t}, \\ c_t - c_* = -\beta_1 Abk_* e^{\xi_1 t} + \beta_2 A e^{\xi_2 t} + \beta_3 A e^{\xi_3 t}, \\ L_t - L_* = -\beta_1 [A + (\rho + a)a] e^{\xi_1 t}, \end{cases}$$

with $A = \sigma(c_*)c_* f''(k_*)$. Since the initial capital stock k_0 is given, it is essential that the initial value of c_0 is chosen as the ordinate corresponding to k_0 on the stable arm. Any other choice would give rise to a path that eventually violates the conditions for an optimum. From the mathematical point of view, choosing c_0 in correspondence to k_0 on the stable arm means choosing the initial conditions in such a way that the arbitrary constant appearing in the term containing the unstable root turns out to be zero. Without loss of generality, assume $\xi_2 < 0$ and $\xi_3 > 0$. Since $e^{\xi_3 t}$ diverges to infinity, it is clear that the solutions will be stable if $\beta_3 = 0$. Thus, the solutions along the stable arm of the saddle-path are given by

$$\begin{cases} k_t - k_* = \beta_1 abk_* e^{-at} + \beta_2 \xi_3 e^{\xi_2 t}, \\ c_t - c_* = -\beta_1 Abk_* e^{-at} + \beta_2 A e^{\xi_2 t}, \\ L_t - L_* = -\beta_1 [A + (\rho + a)a] e^{-at}. \end{cases} \quad (18)$$

5 Speed of convergence

A particularly interesting aspect of the results of the previous section pertains to the eigenvalues. These are crucial in determining the economy's speed of convergence, namely how long it takes for the economy to adjust to the steady state. Many models of growth, including Ramsey model, have the property that the transitional dynamics are determined by a one dimensional stable manifold. As a consequence, all the variables converge to their respective steady states at the same constant speed, which is equal to the magnitude of the unique stable eigenvalue. By contrast, if the stable manifold is two dimensional, as for our model, then the speed of convergence of any variable at any point of time is a weighted average of the two stable eigenvalues. Clearly, over time, the weight of the smaller (more negative) eigenvalue declines, so that the larger of the two stable eigenvalues describes the asymptotic speed of convergence. It is clear that the flexibility provided by the additional eigenvalue allows the system to match some features of the data related with the timing of the key variables and growth rates along the transitional path. Let us now translate the above in mathematical terms. The speed of convergence of a variable z at time t is defined as

$$\xi_t = \frac{\dot{z}_t}{z_t - z_*}. \quad (19)$$

If the stable manifold is one dimensional, this measure equals the magnitude of the unique stable eigenvalue, while if it is two dimensional, then time-varying convergence speeds are generated. In order to have a measure that summarizes the speed of convergence of the overall economy, we define the percentage change in the Euclidean distance

$$\beta_t = \sqrt{(k_t - k_*)^2 + (c_t - c_*)^2 + (L_t - L_*)^2}.$$

This serves as a natural summary measure of the speed of convergence. Log differentiating this formula yields

$$\frac{\dot{\beta}_t}{\beta_t} = \sum_{z \in \{k,c,L\}} \left[\frac{(z_t - z_*)^2}{\beta_t^2} \right] \frac{\dot{z}_t}{z_t - z_*}, \quad (20)$$

which is seen to be a direct generalization of the one dimensional measure (19). In that case, all variables converge at the same rate, and (20) reduces to the single eigenvalue. In our case, (20) indicates that at any instant of time the generalized speed of convergence is a weighted average of the speeds of convergence of the three variables, the weights being the relative square of their distance from the steady state. It is straightforward from (18) to establish that $\lim_{t \to +\infty} \dot{\beta}_t / \beta_t = \xi_i$, with $i = 1$ or $i = 2$, so that asymptotically the system will converge at the rate of the slower growing stable eigenvalue.

6 Analytical solutions

In this section, we will provide a closed form solution for our modified Ramsey model in the special case of a Cobb-Douglas production function (1) and of a CIES utility function (2). First of all, under these assumptions, the system of equations (15), which describes the dynamics of our model, becomes

$$\dot{k}_t = k_t^\alpha - [\delta + n(L_t)] k_t - c_t, \quad (21)$$

$$\dot{c}_t = \sigma c_t \left[f'(k_t) - \delta - \rho \right], \quad (22)$$

$$\dot{L}_t = L_t n(L_t). \quad (23)$$

while the transversality conditions give

$$\lim_{t\to+\infty} e^{-\rho t} c_t^{-1/\sigma} L_t k_t = 0, \quad \lim_{t\to+\infty} \frac{e^{-\rho t} c_t^{-1/\sigma} \dot{k}_t}{n(L_t)} = 0.$$

We are now going to show that such a system can be solved analytically.

Proposition 7. *For all t,*

$$L_t = \frac{ae^{at}}{a - b + be^{at}}. \tag{24}$$

Proof: Equation (23) is separable. Its solution can be derived by first rewriting (23) as

$$\frac{1}{L_t(a - bL_t)}\, dL_t = dt.$$

Subsequently, the left hand side of this equation can be separated in a term with denominator L_t and a term with denominator $a - bL_t$,

$$\left(\frac{1}{aL_t} + \frac{b}{a(a - bL_t)}\right) dL_t = dt.$$

The left and right hand side of this last equation can be integrated between 0 and t to yield

$$\frac{1}{a} \ln\left[\frac{(a - b)L_t}{a - bL_t}\right] = t. \tag{25}$$

Thus, the explicit solution to the logistic growth equation can be obtained exponentiating (25). Notice that the logistic equation is also a Bernoulli equation and so it can also be solved using this technique. □

Remark 8. *The population growth formula* (24) *describes an S-shaped curve, which lies between the two equilibrium solutions, $L_t = 0$, $L_t = a/b$. Moreover, from being $\dot{L}_t > 0$, we get that L_t increases monotonically from $L_0 = 1$ to $L_\infty = \lim_{t\to\infty} L_t = a/b = L_*$.*

Remark 9. *Using* (21)*, we get*

$$\frac{e^{-\rho t} c_t^{-1/\sigma} \dot{k}_t}{n(L_t)} = \frac{e^{-\rho t} c_t^{-1/\sigma} \{k_t^\alpha - \delta k_t - c_t\}}{L_t(a - bL_t)} - e^{-\rho t} c_t^{-1/\sigma} k_t.$$

Since $L_\infty < \infty$, we deduce that our transversality conditions are equivalent to

$$\lim_{t\to+\infty} e^{-\rho t} c_t^{-1/\sigma} k_t = 0, \tag{26}$$

$$\lim_{t\to+\infty} \frac{e^{-\rho t} c_t^{-1/\sigma} \{k_t^\alpha - \delta k_t - c_t\}}{a - bL_t} = 0. \tag{27}$$

Let us introduce the auxiliary variables

$$u_t = k_t^{1-\alpha}, \quad v_t = \frac{c_t}{k_t}.$$

Using these transformations, the system of equations $(21) - (23)$ can be expressed in terms of u_t,v_t as

$$\dot{u}_t = -(1 - \alpha)\left[\delta + n(L_t) + v_t\right] u_t + 1 - \alpha,$$

$$\dot{v}_t = \frac{(\sigma\alpha - 1)v_t}{u_t} + \left[(1 - \sigma)\,\delta - \rho\sigma + n(L_t)\right] v_t + v_t^2, \tag{28}$$

$$\dot{L}_t = L_t n(L_t).$$

with conditions (26), (27) which rewrite as

$$\lim_{t\to+\infty} e^{-\rho t} v_t^{-1/\sigma} u_t^{\frac{1-1/\sigma}{1-\alpha}} = 0,$$

$$\lim_{t\to+\infty} \frac{e^{-\rho t} v_t^{-1/\sigma} u_t^{\frac{1-1/\sigma}{1-\alpha}} \left(u_t^{-1} - \delta - v\right)}{a - bL_t} = 0.$$

Following Smith [24], we assume that $\alpha = 1/\sigma$, i.e. capital's share is equal to the reciprocal of the intertemporal elasticity of substitution. In this case, the term v_t/u_t disappears from equation (28). When the restriction $\alpha = 1/\sigma$ is imposed, the dynamical system, which describes the economy of our modified Ramsey model, reduces to the following set of differential equations

$$\dot{u}_t = -\,(1 - \alpha)\left[\delta + n(L_t) + v_t\right] u_t + 1 - \alpha, \tag{29}$$

$$\dot{v}_t = \left[\left(1 - \frac{1}{\alpha}\right)\delta - \frac{\rho}{\alpha} + n(L_t)\right] v_t + v_t^2, \tag{30}$$

$$\dot{L}_t = L_t n(L_t).$$

plus the transversality conditions

$$\lim_{t\to+\infty} e^{-\rho t} v_t^{-\alpha} u_t = 0, \tag{31}$$

$$\lim_{t\to+\infty} \frac{e^{-\rho t} v_t^{-\alpha} u_t \left(u_t^{-1} - \delta - v_t\right)}{a - bL_t} = 0. \tag{32}$$

This system can be solved recursively.

Proposition 10. *For all t, the time path of the consumption-capital ratio is given by*

$$v_t = -\frac{\dot{g}_t}{g_t}, \tag{33}$$

where we have set

$$g_t = 1 - \frac{c_0}{k_0} \int_0^t e^{[(1-\frac{1}{\alpha})\delta - \frac{\rho}{\alpha}]t} L_t dt. \tag{34}$$

Proof: Equation (30) is a Bernoulli's differential equation. In order to solve it, we take the substitution $z_t = v_t^{-1}$, and convert this into a linear differential equation in z_t, i.e.

$$\dot{z}_t = -\varphi_t z_t - 1,$$

where by definition

$$\varphi_t = \left(1 - \frac{1}{\alpha}\right)\delta - \frac{\rho}{\alpha} + n(L_t).$$

This linear equation is known to be solved by

$$z_t = e^{-\int_0^t \varphi_t dt}\left(z_0 - \int_0^t e^{\int_0^t \varphi_t dt} dt\right). \quad (35)$$

A direct calculation shows

$$\begin{aligned}\int_0^t \varphi_t dt &= \int_0^t \left[\left(1 - \frac{1}{\alpha}\right)\delta - \frac{\rho}{\alpha} + \frac{\dot{L}_t}{L_t}\right] dt \\ &= \left[\left(1 - \frac{1}{\alpha}\right)\delta - \frac{\rho}{\alpha}\right] t + \ln L_t. \quad (36)\end{aligned}$$

Substituting (36) back in (35), the statement will follow by expressing z_t in terms of v_t, and by observing that differentiating (34) with respect to time yields

$$\dot{g}_t = -\frac{c_0}{k_0}\, e^{[(1-\frac{1}{\alpha})\delta - \frac{\rho}{\alpha}]t} L_t. \quad (37)$$

□

Remark 11. *Accordingly to* (37), $\dot{g}_t < 0$. *Thus,* g_t *is monotone decreasing starting from* $g_0 = 1$. *In particular, this implies* $g_t \le 1$.

Proposition 12. *For all t, the time path of the capital-output ratio is*

$$u_t = \left(e^{-\delta t} L_t^{-1} g_t\right)^{1-\alpha} \cdot \cdot \left[k_0^{1-\alpha} + (1-\alpha)\int_0^t \left(e^{-\delta t} L_t^{-1} g_t\right)^{-(1-\alpha)} dt\right].$$

Proof: Equation (29) is a linear differential equation, whose solution is provided by

$$u_t = e^{-(1-\alpha)\int_0^t \psi_t dt} \cdot \cdot \left[u_0 + (1-\alpha)\int_0^t e^{(1-\alpha)\int_0^t \psi_t dt} dt\right],$$

where by definition

$$\psi_t = \delta + n(L_t) + v_t.$$

Since the integral

$$\int_0^t \psi_t dt = \delta t + \ln L_t + \int_0^t v_t dt,$$

and

$$\int_0^t v_t dt = -\int_0^t d(\ln g_t) = -\ln g_t,$$

the statement can now be obtained as a straightforward calculation. □

Corollary 13. *For all t, the time path of capital per effective worker and consumption per effective worker are given by*

$$k_t = -\, e^{-\delta t}\, L_t^{-1} \dot{g}_t \cdot \left[k_0^{1-\alpha} + (1-\alpha)\int_0^t \left(e^{-\delta t} L_t^{-1} g_t\right)^{-(1-\alpha)} dt\right]^{\frac{1}{1-\alpha}},$$

$$c_t = e^{-\delta t}\, L_t^{-1} g_t \cdot \left[k_0^{1-\alpha} + (1-\alpha)\int_0^t \left(e^{-\delta t} L_t^{-1} g_t\right)^{-(1-\alpha)} dt\right]^{\frac{1}{1-\alpha}}.$$

Proof: The statement follows from Propositions 10 and 12, recalling that $k_t = u_t^{\frac{1}{1-\alpha}}$, and $c_t = u_t^{\frac{1}{1-\alpha}} v_t$. □

Lemma 14. *The transversality condition* (31) *will be satisfied if and only if* $\lim_{t\to+\infty} g_t = 0$, *i.e. if*

$$\frac{k_0}{c_0} = \int_0^\infty e^{[-\frac{\rho}{\alpha} - (\frac{1}{\alpha} - 1)\delta]t} L_t^{1-\frac{1}{\alpha}} dt.$$

Proof: Propositions 10 and 12, equation (37), and $L_\infty < \infty$, yield that (31) is equivalent to

$$\lim_{t\to+\infty} g_t h_t = 0, \quad (38)$$

where

$$h_t = k_0^{1-\alpha} + (1-\alpha)\int_0^t \left(e^{-\delta t} L_t^{-1} g_t\right)^{-(1-\alpha)} dt.$$

Remarks 8 and 11 imply that $\int_0^t \left(e^{-\delta t} L_t^{-1} g_t\right)^{-(1-\alpha)} dt$ behaves as $\int_0^t e^{(1-\alpha)\delta t} dt = \left[e^{(1-\alpha)\delta t} - 1\right] / (1-\alpha)\delta$ in the long-run. Thus, it diverges to infinity. Consequently, the statement of our Lemma is immediate in one direction. For the viceversa, rewrite the left hand side of (38) as

$$\lim_{t\to+\infty} \frac{g_t}{h_t^{-1}},$$

and apply Hopital's rule. □

Lemma 15. *The transversality condition* (32) *is always satisfied.*

Proof: The statement is a consequence of the following observations. Remark 11, Lemma 14, and equation (37), yield that $\lim_{t\to+\infty} v_t$ gives rise to an indeterminate form. Then, we can use Hopital's rule, i.e. calculate the limit via finding the derivatives of the numerator and denominator functions, to arrive at

$$v_\infty = \lim_{t\to+\infty} v_t = \left(-1 + \frac{1}{\alpha}\right)\delta + \frac{\rho}{\alpha} > 0.$$

From Proposition 12, and the proof of Lemma 14, we derive that $\lim_{t\to+\infty} u_t = \lim_{t\to+\infty} h_t / \left(e^{\delta t} L_t g_t^{-1}\right)^{1-\alpha}$ is an indeterminate form. Set $u_\infty = \lim_{t\to+\infty} u_t$. Again an application of Hopital's rule yields

$$u_\infty = \lim_{t\to+\infty} \frac{1}{\delta + a - bL_t + v_t} = \frac{1}{\delta + v_\infty} > 0.$$

The statement now follows from being

$$\begin{aligned} &\lim_{t\to+\infty} \frac{e^{-\rho t} v_t^{-\alpha} u_t \left(u_t^{-1} - \delta - v_t\right)}{a - bL_t} \\ &= v_\infty^{-\alpha} u_\infty \left(u_\infty^{-1} - \delta - v_\infty\right) \cdot \lim_{t\to+\infty} \frac{e^{-\rho t}}{a - bL_t} \\ &= 0 \cdot \lim_{t\to+\infty} \frac{e^{-\rho t}}{a - bL_t} = 0. \end{aligned}$$

Note that, if $a \neq \rho$, Hopital's rule proves that

$$\lim_{t\to+\infty} \frac{e^{-\rho t}}{a - bL_t} = \frac{\rho}{a} \lim_{t\to+\infty} \frac{e^{-\rho t}}{a - bL_t}.$$

Therefore

$$\lim_{t\to+\infty} \frac{e^{-\rho t}}{a - bL_t} = 0.$$

If $a = \rho$, then, replacing (24), we get

$$\begin{aligned} \lim_{t\to+\infty} \frac{e^{-at}}{a - bL_t} &= \lim_{t\to+\infty} \frac{(a-b)e^{-at} + b}{a(a-b)} \\ &= \frac{b}{a(a-b)}. \end{aligned}$$

□

7 Conclusion

In this paper we have considered a modified version of the standard Ramsey growth model, obtained by assuming a Benthamite formulation for the utility function and a logistic-type population growth law. As it is well known, the main problem behind the assumption of constant population growth is that as time goes to infinity population size goes to infinity as well, which is clearly unrealistic. Using the logistic, as opposed to the exponential, population growth hypothesis has the advantage that population size tends to a finite saturation level in the very long-run. Under these hypothesis, we have shown that the model is described by a three dimensional dynamical system, whose unique non-trivial steady state equilibrium is saddle point stable. The stable saddle-path has been proved to be two dimensional, thus enriching the transitional adjustment paths relative to that of the standard Ramsey growth model. As well, we have seen that the population growth rate plays no role in determining the long-run equilibrium levels of per capita consumption, capital and output, and these values are greater than the steady states values of the classical Ramsey model. Finally, we have derived an explicit solution to the version of this modified Ramsey model with constant elasticity preferences and Cobb-Douglas technology. The solution can only be obtained when capital's share of GDP is equal to the reciprocal of the intertemporal elasticity of substitution. Although this is admittedly a special case, it provides a concrete example that makes the dynamic workings of the model transparent. Moreover, it is not intended to supplant graphical and numerical analysis, but to serve as a complementary tool that helps reveal the economic properties of the model. For future research it would be interesting to include in our discussion tables and real experiments.

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