About Solutions of Renewal Equations and Determinations of Failure Moments of a System as Equilibrium Points

ILIE MITRAN
The Mathematical Department
University of Petroşani
Str. Universităţii, Nr.20, Petroşani
ROMÂNIA
iliemitran@gmail.com
http://www.upet.ro/facultati/stiinte/catedre/mate/CVISTORIC/mitran.html

Abstract: - The results presented in this paper lead to solving main problems of the reliability theory: specifying the failure moments of a failure and determining the solutions of renewal equations. We analyze the following situations: 1. the system structure is not taken into consideration; 2. the system structure is known. In the first case we presume that the adopted efficiency function is the average operation time and, by using specific methods of the theory of games, we can prove that there is no equilibrium type solution, so the failure moment of the system cannot be precisely determined. By solving some specific problems, type maximum or minimax, we can only get the interval where the failure point of the system is found. The optimal problem type maximum is solved by specific methods from the theory of games while the optimal problem type minimax is solved by using the maximum principle of Pontriaghin. In the second case we start from the graph structure associated to a system with renewal operations and we build immediately the equation system with finite differences and the system of differential equations associated to this graph. Applying the Laplace transformation it is determined the system availabilities and unavailabilities caused by its subsystems. The failure moments of the system are determined as equilibrium points but the difficulties in calculations lead to obtaining only an approximate solution. Knowing the failure moments of the analyzed system lead to the reconsideration of the renewal policies of the system. Practically, there are determined the approximate solutions of the renewal equations and their separation curve. Having these elements we can completely analyze the renewal process; this analysis being based both on the failure moments of the system and on the renewal costs of the analyzed system.

Key-Words: - reliability function, Markov process, Laplace transformation of availability, replacement times

1 Determining the failure moments of a system without taking into consideration its structure

We start from the problem of determining the failure moment of a system S, when we know (statistically) the average value m and the dispersion D of the operation time. The efficiency function f is considered the average operation time, the variables being represented by the failure moment x and the repartition of the failure probability p.

For this efficiency function we will consider the corresponding problems type maximum and minimax, which are used to determine the optimal guaranteed strategies.

The problem is to determine the probability distributions for which the average value m and dispersion D are known as solutions to the following problems (Ghermeier 1971, Mitran 1992):

$$\textbf{(P}_1\textbf{)}\ \max_{x}\min_{p(t)} f\left(x,p(t)\right)\ \textbf{(P}_2\textbf{)}\ \min_{p(t)}\min_{x} f\left(x,p(t)\right) \quad (1)$$

where:

$$f:(0,\infty)\times M\rightarrow R,\ f\left(x,p(t)\right)=\int_0^x p(t)\,dt+mp(x) \quad (2)$$

(M represents the set of all the probability repartitions with the same average value m and the same dispersion D). It is obvious that for the considered efficiency function f, the following inequalities are achieved (Ghermeier 1971, Mitran 1992):

$$\max_{x}\min_{p(t)} f\left(x,p(t)\right)\leq\min_{p(t)}\max_{x} f\left(x,p(t)\right) \quad (3)$$

but only on the solutions of problems (P_1) and (P_2) and. implicitly, on the values:

$$\overline{v}_1 = \max_x \min_{p(t)} f\left(x, p(t)\right)$$
$$\overline{v}_2 = \min_{p(t)} \max_x f\left(x, p(t)\right) \tag{4}$$

The explanation of these choices is connected to the significance of the variables of efficiency function f : x represents the replacement moment of a machine and p represents the probability function of flawless operation.

If m and D are exactly known and $m^2 > D$ (the most frequent case in practice), f has no saddle points and consequently problems (P1) and (P2) must be separately solved. The solution's of these problems represent the guaranteed optimal solutions searched for.

We consider problem (P1): $\max\limits_x \min\limits_{p(t)} f\left(x, p(t)\right)$.

The maxmin optimal strategy is the pair $\left(\overline{x}_0, \overline{p}_0(t)\right)$, where (Ghermeier 1971, Mitran 1992):

a) $\overline{x}_0$ is the solution of the equation $(m-x)^4 - 2mx(m-x) + D^2 = 0$.

On note $f(x) = (m-x)^4 - 2mx(m-x) + D^2$. For the equation, $f(x) = 0$, we have $x_1 = m - \sqrt{\frac{D}{2}}$; $x_2 = m$, $x_3 = m + \sqrt{\frac{D}{2}}$; immediately results that $f(x_1) = \frac{D}{4}\left(-4m^2 + 7D\right)$, $f(x_2) = D\left(-m^2 + D\right)$, $f(x_3) = \frac{D}{4}\left(-4m^2 + 7D\right)$.

The following possibilities exist:

a1) $m^2 < D$. In this case the above equation has no solution because $f(x_1) > 0$, $f(x_2) > 0$, $f(x_3) > 0$.

a2) $m^2 > D$. The equation $f(x) = 0$ has two roots:

a2.1) If $D < m^2 < \frac{7}{4}D$, the equation has the solution $x_1^* \in \left(m - \sqrt{\frac{D}{2}}, m\right)$, $x_2^* \in \left(m, m + \sqrt{\frac{D}{2}}\right)$, because $f(x_1) > 0$, $f(x_2) > 0$, $f(x_3) > 0$.

a2.2) If $m^2 > \frac{7}{4}D$, the equation has the solution $x_1^* \in \left(0, m - \sqrt{\frac{D}{2}}\right)$, $x_2^* \in \left(m + \sqrt{\frac{D}{2}}, m + D\right)$.

The effectively determining of the solution x_1^*, x_2^* in the cases a2.1) and a2.2) on obtain by Banach principle.

From the $f(x) = 0$ immediately results $x = m \pm \sqrt{Dx(2m-x) - D^2}$

On note $f_1(x) = m - \sqrt{Dx(2m-x) - D^2}$, $f_2(x) = m + \sqrt{Dx(2m-x) - D^2}$.

Immediately results f_1, f_2 have the properties to be the contractions.

Therefore, if $x_n^1 = f_1\left(x_{n-1}^1\right)$, $x_n^2 = f_2\left(x_{n-1}^2\right)$, $x_0^1 > x_0^2 > 0$ every, we obtain $x_1^* = \lim\limits_n x_n^1$, $x_2^* = \lim\limits_n x_n^2$; x_1^*, x_2^* are the solutions of the equation $\left(x^*\right)^2 + Dx^* + D^2 - Dm^2 = 0$

On obtain $x_1^* = m - \sqrt{\frac{-D - \sqrt{D\left(4m^2 - 3D\right)}}{2}}$, $x_2^* = m + \sqrt{\frac{-D - \sqrt{D\left(4m^2 - 3D\right)}}{2}}$.

The optimal solution reached for $\overline{x}_0$ is the smallest of them (therefore $\overline{x}_0 = x_1^*$) and so: $\overline{x}_0 \in \left(m - \sqrt{\frac{D}{2}}, m\right)$ if $D < m^2 < \frac{7}{4}D$, $\overline{x}_0 \in \left(0, m - \sqrt{\frac{D}{2}}\right)$ if $m^2 > \frac{7}{4}D$.

Remark 1:

a)Always $D\left(4m^2 - 3D\right) > 0$, $-D + \sqrt{D\left(4m^2 - 3D\right)} > 0$ and, because, the solutions x_1^*, x_2^* exist.

b) If D is very small, it can be shown that $\overline{x}_0 \approx m - \sqrt[4]{m^2 D}$.

For Ghemieier, the solution (on note x_G^*) is $\overline{x}_0 \approx m - \sqrt[3]{2mD}$, if D is very small. The following results take place: if $m^2 > 16D$ result $\overline{x}_0 < \overline{x}_G^* < m - \sqrt{\frac{D}{2}}$; if $\frac{7}{4}D < m^2 < 16D$ result $x_G^* < \overline{x}_0 < m - \sqrt{\frac{D}{2}}$; if $D < m^2 < \frac{7}{4}D$ result $m - \sqrt{\frac{D}{2}} < x_G^* < \overline{x}_0 < m$.

c) $\overline{p}_0$ is of exponential type, more precisely:

$$\overline{p}_0(t)=\begin{cases} e^{-\frac{t}{m}}, & t\in[0,\overline{x}_0] \\ 0, & t>\overline{x}_0 \end{cases} \tag{5}$$

The maxmin optimal value is:

$$f(\overline{x}_0,\overline{p}_0(t))= \\ =2m\left\{1-\left(\frac{D}{4m^2}\right)^{\frac{1}{3}}-\frac{\left(\frac{D}{4m^2}\right)^{\frac{1}{3}}}{2\left[1+\left(\frac{D}{4m^2}\right)^{\frac{1}{3}}\right]}\right\} \tag{6}$$

If $\overline{x}_0$ is very close to 0, then:

$$f(\overline{x}_0,\overline{p}_0(t))\approx m-\frac{mD}{m^2+D} \tag{7}$$

We consider problem (P2): $\min\limits_{p(t)}\max\limits_{x} f(x,p(t))$.

The minmax optimal strategy is the pair $\left(\overline{\overline{x}}_0,\overline{\overline{p}}_0(t)\right)$ where:

a) $\overline{\overline{x}}_0\in[\overline{x}_1,\overline{x}_2]$, $\overline{x}_1=me^{-z}$, $\overline{x}_2=me^{-z}+mz$, z being a root of the equation:

$$e^{-2z}+2ze^{-z}+\frac{D}{m^2}-1=0 \tag{8}$$

It can be noticed that, if we note $g(z)=e^{-2z}+2ze^{-z}+\frac{D}{m^2}-1$, the equation $g'(z)=0$ has two roots z_1 and z_2, a negative and a positive one, placed in a neighborhood of the origin and with $z_2\in(0,1)$. Therefore, developing $g(z)$ into a Taylor series and excluding the terms of superior order, we obtain $g(z)\approx -z^3+\frac{D}{m^2}$.

Then, the equation has the approximate solution $z\approx\left(\frac{D}{m^2}\right)^{\frac{1}{3}}$ and we obtain that $\overline{x}_1\approx\left(\frac{D}{m^2}\right)^{\frac{1}{3}}$, $\overline{x}_2\approx m\left[\left(\frac{D}{m^2}\right)^{\frac{1}{3}}+e^{-\left(\frac{D}{m^2}\right)^{\frac{1}{3}}}\right]$.

b) $$\overline{\overline{p}}_0(t)=\begin{cases} 1, & t<\overline{x}_1 \\ e^{-\frac{t-\overline{x}_1}{m}}, & \overline{x}_1\le t\le\overline{x}_2 \\ 0, & t>\overline{x}_2 \end{cases}$$

The minmax value of the efficiency function is $f\left(\overline{\overline{x}}_0,\overline{\overline{p}}_0(t)\right)=m\left(1+e^{-z}\right)\approx m\left(1+e^{\left(\frac{D}{m^2}\right)^{\frac{1}{3}}}\right)$.

Because $\frac{D}{m^2}<\frac{1}{2}$, immediately results that $\overline{x}_1<\frac{m}{2}$, $\overline{x}_2<1{,}25m$.

$$\overline{x}_2-\overline{x}_0\approx m\left[\left(\frac{D}{m^2}\right)^{\frac{1}{3}}+e^{-\left(\frac{D}{m^2}\right)^{\frac{1}{3}}}-1\right]+(2mD)^{\frac{1}{3}} \tag{9}$$

For practical reasons, it is recommended to consider that solution $\overline{\overline{x}}_0$ can be approximated with (or even equal to) x_1. In this case, the breakdown moment, x^* verifies the inequalities:

$$m-(2mD)^{\frac{1}{3}}\approx\overline{x}_0<x^*\le\overline{x}_1\approx m\left(\frac{D}{m^2}\right)^{\frac{1}{3}} \tag{10}$$

and the bandwidth to which the real breakdown curve belongs varies as in the previous case from the interval $[0,\overline{x}_1]$ in the upper part and $[\overline{x}_0,\overline{x}_1]$ in the lower part (see figure l).

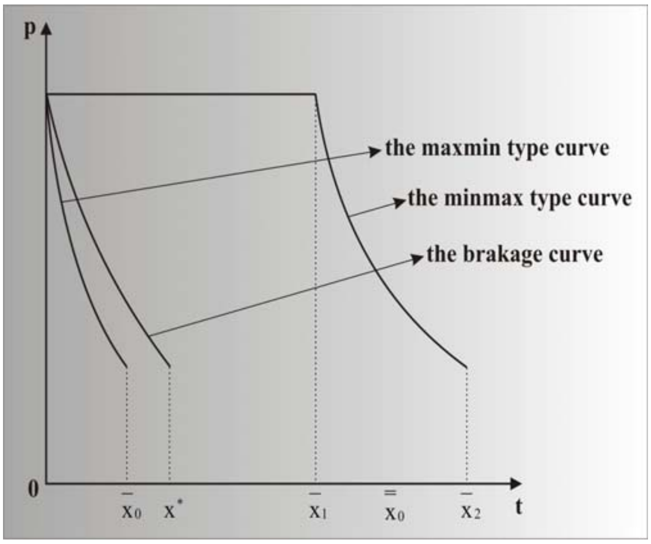

Fig. 1 The approximation curves

2 The Problem of Replacement Timing and Optimal Renewal Strategies

2.1 Considerations about Optimal Renewal Strategies

The renewal strategies goal is to determine the optimal replace time and they can be divided in two categories:

1) Periodical (warning) renewal strategies;
2) Non-periodical renewal strategies.

2.1.1 Periodical Renewal Strategies

This kind of renewals has a determinist nature and they are characterized by a constant time between two consecutive warning renewals.

BRP strategy (Block Replacement Policy)

At this kind of strategy the system is renewed after a time equal kT, where $k = 1, 2, ...$, and T is the time period (T is unknown). In order to determine T we use the following criteria:

a) The reaching of a certain level of reliability: $R(T) \geq R_0$, R_0 being value;

b) The minimization of the mean cost of the system in the period T:

$$K_{BRP} = \frac{H(T) + b}{T} \to \min \tag{11}$$

where $H(T)$ is the mean number of failures in the time period $(0, T)$ or the **renewal function** and b is the **cost of warning renewal**. The effective determination of the cost K_{BRP} presumes the calculation of H function. This can be done in the following way:

1) We start from the reliability function of the system R and we determine the probability density;

2) We determine the Laplace Transform $\mathfrak{L}^*(s)$ of the probability density:

$$f(t) = -\frac{dR(t)}{dt} \tag{12}$$

3) Next, we compute the Laplace transform of the renewal density:

$$h^*(s) = \frac{f^*(s)}{1 - f^*(s)} \tag{13}$$

4) We apply the inverse transform and we determine the renewal density $h(t)$;

5) We determine the renewal function corresponding to the time period T:

$$H(T) = \int_0^T h(t)\,dt \tag{14}$$

From the minimum condition results immediately the renewals equation whose solving leads to the determination of $T^* = T_{optim}$:

$$T^* h(T^*) - H(T^*) = b \tag{15}$$

Let us note, excepting the difficulty in solving the equation, the asymptotic behavior of the K_{BRP} cost:

$$\lim_{t \to \infty} K_{BRP} = \frac{1}{m} \tag{16}$$

DRP strategy (Delayed Replacement Policy)

To evaluate the mean cost of the system maintenance it is necessary to consider, excepting the b-cost of the warning renewals, the d-**cost of the system stagnation** (or inappropriate functioning) in the time unit. The mean cost in a time period T includes the following:

a) the warning renewal cost b;

b) the cost of the stagnation mean time:

$$d \int_0^{T_D} x h(T - x)\,dx \tag{17}$$

where d is the stagnation cost and T_D the occurrence time of the defect;

c) the proper renewals cost:

$$H(T - T_D) \tag{18}$$

When adopting the DRP strategy, the essential idea is that if the renewals were not executed in kT time, $k = 1, 2, ...$ and the dissertation occurred in the interval $(kT - T_D, kT)$, the system would not be renewed and it would wait for the first warning renewal.

In this case the minimal cost per time is:

$$K_{DRP}\left(T;T_D\right)= \\ =\frac{1}{T}\left[b+H\left(T-T_D\right)+d\int_0^{T_D} xh\left(T-x\right)dx\right] \tag{19}$$

By minimizing K_{DRP} reported to T_D, results immediately the optimal time T_D^*:

$$T_D^*=\frac{1}{d} \tag{20}$$

By replacement, for $T_D^* \ll T$, we obtain the minimal cost:

$$K_{DRP}\left(T,T_D^*\right)=K_{BRP}\left(t\right)-\frac{h\left(T\right)}{2dT} \tag{21}$$

Now, considering the optimal time period of the warning renewals, we can express the minimal medium cost K_{DRP}^*:

$$K_{DRP}^*=K_{DRP}\left(T^*,T_D^*\right)=K_{BRP}\left(T^*\right)-\frac{h\left(T^*\right)}{2dT^*}= \\ =h\left(T^*\right)\left(1-\frac{1}{2dT^*}\right)=h\left(T^*\right)\left(1-\frac{T_D^*}{2T^*}\right) \tag{22}$$

2.1.2 Non-periodical renewals strategies

These kinds of strategies have a random nature and they are elaborated taking into account the system age, its wear and other random variables which describe the system evolution.

ARP strategy (Age Replacement Policy)

It is the simplest non-periodical strategy and it considers a certain age   reached by the system. Due to the non-periodical nature of renewals, the effective achievements of the strategy are more difficult and this can be expressed by a cost, associated to an ARP renewal, greater than the cost corresponding to a warning renewal.

Mainly, the criteria for determining the system age for a renewal are the following:

1) The guarantee of a certain reliability (or availability) level;

2) An extreme condition for an x-depending variable;

3) The minimization of the minimal cost of the system maintenance per time unit.

The most important criterion of projecting the ARP strategy is the minimization of the mean cost of the system maintenance per time unit:

$$K_{ARP}\left(x\right)=\frac{F\left(x\right)-a^*R\left(x\right)}{\int_0^x R\left(t\right)dt}\to \min \tag{23}$$

The numerator of the ARP cost means the cost of the system maintenance for its entire life. We noted a^* the cost of the warning renewal. From the extreme condition $\frac{dK}{dx}=0$, we obtain immediately the equation of the optimal age for executing the system renewal:

$$z\left(x^*\right)\int_0^{x^*} R\left(t\right)dt+R\left(x^*\right)=\frac{1}{1-a^*} \tag{24}$$

We obtain the minimal mean value of the cost by replacing the solution x^* of optimal age equation in the analytical expression of the ARP cost:

$$K_{ARP}\left(x^*\right)=\left(1-a^*\right)z\left(x^*\right) \tag{25}$$

2.2 The Determination of the System and Subsystem Components Function Probabilities

Let us consider a system described by a logical model of series type formed by n elements with rates of damages and of renewals constant and equal with a_i and r_i, $i=\overline{1,n}$. The mean timings of function and renewal are:

$$m_{1i}=\frac{1}{a_i},\ m_{2i}=\frac{1}{r_i},\ i=\overline{1,n} \tag{26}$$

Analyze of the system's reliability presumes two stages:

2.2.1 Calculating the reliability indicators, the reliability function and the well functioning time average

The determination of the reliability function can be done directly by using the logical model, because the renewals of the system's elements don't influence its behavior till the damage:

$$P_0=e^{-a_0 t},\ a_0=\sum_{i=1}^{n} a_i \tag{27}$$

(we noted with P_0 the maintenance function of the system). From this results immediately the time functioning average:

$$m_1 = \frac{1}{a_0} \tag{28}$$

Remark 2. The reliability function can be determined using the Markov process model which has the following graph (Figure 2).

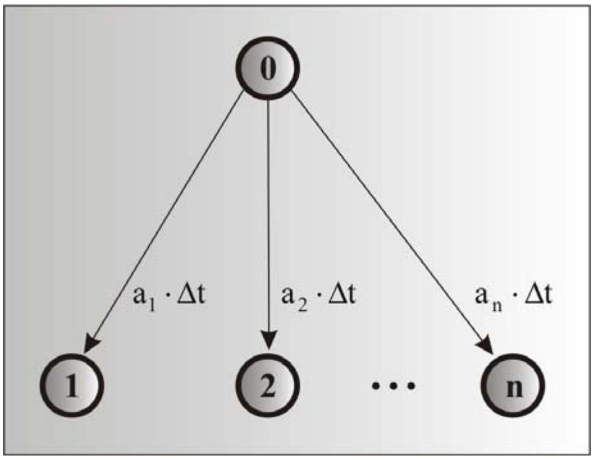

Fig.2 The Markov process model

Here 0 represents the state of good functioning and $1,2,\ldots,n$ represent the states of damage corresponding to the damages of the system's elements.

Because at this stage we are interested only in the system's behavior till the damage, we ignore the possibilities of renewing from the state of damage to the state of good functioning. The equation with finite difference which, characterizes the process is:

$$P_0(t+\Delta t) = P_0(t)\cdot(1-a_0\cdot\Delta t) \tag{29}$$

It results immediately the following differential equation:

$$\frac{dP_0(t)}{dt} = -a_0\cdot P_0(t), \quad P_0(0)=1 \tag{30}$$

Its solution gives us the passing matrix. The system being presumed without renewal, it can be calculated immediately the reliability indicators required by safe functioning analyzes.

Comparing the two methods of calculus it results that the Markov process model does not present any calculation advantages in structural analyze of the system's reliability (without renewal).

2.2.2 The determination of the renewal specific indicators

First of all it is used the global model of renewal process alternated with the Markov process model:- we apply the global model of alternate renewal.

This model presumes that we know the distribution of the system's functioning and renewal duration. The density of the renewal duration at the system's level is generally calculated by a structural analyze which is the real disadvantage in using the global model. The system's renewal duration, by the rehabilitation of the element i, is distributed with the density of probability f_1, given by: $f_1(t) = -P_0'(t) = a_0 e^{-a_0 t}$. The probability of the element's i damaging to be the cause of the system's damage, knowing that it took place, is $\frac{a_i}{a_0}$. It results immediately that:

$$f_2(t) = \sum_{i=1}^{n} \frac{a_i}{a_0} r_i e^{-r_i t} \tag{31}$$

The average of the system's renewal duration is:

$$m_2 = \int_0^\infty t f_2(t)\,dt = \sum_{i=1}^{n} \frac{a_i}{a_0 r_i} = m_1 \sum_{i=1}^{n} \frac{m_{2i}}{m_{1i}} \tag{32}$$

With the help of the density of probability f_2 given by (31), we can calculate immediately the Laplace transformation of the renewal densities:

$$h_1^*(s) = \frac{f_1^*(s)}{1 - f_1^*(s)\cdot f_2^*(s)} = \frac{\dfrac{a_0}{a_0+s}}{1 - \dfrac{a_0}{a_0+s}\displaystyle\sum_{i=1}^{n}\frac{a_i r_i}{a_0(s+r_i)}} = \frac{a_0}{s\left(1+\displaystyle\sum_{i=1}^{n}\frac{a_i}{s+r_i}\right)} \tag{33}$$

$$h_2^*(s) = \frac{f_1^*(s)\cdot f_2^*(s)}{1 - f_1^*(s)\cdot f_2^*(s)} = \frac{a_0}{a_0+s}\sum_{i=1}^{n}\frac{a_i\cdot r_i}{(s+r_i)\cdot a_0} = \frac{\displaystyle\sum_{i=1}^{n} a_i\cdot\frac{r_i}{s+r_i}}{s\left(1+\displaystyle\sum_{i=1}^{n}\frac{a_i}{s+r_i}\right)} \tag{34}$$

Remark 3. (**regarding the asymptotic behavior of the system**). The renewal densities attend to the same asymptotical value equal to the inverse of a functioning renewal cycle average:

$$\lim_{t\to\infty} h_1(t) = \frac{m_2}{m_1 + m_2}, \; \lim_{t\to\infty} h_2(t) = \frac{1}{m_1 + m_2} \quad (35)$$

The Laplace transformation of availability is:

$$\lim_{t\to\infty} h_1(t) = \lim_{s\to\infty} s h_1^*(s) = \frac{a_0}{1 + \sum_{i=1}^{n} \frac{a_i}{r_i}} = \\ = \frac{1}{m_1 + m_1 \sum_{i=1}^{n} \frac{m_{2i}}{m_{1i}}} = \frac{m_1}{m_1 + m_2} \quad (36)$$

$$\lim_{t\to\infty} h_2(t) = \lim_{s\to\infty} s h_2^*(s) = \frac{1}{m_1 + m_2} \quad (37)$$

$$A^*(s) = P_0^*(s) \cdot \left(1 + h_2^*(s)\right) = \frac{1}{s\left(1 + \sum_{i=1}^{n} \frac{a_i}{s} + r_i\right)} \quad (38)$$

The availability coefficient, the asymptotic value in stationary state is obtained by the limit:

$$A = \lim_{t\to\infty} A(t) = \lim_{s\to 0} s A^*(s) = \frac{1}{1 + \sum_{i=1}^{n} \frac{a_0}{r_i}} = \\ = \frac{1}{1 + \sum_{i=1}^{n} \frac{m_{2i}}{m_{1i}}} = \frac{m_1}{m_1 + m_1 \sum_{i=1}^{n} \frac{m_{2i}}{m_{1i}}} = \frac{m_1}{m_1 + m_2} \quad (39)$$

The second stage of reliability (figure 3) analyze it's done by using the Markov processes, taking into account the system's returning possibilities from the states of damage associated with the transition probabilities $r_i \cdot \Delta t$.

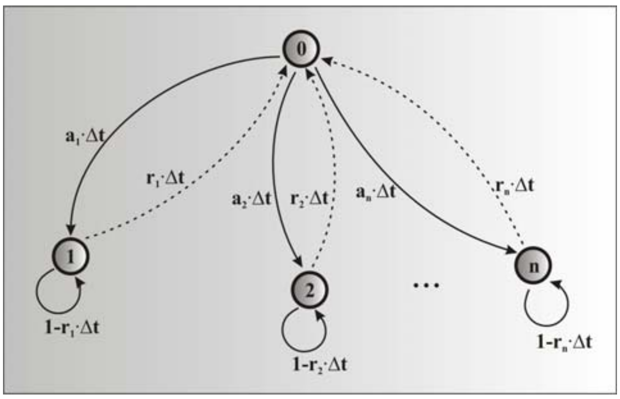

Fig.3 The second stage of reliability

The processes state probabilities are determined with the help of the system of equations with finite differences:

$$P_i(t + \Delta t) = P_0(t) a_1 \Delta t + P_i(t)(1 - r_i \Delta t), \; i = \overline{1,n} \quad (40)$$

Adding the following relation:

$$P_0(t) + \sum_{i=1}^{n} P_i(t) = 1 \quad (41)$$

we get the following system of differential equations:

$$\frac{dP_i(t)}{dt} = -r_i P_i(t) + a_i P_0(t), \; i = \overline{1,n}, \; P_0(0) = 1 \quad (42)$$

Using the Laplace transformation in (41) and (42), we obtain:

$$\sum_{i=1}^{n} P_i^*(s) = \frac{1}{s} - P_0^*(s) = \sum_{i=1}^{n} \frac{a_i}{s + r_i} P_0^*(s) \quad (43)$$

From this relation results immediately $P_0^*(s)$, which is the Laplace transformation of the system's availability:

$$A^*(s) = P_0^*(s) = \frac{1}{s\left(1 + \sum_{i=1}^{n} \frac{a_i}{s + r_i}\right)} \quad (44)$$

From (42) and (43) it results the Laplace transformation of the passing probabilities:

$$P_i^*(s) = \frac{a_i}{s + r_i} P_0^*(s) = \frac{\frac{a_i}{s + r_i}}{s\left(1 + \sum_{i=1}^{n} \frac{a_i}{s + r_i}\right)} \quad (45)$$

The P_i probability represents the system's unavailability because of the element i. Its asymptotical value is:

$$\lim_{t\to\infty} P_i(t) = \lim_{s\to 0} s P_i^*(s) = \frac{\frac{a_i}{r_i}}{1 + \sum_{i=1}^{n} \frac{a_i}{r_i}} = \frac{\frac{m_{2i}}{m_{1i}}}{1 + \sum_{i=1}^{n} \frac{m_{2i}}{m_{1i}}} \quad (46)$$

From practical grounds s_0 is taken m and only the first two terms of the developing are taken. We have:

$$P_i^*(s) = \frac{a_i}{B_i}\left(\frac{1}{s} - \frac{A_i}{A_i s + B_i}\right) \quad (47)$$

where:

$$A_i = 1 + \sum_{k=1}^{n} a_k \frac{r_k - r_i}{(s_0 + r_k)^2},\ i = \overline{1,n} \quad (48)$$

$$B_i = r_i + \sum_{k=1}^{n} a_k \left[\frac{s_0 + r_i}{s_0 + r_k} + \frac{s_0(r_i - r_k)}{(s_0 + r_k)^2}\right],\ i = \overline{1,n} \quad (49)$$

It results that:

$$P_i(t) = \frac{a_i}{B_i}\left(1 - e^{-\frac{B_i}{A_i}t}\right),\ i = \overline{1,n} \quad (50)$$

The determination of maintenance functions for the n components of the analyzed system allows the computation of the renewal unities h_i and the renewal functions H_i, $i = \overline{1,n}$:

$$f_i(t) = -\frac{dP_i(t)}{dt} = \frac{a_i}{A_i} e^{-\frac{B_i}{A_i}t} \quad (51)$$

$$h_i(t) = \frac{a_i}{A_i} e^{-\frac{B_i - a_i}{A_i}t} \quad (52)$$

$$H_i(t) = \int_0^t h_i(x)\,dx = \frac{a_i}{B_i - a_i}\left(e^{-\frac{B_i - a_i}{A_i}t} - 1\right) \quad (53)$$

Using these elements can be analyzed the safe in functionality for the analyzed system as well the component subsystems for every period of time t fixed.

2.3 The Problem of Option Between Different Renewal Strategies

The option between different renewal strategies has to be done by using an unitary criterion and considering the optimal variant of different strategies. Let us consider for the beginning the ARP strategies when no FRP warning renewal is done. The criterion is determined by the minimal cost of the maintenance per unit. As the optimal values of the K_{ARP} and K_{BRP} costs are raising reported to the warning renewal cost, there will be the unique values $a_0, b_0 > 0$ as to have the inequalities: $K_{ARP}^*(a) > K_{FRP}$, $(\forall)a > a_0$; $K_{BRP}^*(b) > K_{FRP}$, $(\forall)b > b_0$

It results that in the situations in which $a > a_0$ and $b > b_0$ the strategy FRP will be adopted. If $a > a_0$ and $b < b_0$, the strategy BRP is adopted and if $a < a_0$, $b > b_0$, the ARP strategy is adopted. When $a < a_0$, $b < b_0$, the ARP and BRP costs have to be directly compared between them. As both of them are increased functions of a, respectively of b, it results that for a fixed a, there is an unique $b^*(a)$ for which the following equality is achieved:

$$K_{BRP}^*\left(b^*(a)\right) = K_{ARP}^*(a) \quad (54)$$

We can easily notice that $b^*(a) < b_0$ and it is monotonously increasing in comparison with a. Knowing the $b^*(a)$ function, we can decide immediately decide over the strategy that have to be adopted.

All the conclusions regarding the choice of the renewal strategies are synthesized in the figure 4.

Using the results from 2.1 and 2.2, the following elements can be determined for each subsystem i, $1 \le i \le n$, of the analyzed system:

- the values a_0^i, b_0^i which border the strategy ARP by the ARP and BRP strategies;
- the separation curve $b_i = b^*(a)$ of the AR.P and BRP strategies;
- the replacement optimal times T_i^* and z_i^* in the case of using the BRP strategy, respectively the ARP strategy.

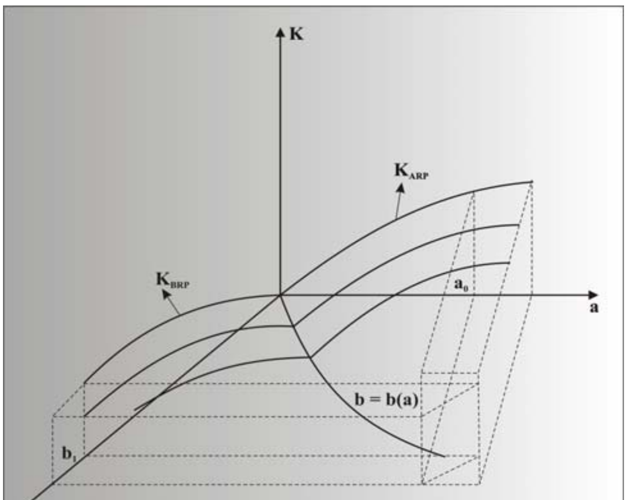

Fig. 4 The choice of the renewal strategies

2.3.1 The determination of the separation curve of the BRP and ARP strategies

From the equality $h_i(b)=(1-a)z_i(a)$, it results that:

$$\frac{H_i(b)+b}{b}>\frac{1}{m_i}\Rightarrow\frac{a_i}{B_i-a_i}\frac{e^{-\frac{B_i-a_i}{A_i}b}}{b}\geq\frac{1}{m_i}-1\Rightarrow$$
$$\Rightarrow e^{-\frac{B_i-a_i}{A_i}b}\geq\frac{B_i-a_i}{a_i}\frac{1-m_i}{m_i}b \quad (55)$$

Using the equality: $e^x=x+1$, for x very small; we immediately obtain the approximately value for the searched b_0^i:

$$b_0^i\approx\frac{1}{\frac{B_i-a_i}{A_i}+\frac{B_i-a_i}{a_i}\left(\frac{1-m_i}{m_i}\right)} \quad (56)$$

The determination of a_0^i

A straightforward calculus leads to:

$$z_i(t)\approx\frac{a_i\left(1-\frac{B_i}{A_i}t\right)}{A_i\left(1-\frac{a_i}{A_i}t\right)} \quad (57)$$

The equation $(1-a)z_i(a)=\frac{1}{m_i}$ leads to the following second order equation:

$$a^2\frac{B_i}{A_i}-a\left(1+\frac{B_i}{A_i}+\frac{a_i}{mA_i}\right)+1-\frac{1}{m_i}=0 \quad (58)$$

For this equation we have:

$$\Delta=\left(1+\frac{B_i}{A_i}+\frac{a_i}{mA_i}\right)^2-4\frac{B_i}{A_i}\left(1-\frac{1}{m_i}\right)=$$
$$=\left(1-\frac{B_i}{A_i}\right)^2+\frac{a_i^2}{m^2A_i^2}+\frac{2a_i}{mA_i}+\frac{2a_iB_i}{mA_i^2}+\frac{4B_i}{mA_i}>0 \quad (59)$$

The solution is:

$$a_0^i=\frac{1+\frac{B_i}{A_i}+\frac{a_i}{mA_i}+\sqrt{\Delta}}{2} \quad (60)$$

2.3.2 The Determination of the Optimal Replacement Times for the BRP Strategy

The optimal replacement time T_i^* is the solution of the equation:

$$T_ih_i(T_i)-H_i(T_i)=b \quad (61)$$

We have:

$$T_i\frac{a_i}{A_i}e^{-\frac{B_i-a_i}{A_i}T_i}-\frac{a_i}{B_i-a_i}e^{-\frac{B_i-a_i}{A_i}T_i}=b \quad (62)$$

Developing in series Mac-Laurin and taking only the first two terms, the equation becomes:

$$\left(1-\frac{B_i-a_i}{A_i}T_i\right)\left(T_i\frac{a_i}{A_i}-\frac{a_i}{B_i-a_i}\right)=b\Rightarrow$$
$$\alpha\beta T_i^2+T_i(\alpha\gamma-1)+b-\gamma=0 \quad (63)$$

where:

$$\alpha=\frac{B_i-a_i}{A_i},\ \beta=\frac{a_i}{A_i},\ \gamma=\frac{a_i}{B_i-a_i} \quad (64)$$

We obtain that:

$$T_i^*=\frac{1-\alpha\gamma+\sqrt{\Delta}}{2\alpha\beta} \quad (65)$$

where:

$$\Delta=(\alpha\gamma-1)^2-4\alpha\beta(\beta-\gamma) \quad (66)$$

Remark 4. It can be proven that α is very small. For $\alpha\approx 0$ results:

$$T_i^*\approx\left(b+\frac{a_i}{B_i-a_i}\right)\frac{A_i}{a_i} \quad (67)$$

2.3.3 The Determination of the Optimal Replacement for the ARP Strategy

The optimal replacement time x_i^* is the solution of the equation:

$$z(x_i)\int_0^{x_i}P_o(t)\,dt+P_0(x_i)=\frac{1}{1-a_i^*} \quad (68)$$

This leads us to the equation:

$$\frac{\frac{a_i}{A_i}\left(1-\frac{B_i}{a_i}x_i\right)}{1-\frac{a_i}{A_i}x_i}\left(-\frac{A_i}{B_i}e^{-\frac{B_i}{A_i}x_i}+1\right)+ \\ +1-\frac{a_i}{B_i}\left(1-e^{-\frac{B_i}{A_i}x_i}\right)=\frac{1}{1-a_i^*} \quad (69)$$

Because $1-e^{-\frac{B_i}{A_i}x_i}\approx\frac{B_i}{A_i}x_i$, we obtain the following equation:

$$x_i^2\frac{a_i}{A_i^2}(a_i-B_i)-\frac{a_i}{A_i}\left(\frac{B_i}{A_i}-\frac{1}{1-a_i^*}\right)x_i+ \\ +\frac{a_i(B_i-A_i)}{A_iB_i}+1-\frac{1}{1-a_i^*}=0 \quad (70)$$

From this second order equation we obtain the searched solution x_i^* (from practical point of view, the solution which we are looking for is the greatest root of the above equation).

2.4 The Determination of the Optimal Replacement Times

The points $t_1,t_2,\dots,t_n$ of the damage of the system corresponding to the damages of the system's elements $S_1,S_2,\dots,S_n$ can be calculated like equilibrium points (the point t_i is the solution of the equation $R(t)=P_i(t),\quad i=\overline{1,n}$).

Because, $R(t)=e^{\frac{t}{m_1}}$, $\frac{a_i}{B_i}=\frac{a_i}{r_i}\cdot\frac{m_1}{m_1+m_2}$, $\frac{B_i}{A_i}=\frac{r_i\cdot(m_1+m_2)}{1+m_1-r_i\cdot m_2}$, t_i is the solution of the equation (figure 5):

$$e^{-\frac{t}{m_1}}=\frac{a_i}{r_i}\cdot\frac{m_1}{m_1+m_2}\left(1-e^{-\frac{r_i\cdot(m_1+m_2)}{1+m_1-r_i\cdot m_2}t}\right),\quad i=\overline{1,n} \quad (71)$$

Developing in Taylor series (in the point $t_0=m_1$) and taking only the first two terms, the solution of the equation is:

$$t_i=\left(\frac{e^{-\frac{t_0}{m_1}}+\frac{t_0}{m_1}\cdot e^{-\frac{t_0}{m_1}}}{\alpha}\right)- \\ -\left(\frac{-\frac{a_i}{B_i}+\frac{a_i}{B_i}\cdot e^{-\frac{B_i}{A_i}t_0}}{\alpha}\right)+\left(\frac{\frac{a_i}{A_i}\cdot e^{-\frac{B_i}{A_i}t_0}}{\alpha}\right) \quad (72)$$

When $\alpha=\frac{e^{-\frac{t_0}{m_1}}}{m_1}+\frac{a_i}{A_i}\cdot e^{-\frac{B_i}{A_i}t_0}$.

For t_0 very small on obtain:

$$t_i=\frac{m_1\cdot(1+m_1-r_i\cdot m_2)}{a_i\cdot m_1^2+1+m_1-r_i\cdot m_2} \quad (73)$$

We assume $\frac{a_1}{r_1}>\frac{a_2}{r_2}>\dots>\frac{a_n}{r_n}$.

Practically, the solution (73) was obtained by leveling the equation (74) obviously representing an approximate solution.

The equation $R(t)=P_i(t)$ can be solved, relatively easy by using the method of successive approximates, leading to a lower error solution.
Starting with the equation:

$$e^{-\frac{t}{m_1}}=\frac{a_i}{B_i}\cdot\left(1-e^{-\frac{B_i}{A_i}\cdot t}\right) \quad (74)$$

We note $x=e^{-\frac{t}{m_1}}$ which leads to solving the equation:

$$x=\frac{1}{C}\cdot\left(1-x^D\right) \quad (75)$$

where $C=\frac{B_i}{a_i}$, $D=\frac{B_i}{A_i}\cdot m_1$.

Applying the method of successive approximates, after a relatively easy calculation, we obtain the solution of the equation (75).

$$x^* = \frac{1}{1 + \frac{A_i}{a_i \cdot m_1}} \tag{76}$$

which leads to the solution of the equation (74):

$$t_i = m_1 \cdot \ln\left(1 + \frac{A_i}{a_i \cdot m_1}\right) \tag{77}$$

Practically, the failure moments of the system are obtained by solving, for each subsystem, an equation type (77).

Practically, the failure moments of the system are obtained by solving, for each subsystem, an equation type (32).

If we take into consideration that $A_i \approx B_i \cdot m_1$, from (32) it results:

$$t_i \approx m_1 \cdot \ln\left[1 + \frac{r_i}{a_i}\left(1 + \frac{m_2}{m_1}\right)\right] \tag{78}$$

It is obvious that relation (78) shows the dependence between i component moment of failure elements m_1, m_2 specific for the reliability of the analyzed system and elements a_i, r_i specific for reliability of i subsystem.

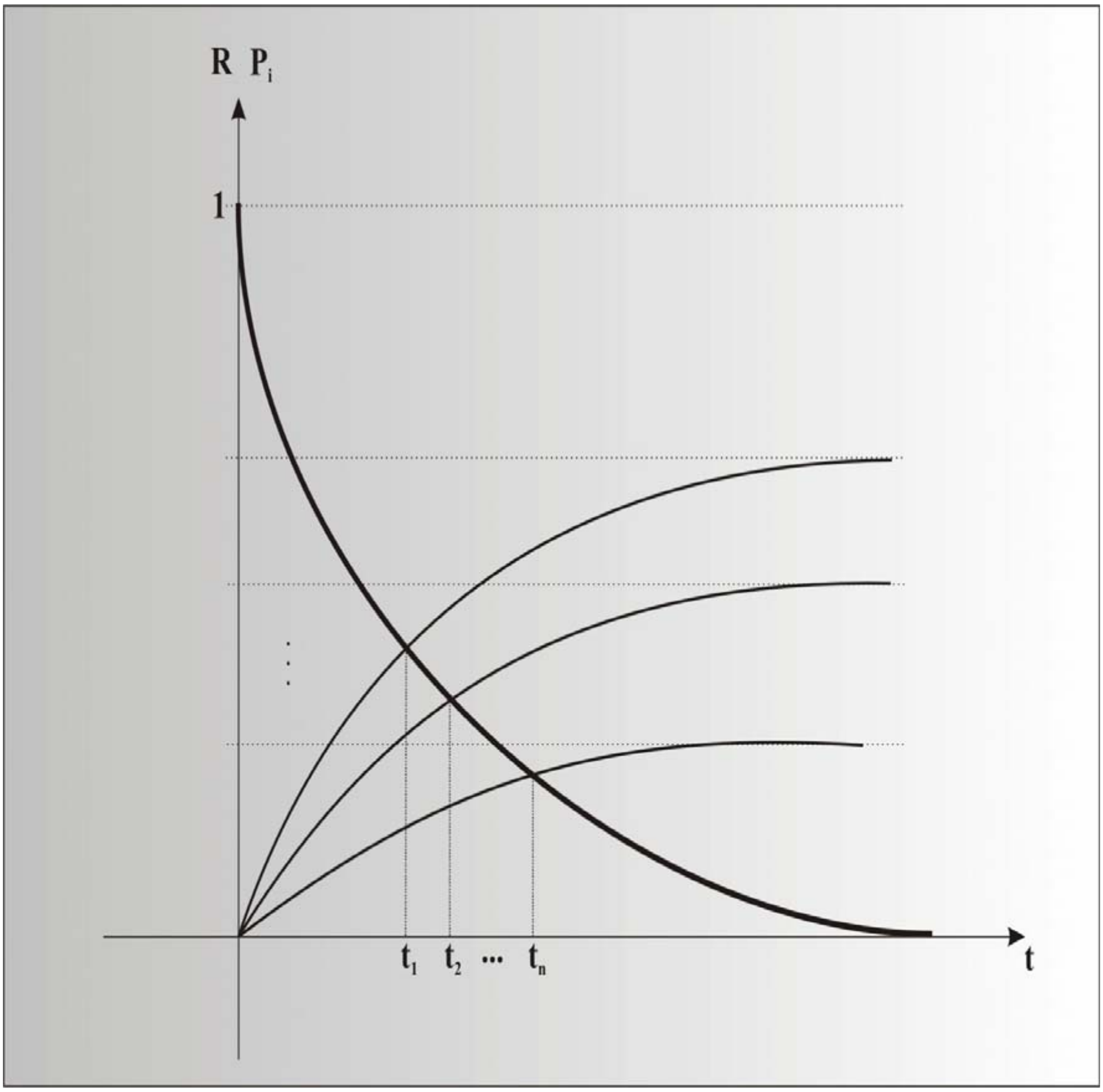

Fig.5 The equilibrium points

3 Conclusions

From technical data it can't be precisely determined the breakdown times for the subsystems of the analyzed system. Only the intervals in which the breakdowns take place can be determined.

The failure moments of the analyzed system components were calculated approximately, and it is obviously needed to study the errors. Practically, it is needed to determine the error from the calculation of $P_i^*(s)$ but also from the calculation of the failure moments $t_i, i = 1, \overline{n}$.

Introducing elements with economic aspect (e.g. the costs}, it can be determined the renewal moments as well as the renewal strategy which has to be adopted.

The use of the results presented in this paper can lead to the following advantages:

- Calculation, aren't so difficult and allow to obtain results which are easy to apply in practice;
- In specialized literature, theoretical findings generally use asymptotic properties and for this reason the results are difficult to apply or lead to erroneous results;
- The results presented in the paper can always be applied because every probability distribution associated to the system's reliability can be estimated through a succession of exponential distributions.

Actually, introducing elements related to the intervention costs, the concept of determining the intervention moments of the systems, influenced by the renewal processes, is changed fundamentally.

References:

[1] Ascher, H.E., Kobbacy, A.H., *Modelling preventive maintenance for deteriorating repairable systems*, IMA Journal of Management Mathematics 1995 6(1):85-99;

[2] Ansell, J.I., Phillips, M.J., 1990. *Practical reliability data analysis*, Reliability Engineering and System Safety, 28, 237-56;

[3] Barlow, R.E., Proschan, F., *Mathematical Theory of Reliability*, S.I.A.M., 1996;

[4] Blischke, W.R., Hawkins, B.D., *Case Studies in Reliability and Maintenance*, John Wiley and Sons, 2003;

[5] Blischke, W.R., Hawkins, B.D., *Reliability: Modeling, Prediction, and Optimization*, Wiley-IEEE, 2000;

[6] Cătuneanu, M.V., Mihalache, A. – *Bazele teoretice ale fiabilitatii (Theoretical Fundamentals of Reliability)*, Ed. Academiei, Bucuresti, 1983;

[7] Cox, D.R. - *Renewal Theory*, John Willey, Now York, 1962;

[8] Ghermeier, L.B. - *Introduction in the Operational Research Theory*, Nauka,Moscow,1971;

[9] Hiriart-Urruty, J.B. , Lemaréchal, C., *Convex Analysis And Minimization Algorithms II*, Grundlehren Math. Wiss 306, Springer Verlag, Berlin, 1996;

[10] Leadbetter, M.R., Lindgren, G., Rootzen, H., *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, N.Y., 1983;

[11] Mitran, I. - *Optimalité Minmax, monographs*, AMSE, France, 1992;

[12] Mitran I., *On Some numeric Method to Determinate the Guaranteed Optimal Values*, Studia Univ. "Babes-Bolyai" , Mathematica, Vol. LII, 1, March 2007, 79-97;

[13] M. Shuaib Khan, G.R Pasha, Ahmed Hesham Pasha, *Theoretical Analysis of Inverse Weibull Distribution*, WSEAS TRANSACTIONS on MATHEMATICS, Issue 2, Volume 7, February 2008, ISSN: 1109-2769, p. 30…38;

[14] Mobley, R.K., *Maintenance fundamentals*, Butterworth-Heinemann, 2004;

[15] Radu-Viorel Ursianu, Emiliana Ursianu, Victor Ursianu, *Mathematical Model for Deterioration process of an Electrical Station with Multiple Components*, Proceedings of the 10th WSEAS International Conference on MATHEMATICAL and COMPUTATIONAL METHODS in SCIENCE and ENGINEERING (MACMESE'08), ISSN: 1790-2769, ISBN: 978-960-474-019-2;

[16] Smith, R., Hawkins, B., *Lean Maintenance: Reduce Costs, Improve Quality, and Increase Market Share*, Butterworth-Heinemann, 2004;

[17] Vassilios C. Moussas, *Structural Reliability Improvement using Non-Linear and Adaptive Multi-Model Techniques*, Proceedings of the 10th WSEAS International Conference on Mathematical Methods, Computational Techniques, Non-Linear Systems, Intelligent Systems, (MAMECTIS '08) Corfu, Greece, October 26-28, ISBN: 978-960-474-012-3, ISSN: 1790-2769, p. 149…155;

[18] Wireman, T., *Developing Performance Indicators for Managing Maintenance*, Industrial Press Inc. 2005