Neimark-Sacker and flip bifurcations in a discrete-time dynamic system for Internet congestion

GABRIELA MIRCEA, DUMITRU OPRIS
West University of Timisoara
ROMANIA
gabriela.mircea@fse.uvt.ro, opris@math.uvt.ro

Abstract: - The aim of this paper is to study the Neimark-Sacker and flip bifurcations for the discrete-time dynamic system, which describes the Internet congestion, with a single link and two sources. We describe the algorithm in order to determine the Neimark-Sacker bifurcation and the normal form. We establish the existence of a flip bifurcation for the case when the model's two parameters depend on the real parameter *a*, which influences the existence of the bifurcation. The numerical simulations verify the theoretical results.

Key-Words: Internet model, Neimark-Sacker bifurcation, flip bifurcation, feedback delay, numerical simulations

1 Introduction

Congestion control mechanisms and active queue management schemes (AQM) for the Internet have been extensively studied since the work of Kelly et all [2]. In [7], the Hopf bifurcation has been studied for the model of an Internet network with $r\,(r>1)$ link and single source, which can be formulated as:

$$\dot{x}_i(t)=k\Big(w-af(x_i(t-\tau))-b\sum_{\substack{j=1\\ j\neq i}}^{r}x_j(t-\tau)f(x_i(t-\tau))\Big), \quad (1)$$
$$i=1,\dots,r$$

where $x_i(t)$ is the sending rate of the source i at time, k is a positive gain parameter, τ is the sum of forward and return delays, w is a target (set –point), and the congestion indication function $f(x)$ is increasing, nonnegative, which characterizes the congestion.

The model obtained from discretizing the system (1) is given by:

$$x_i(n+1)=x_i(n)+k(w-af(x_i(n-q))-b\sum_{\substack{j=1\\ j\neq i}}^{r}x_j(n-q)f(x_j(n-q))), \quad (2)$$
$$i=1,\dots,r, \quad n,q\in N$$

And it represents the dynamical system with discrete-time for Internet congestion with r link and single source.

In [7] the system (2) was analyzed, considering k to be a parameter, $q=1$ and $q=2$. We determine the value of the parameter k for which the Neimark-Sacker bifurcation takes place. For different values of the parameters, we carry out a numerical simulation.

The dynamic model with a single link and r sources can be described by:

$$\begin{aligned}\dot{x}_i(t)&=k_i x_i(t-\tau_i)\left(\frac{\alpha_i}{x_i(t)}-\beta_i x_i(t)p(t)\right)\\ \dot{p}(t)&=kp(t)\left(\sum_{i=1}^{r}x_i(t-\tau_i)-c\right),\ i=1,\dots,r\end{aligned} \quad (3)$$

where, $x_i(t)$ is the rate at which source i transmits data at the time t, α_i and β_i are positive real numbers, $p(t)$ is the loss probability function, τ_i is round-tripe delay for source i, c is the capacity, k_i, k are gain parameters.

The model obtained from discretizing the system (3) is given by:

$$\begin{aligned}x_i(n+1)&=x_i(n)+k_i x_i(n-q_i)\left(\frac{\alpha_i}{x_i(n)}-\beta_i x_i(n)p(n)\right)\\ p(n+1)&=p(n)+kp(n)\left(\sum_{i=1}^{r}x_i(n-q_i)-c\right),\ i=1,\dots,r\end{aligned} \quad (4)$$

where $n,\ q_i\in N$.

In this paper we will focus on the local stability, the Neimark-Sacker and flip bifurcations, if the parameters k_i, k satisfy relations that are obtained by using the Schur criterion.

The rest of the paper is organized as follows: In section 2, by analyzing the model (4) for $r=2$ and $q_1=0$, $q_2=0$, we establish the relations that satisfy the k_1, k_2, k parameters so that the equilibrium point is asymptotically stable. We prove that there is a value k_0 for which a Neimark-Sacker bifurcation takes place. We determine the normal form on the central manifold corresponding to the value k_0, and the associated Lyapunov coefficient. For fixed values of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, c, k_1, k_2$, we determine the value of k_0 and we visualise the corresponding orbits. In section 3, we study the local stability and the Neimark-Sacker bifurcation for $q_1 = 0$, $q_2 = 1$ and $q_1 = 1$, $q_2 = 0$. We determine the k parameter in relation to k_1 and k_2, and we determine the value k_{20} for which a Neimark-Sacker bifurcation takes place. For this value, we establish the normal form on the central manifold as well as the Lyapunov coefficient. For fixed values of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, c, k_1, k_2$, we determine the Lyapunov coefficient and we visualize the orbits $(n, p(n))$, $(x_1(n), p(n))$, $(x_2(n), p(n))$. In section 4, we analyze the model (4) for $q_1 = q,\ q \geq 1,\ q_2 = 0$ and $q_1 = 0,\ q_2 = q,\ q \geq 1$. For analysis, we will consider that parameters k_1, k_2, k depend on the fixed parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and the variable parameter a. For $q_1 = q,\ q_2 = 0,\quad q \geq 0$, we establish the existence of a flip bifurcation. For fixed values of $\alpha_1, \alpha_2, \beta_1, \beta_2, c$, we visualize the orbits $(a, x_1(n))$, $(a, x_2(n))$, $(a, x_3(n))$, where a is the real parameter that characterizes the bifurcation.

2 Local stability and the Neimark-Sacker bifurcation analysis for q_i=0, i=1,2.

In this section, we consider the model with a single link and two sources with $q_i = 0,\ i = 1,2$. The model can be described by:

$$\begin{aligned} x_i(n+1) &= x_i(n) + k_i x_i(n)\left(\frac{\alpha_i}{x_i(n)} - \beta_i x_i(n) p(n)\right) \\ p(n+1) &= p(n) + kp(n)\left(\sum_{i=1}^{2} x_i(n) - c\right),\ i = 1,2 \end{aligned} \tag{5}$$

where $k_i > 0$, $i = 1,2$, $k > 0$, $\alpha_i > 0$, $\beta_i > 0$, $i = 1,2$.

Let (x_{10}, y_{20}, p_0) be the non-zero equilibrium point of the system (5). Hence it satisfies the following equation:

$$\begin{aligned} x_{10} &= \frac{c\sqrt{\alpha_1\beta_2}}{\sqrt{\alpha_1\beta_2} + \sqrt{\alpha_2\beta_1}} \\ x_{20} &= \frac{c\sqrt{\alpha_2\beta_1}}{\sqrt{\alpha_1\beta_2} + \sqrt{\alpha_2\beta_1}} \\ p_0 &= \frac{1}{c^2}\left(\sqrt{\frac{\alpha_1}{\beta_1}} + \sqrt{\frac{\alpha_2}{\beta_2}}\right)^2. \end{aligned} \tag{6}$$

If we linearize the system (5) and if the equilibrium satisfies (6), we can obtain:

$$\begin{aligned} y_1(n+1) &= a_{11}y_1(n) + a_{13}y_3(n) \\ y_2(n+1) &= a_{22}y_2(n) + a_{23}y_3(n) \\ y_3(n+1) &= a_{31}y_1(n) + a_{32}y_2(n) + a_{33}y_3(n) \end{aligned} \tag{7}$$

where

$$\begin{aligned} &a_{11} = 1 - 2k_1\beta_1 x_{10} p_0,\quad a_{13} = -k_1\beta_1 x_{10}^2 \\ &a_{22} = 1 - 2k_2\beta_2 x_{20} p_0,\quad a_{23} = -k_2\beta_2 x_{20}^2 \\ &a_{31} = kp_0,\quad a_{32} = kp_0,\quad a_{33} = 1 \end{aligned} \tag{8}$$

The characteristic equation of the system (7) is given by:

$$\lambda^3 - A_1\lambda^2 + A_2\lambda - A_3 = 0 \tag{9}$$

where

$$\begin{aligned} A_1 &= a_{11} + a_{22} + a_{33} \\ A_2 &= a_{33}(a_{11} + a_{22}) + a_{11}a_{22} - kp_0(a_{13} + a_{23}) \\ A_3 &= a_{11}a_{22}a_{33} - kp_0(a_{13}a_{22} + a_{23}a_{11}) \end{aligned} \tag{10}$$

It is well known that the trivial solution of (7) is locally asymptotically stable if all the roots of the characteristic equation are in modulus less than 1 and unstable if one root is in modulus greater than 1. Therefore, in order to study the local asymptotical stability of the equilibrium of the system (5), we need to investigate the distribution of the roots of equation (9) with the function of parameters k_1, k_2, k.

For fixed k_1, k_2, let k_0 be the positive root of equation:

$$\begin{aligned} &k^2 p_0(a_{13}a_{22} + a_{23}a_{11})^2 + kp_0(A_1(a_{13}a_{22} + a_{23}a_{11}) - \\ &a_{11}a_{22}a_{33}(a_{13}a_{12} + a_{23}a_{11})) + a_{33}(a_{11} + a_{22}) + \\ &a_{11}a_{22} - 1 + A_1 a_{11}a_{22}a_{33} + a_{11}^2 a_{22}^2 a_{33}^2 = 0 \end{aligned} \tag{11}$$

Using the Schur criteria [4], [5] we obtain:

Proposition 1:

a) If k_1, k_2, k satisfy the relations:

$$|A_3|<1,\ |A_1-A_3|<2,\ 1-A_2+A_1A_3-A_3^2<0$$

where A_1, A_2, A_3 are given by (10), then the equilibrium of the system (5) is locally asymptotically stable.

b) If k_0 is the positive root of the equation (11) and k_1, k_2, k satisfy the relations:

$$|A_3|<1,\ |A_1-A_3|<2,$$

then the equation (9) has one root in modulus less than 1 and two roots in modulus greater than 1.

c) The value k_0 is a Neimark-Sacker bifurcation, namely there exists an $\alpha>0$ sufficiently small so that, for $k=k_0-\alpha$, the equation (9) has the roots in modulus less than 1 and, for $k=k_0+\alpha$, it has two roots in modulus greater than 1.

In what follows we determine the normal form for system (5) on the central manifold corresponding to the value k_0 for bifurcating parameter k.

Let $\mu=\mu(\alpha)$ be one root of the characteristic equation (9) for $k=k_0+\alpha$.

The next propositions hold:

Proposition 2:

a) The eigenvector corresponding to the eigenvalue μ, the solution of system $Al=\mu l$ has the components:

$$l_1=-a_{13}(\mu-a_{22}),\quad l_2=a_{23}(\mu-a_{11}),\quad l_3=(\mu-a_{11})(\mu-a_{22}). \tag{12}$$

where

$$A=\begin{pmatrix} a_{11} & 0 & a_{13}\\ 0 & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}.$$

b) The eignevector corresponding to the eigenvalue $\bar{\mu}$, the nontrivial solution of system $A^T m=\bar{\mu} m$ has the components:

$$m_1=\frac{a_{31}}{(\mu-a_{11})V},\ m_2=\frac{a_{32}}{(\mu-a_{22})V},\ m_3=\frac{1}{V} \tag{13}$$

where

$$V=\frac{\bar{l}_1 a_{31}}{\bar{\mu} a_{11}}+\frac{\bar{l}_1 a_{31}}{\bar{\mu} a_{11}}+\bar{l}_3. \tag{14}$$

In order to determine the normal form of the system (5), by applying the method from [4], [5], we obtain the following coefficients:

$$\begin{aligned}
B_{120}&=-2k_1\beta_1 p_0 l_1^2-4k_1\beta_1 x_{10} l_1 l_3,\\
B_{111}&=-2k_1\beta_1 p_0 l_1 \bar{l}_1-2k_1\beta_1 x_{10}(\bar{l}_1 l_3+l_3\bar{l}_1),\\
B_{102}&=\bar{B}_{120},\\
B_{220}&=-2k_2\beta_2 p_0 l_2^2-4k_2\beta_2 x_{20} l_2 l_3,\\
B_{211}&=-2k_2\beta_2 p_0 l_2 \bar{l}_2-2k_2\beta_2 x_{20}(\bar{l}_2 l_3+l_2\bar{l}_3),\\
B_{202}&=\bar{B}_{220},\\
B_{320}&=2k(l_1 l_3+l_2 l_3),\\
B_{311}&=k\beta(\bar{l}_1 l_3+l_1\bar{l}_3+\bar{l}_2 l_3+l_2\bar{l}_3),\\
B_{302}&=\bar{B}_{320}.
\end{aligned} \tag{15}$$

$$\begin{aligned}
g_{20}&=B_{120}m_1+B_{220}m_2+B_{320}m_3,\\
g_{11}&=B_{111}m_1+B_{211}m_2+B_{311}m_3,\\
g_{02}&=B_{102}m_1+B_{202}m_2+B_{302}m_3,\\
k_{20}&=B_{120}\bar{m}_1+B_{220}\bar{m}_2+B_{320}\bar{m}_3,\\
k_{11}&=B_{111}\bar{m}_1+B_{211}\bar{m}_2+B_{311}\bar{m}_3,\\
k_{02}&=B_{102}\bar{m}_1+B_{202}\bar{m}_2+B_{302}\bar{m}_3.
\end{aligned} \tag{16}$$

$$\begin{aligned}
h_{i20}&=B_{i20}m_1-g_{20}l_i-k_{20}\bar{l}_i,\ i=1,2,3\\
h_{i11}&=B_{i11}m_1-g_{11}l_i-k_{11}\bar{l}_i,\ i=1,2,3\\
h_{i02}&=B_{i02}m_1-g_{02}l_i-k_{02}\bar{l}_i,\ i=1,2,3.
\end{aligned} \tag{17}$$

$$\begin{aligned}
w_{20}&=A(\mu^2)^{-1}h_{20},\\
w_{11}&=A(\mu^2)^{-1}h_{11},\\
w_{02}&=A(\mu^2)^{-1}h_{02}.
\end{aligned} \tag{18}$$

where

$$A(\mu^2)=\begin{pmatrix}\mu^2-a_{11} & 0 & -a_{13}\\ 0 & \mu^2-a_{22} & -a_{23}\\ -a_{31} & -a_{32} & \mu^2-a_{33}\end{pmatrix},$$

$$A(1)=\begin{pmatrix}1-a_{11} & 0 & -a_{13}\\ 0 & 1-a_{22} & -a_{23}\\ -a_{31} & -a_{32} & 1-a_{33}\end{pmatrix}, \tag{19}$$

$$A(\bar{\mu}^2)=\bar{A}(\mu^2).$$

$$\begin{aligned}
h_{20}&=(h_{120},h_{220},h_{320})^T,\\
h_{11}&=(h_{111},h_{211},h_{311})^T,\\
h_{02}&=(h_{102},h_{202},h_{302})^T.
\end{aligned} \tag{20}$$

$$\begin{aligned}
g_{21} = &(-2k_1\beta_1 p_0 \bar{l}_1 w_{120} - 2k_1\beta_1 x_{10}(\bar{l}_1 w_{320} + \bar{l}_3 w_{120}),\\
&-2k_2\beta_2 p_0 \bar{l}_2 w_{220} - 2k_1\beta_1 x_{20}(\bar{l}_2 w_{320} + \bar{l}_3 w_{220}),\\
&k(\bar{l}_1 w_{320} + \bar{l}_3 w_{120} + \bar{l}_2 w_{320} + \bar{l}_3 w_{220})m+\\
&+2(-2k_1\beta_1 p_0 \bar{l}_1 w_{111} - 2k_1\beta_1 x_{10}(\bar{l}_1 w_{311} + \bar{l}_3 w_{111}),\\
&-2k_2\beta_2 p_0 \bar{l}_2 w_{211} - 2k_2\beta_2 x_{20}(\bar{l}_2 w_{311} + \bar{l}_3 w_{211}),\\
&-2k_2\beta_2 x_{20}(\bar{l}_2 w_{311} - \bar{l}_3 w_{211}),\\
&k(\bar{l}_1 w_{311} + \bar{l}_3 w_{220} \bar{l}_2 w_{111} + \bar{l}_2 w_{311} + \bar{l}_3 w_{211})m+\\
&+(-2k_1\beta_1 p_0 l_1^2 \bar{l}_3, -2k_2\beta_2 l_2^2 \bar{l}_3, 0)l.
\end{aligned} \quad (21)$$

where $l = (l_1, l_2, l_3)$, $m = (m_1, m_2, m_3)^T$.

Proposition 3:

a) The normal form for the system (5) is:

$$z(n+1) = \mu z(n) + \frac{1}{2} g_{20} z(n)^2 + g_{11} z(n)\bar{z}(n) + \frac{1}{2} g_{02}\bar{z}(n)^2 + \frac{1}{2} g_{21} z(n)^2 \bar{z}(n) \quad (22)$$

where $z(n) \in C$, $n \in N$ and the coefficients are given by (12), (21).

b) The system (5) in the neighbourhood of the equilibrium point (x_{10}, x_{20}, p_0) is:

$$x_i(n) = x_{i0} + l_i z(n) + \bar{l}_i \bar{z}(n) + \frac{1}{2} w_{i20} z(n)^2 + w_{i11} z(n)\bar{z}(n) + \frac{1}{2} w_{i02}\bar{z}(n)^2 \quad (23)$$

$i = 1,2,3$, $n \in N$, $x_3(n) = p(n)$ and $z(n)$ is a solution for (22) and the coefficients are given by (18).

c) The Lyapunov coefficient associated to the normal form (22) is given by:

$$L_{yap}(\alpha) = \frac{g_{20}(\alpha) g_{11}(\alpha)(\bar{\mu}(\alpha) - 3 - 2\mu(\alpha))}{2(\mu(\alpha)^2 - \mu(\alpha))(\bar{\mu}(\alpha) - 1)} + \frac{|g_{11}(\alpha)|^2}{1 - \bar{\mu}(\alpha)} + \frac{|g_{02}(\alpha)|^2}{2(\mu^2(\alpha) - \bar{\mu}(\alpha))} + \frac{g_{21}(\alpha)}{2} \quad (24)$$

d) Let $\theta_0 = \arg(\mu(0))$ and $L_0 = \mathrm{Re}(e^{-i\theta_0} L_{yap}(0))$. If $L_0 < 0$ $(L_0 > 0)$, there is an invariant, stable (unstable) orbit in the neighbourhood of the equilibrium (x_{10}, x_{20}, p_0).

For the simulation, we consider the values: $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\beta_1 = 0.1$, $\beta_2 = 0.1$, $c = 1$, $k_1 = 0.3$, $k_2 = 0.1$.

Using a program in Maple 12, we obtain the orbits presented in the figures below.

The roots for the characteristic equation are:

$\mu_1 = -0.497$, $\mu_2 = -0.006 + 0.991$, $\mu_3 = \bar{\mu}_2$, with $|\mu_1| < 1$, $|\mu_2| = |\bar{\mu}_3| = 1$, $L_0 = 0.9562$.

Because $L_0 > 0$, we obtain an unstable orbit. Fig.1 represents a visualisation of the orbit $(n, p(n))$, for $\mu(\alpha) = \mu_1 - \alpha$, $\alpha = 0.01$. Fig.2 represents a visualisation of the orbit $(n, p(n))$ for $\mu(\alpha) = \mu_1 + \alpha$, $\alpha = 0.01$. Fig.3 and Fig. 4 represent visualisations of the orbit $(x_1(n), p(n))$, and $(x_2(n), p(n))$ respectively.

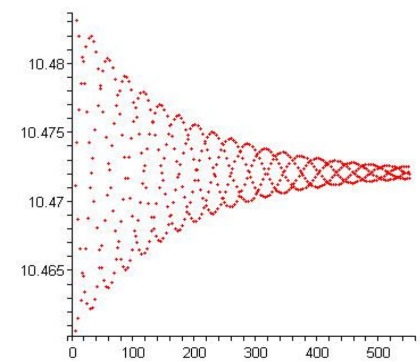

Fig.1 The orbit $(n, p(n))$, for $\mu(\alpha) = \mu_1 - \alpha$, $\alpha = 0.01$

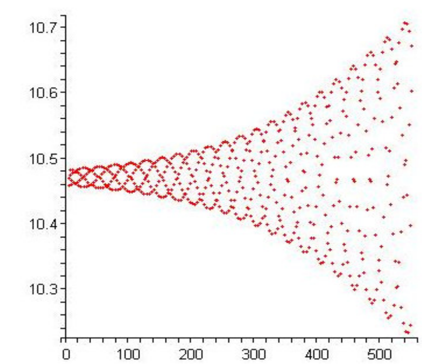

Fig.2 The orbit $(n, p(n))$ for $\mu(\alpha) = \mu_1 + \alpha$, $\alpha = 0.01$

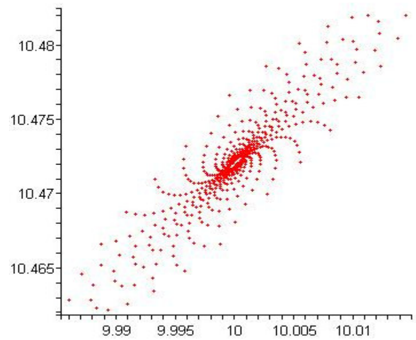

Fig.3 The orbit $(x_1(n), p(n))$, for $\mu(\alpha) = \mu_1 - \alpha$, $\alpha = 0.01$

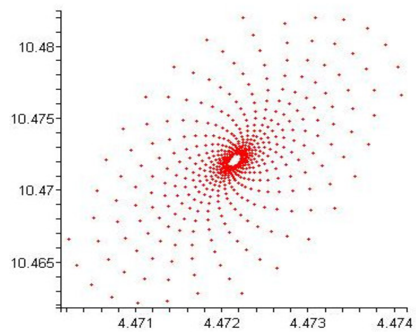

Fig.4 The orbit $(x_2(n), p(n))$ for $\mu(\alpha) = \mu_1 + \alpha$, $\alpha = 0.01$

3 Local stability and the Neimark-Sacker bifurcation analysis for q1=0, q2=1 and q1=1, q2=0

In this section, we consider the model with a single link and two sources with $q_1 = 0, q_2 = 1$. The model is described by:

$$\begin{aligned} x_0(n+1) &= x_2(n) \\ x_1(n+1) &= x_1(n) + k_1 x_1(n)\left(\frac{\alpha_1}{x_1(n)} - \beta_1 x_1(n) p(n)\right) \\ x_2(n+1) &= x_2(n) + k_2 x_0(n)\left(\frac{\alpha_{21}}{x_2(n)} - \beta_2 x_2(n) p(n)\right) \\ p(n+1) &= p(n) + kp(n)(x_1(n) + x_0(n) - c) \end{aligned} \tag{25}$$

The non-zero equilibrium point of the system (25) is $(x_{00} = x_{20}, x_{10}, x_{20}, p_0)$ where x_{10}, x_{20}, p_0 is given by (6).

If we linearize the system (25) and obtain:

$$\begin{aligned} y_0(n+1) &= y_2(n) \\ y_1(n+1) &= b_{11} y_1(n) + b_{13} y_3(n) \\ y_2(n+1) &= b_{22} y_2(n) + b_{23} y_3(n) \\ y_3(n+1) &= b_{30} y_0(n) + b_{31} y_1(n) + b_{33} y_3(n) \end{aligned} \tag{26}$$

where

$$\begin{aligned} & b_{11} = 1 - 2k_1\beta_1 x_{10} p_0, \; b_{13} = -k_1\beta_1 x_{10}^2, \\ & b_{22} = 1 - 2k_2\beta_{21} x_{20} p_0, \; b_{23} = -k_2 p_2 x_{20}^2, \\ & b_{30} = kp_0, \; b_{31} = kp_0, b_{33} = 1. \end{aligned} \tag{27}$$

The characteristic equation of the system (26) is given by:

$$\lambda^4 - B_1\lambda^3 + B_2\lambda^2 - B_3\lambda - B_4 = 0. \tag{28}$$

where

$$\begin{aligned} B_1 &= b_{11} + b_{22} + b_{33} \\ B_2 &= b_{11}b_{22} + b_{33}(b_{11} + b_{22}) - b_{13}b_{31} \\ B_3 &= b_{33}b_{11}b_{22} + b_{13}b_{31}b_{22} + b_{30}b_{23} \\ B_4 &= b_{33}b_{11}b_{22} - b_{30}b_{23}b_{31} + b_{11} \end{aligned} \tag{29}$$

For fixed k_1, k_2, let k_0 given by

$$k_0 = \frac{b_{11}b_{22}}{p_0(b_{11}b_{13} - b_{23})} \tag{30}$$

Using the Schur criteria, we obtain:

Proposition 4:

a) If k_1, k_2 satisfy the relations:

$|B_4| < 1$, $|B_1| < 2((1 + B_4)$, $(B_2 - 1 + B_4)(1 + B_4)^2 < B_1^2 B_4$, where B_1, B_2, B_4 are given by (28) and $k = k_0$ given by (30), then the equilibrium of the system (25) is locally asymptotically stable.

b) If k_{20} is the positive root of the equation $(B_2 - 1 + B_4)(1 + B_4)^2 - B_1^2 B_4 = 0$, and k_1, satisfy the relations $|B_4| < 1$, $|B_1| < 2(1 + B_4)$ then k_{20} is a Neimark-Sacker bifurcation, namely there exists a $\alpha > 0$ sufficiently small so that, for $k_2 = k_{20} - \alpha$, the equation (28) has the roots in modulus less than 1 and for $k_2 = k_{20} + \alpha$, it has the roots in modulus greater than 1.

In what follows, we determine the normal form of the system (25) on the central manifold corresponding to the value k_{20} of the bifurcation parameter k_2.

Let $\mu = \mu(\alpha)$ be one root of characteristic equation (28) corresponding to the value $k_2 = k_{20} + \alpha$.

The next proposition holds:

Proposition 5:

a) If the eigenvector corresponding to the eigenvalue μ, the solution of the system $Bl=\mu l$ where:

$$B=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & b_{11} & 0 & b_{13} \\ 0 & 0 & b_{22} & b_{23} \\ b_{30} & b_{31} & 0 & b_{33} \end{pmatrix} \tag{31}$$

has the components:

$$l_0=\frac{b_{23}(\mu-b_{11})}{\mu},\ l_1=b_{13}(\mu-b_{22}),\quad l_2=b_{23}(\mu-b_{11}),\ l_3=(\mu-b_{11})(\mu-b_{22}). \tag{32}$$

b) The eigenvector corresponding to the eigenvalue $\bar{\mu}$, the solution of the system $B^T m=\bar{\mu} m$ has the components:

$$m_0=\frac{1}{V},\quad m_1=\frac{b_{31}\bar{\mu}}{b_{30}(\bar{\mu}-b_{11})V},\quad m_2=-\frac{1}{(\bar{\mu}-b_{22})V},\quad m_3=\frac{\bar{\mu}}{b_{30}V}. \tag{33}$$

where

$$V=\bar{l}_0+\frac{b_{31}\bar{l}_1\bar{\mu}}{b_{30}(\bar{\mu}-b_{11})}-\frac{\bar{l}_2}{\bar{\mu}-b_{22}}+\frac{\bar{l}_3\bar{\mu}}{b_{30}}. \tag{34}$$

In order to determine the normal form of the system (25), by applying the method from [4], [5], we obtain the following coefficients:

$$\begin{aligned}
B_{120}&=-2k_1\beta_1 p_0 l_1^2-4k_1\beta_1 x_{10} l_1 l_3,\\
B_{111}&=-2k_1\beta_1 p_0 l_1\bar{l}_1-2k_1\beta_1 x_{10}(\bar{l}_1 l_3+l_3\bar{l}_1),\\
B_{102}&=\bar{B}_{120},\\
B_{220}&=-2\frac{k_2\alpha_2}{x_{20}^2}l_0 l_2+2\frac{k_2\alpha_2}{x_{20}^2}l_2^2,\\
B_{211}&=-2\frac{k_2\alpha_2}{x_{20}^2}(l_0\bar{l}_2+\bar{l}_0 l_2)+2\frac{k_2\alpha_2}{x_{20}^2}l_2\bar{l}_2,\\
B_{202}&=\bar{B}_{220},\\
B_{320}&=2k_0 l_0 l_3,\\
B_{311}&=k_0(\bar{l}_0 l_3+l_0\bar{l}_3),\\
B_{302}&=\bar{B}_{320}.
\end{aligned} \tag{35}$$

$$\begin{aligned}
g_{20}&=B_{120}m_1+B_{220}m_2+B_{320}m_3,\\
g_{11}&=B_{111}m_1+B_{211}m_2+B_{311}m_3,\\
g_{02}&=B_{102}m_1+B_{202}m_2+B_{302}m_3,\\
k_{20}&=B_{120}\bar{m}_1+B_{220}\bar{m}_2+B_{320}\bar{m}_3,\\
k_{11}&=B_{111}\bar{m}_1+B_{211}\bar{m}_2+B_{311}\bar{m}_3,\\
k_{02}&=B_{102}\bar{m}_1+B_{202}\bar{m}_2+B_{302}\bar{m}_3.
\end{aligned} \tag{36}$$

$$\begin{aligned}
h_{020}&=-g_{20}l_0-k_{20}\bar{l}_0,\\
h_{011}&=-g_{11}l_0-k_{11}\bar{l}_0,\\
h_{002}&=-g_{02}l_0-k_{02}\bar{l}_0,\\
h_{i20}&=B_{i20}m_1-g_{20}l_i-k_{20}\bar{l}_i,\quad i=1,2,3\\
h_{i11}&=B_{i11}m_1-g_{11}l_i-k_{11}\bar{l}_i,\quad i=1,2,3\\
h_{i02}&=B_{i02}m_1-g_{02}l_i-k_{02}\bar{l}_i,\quad i=1,2,3.
\end{aligned} \tag{37}$$

$$\begin{aligned}
w_{20}&=B(\mu^2)^{-1}h_{20},\\
w_{11}&=B(1)^{-1}h_{11},\\
w_{02}&=B(\bar{\mu}^2)^{-1}h_{02}.
\end{aligned} \tag{38}$$

where

$$B(\mu^2)=\begin{pmatrix} \mu^2 & 0 & -1 & 0 \\ 0 & \mu^2-b_{11} & 0 & -b_{13} \\ 0 & 0 & \mu^2-b_{22} & -b_{23} \\ -b_{30} & -b_{31} & 0 & \mu^2-b_{33} \end{pmatrix},$$

$$B(1)=\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1-b_{11} & 0 & -b_{13} \\ 0 & 0 & 1-b_{22} & -b_{23} \\ -b_{30} & -b_{31} & 0 & 1-b_{33} \end{pmatrix}, \tag{39}$$

$$B(\bar{\mu}^2)=\bar{B}(\mu^2).$$

$$\begin{aligned}
g_{21}=&(0,-2k_1\beta_1 p_0\bar{l}_1 w_{120}-2k_1\beta_1 x_{10}(\bar{l}_1 w_{320}+\bar{l}_3 w_{120}),\\
&-\frac{k_2\alpha_2}{x_{20}^2}(\bar{l}_0 w_{220}+\bar{l}_2 w_{020})+2\frac{k_1\alpha_2}{x_{20}^2}\bar{l}_2 w_{220},\\
&k_0(\bar{l}_0 w_{320}+\bar{l}_3 w_{020}))m+2(0,-2k_1\beta_1 p_0 l_1 w_{111}\\
&-2k_1\beta_1 x_{10}(l_1 w_{311}+l_3 w_{111}),-\frac{k_2\alpha_2}{x_{20}^2}(l_2 w_{211}+l_2 w_{011}))m\\
&+(0,-2k_1\beta_1 p_0 l_1^2\bar{l}_3,-2\frac{1}{x_{20}^2}l_0 l_2\bar{l}_2,0)l.
\end{aligned} \tag{40}$$

where $l=(l_0,l_1,l_2,l_3)^T$, $m=(m_0,m_1,m_2,m_3)^T$.

The normal form for the system (25) is given by (22) with the coefficients from (35). The system (25) in the neighbourhood of the equilibrium point $(x_{20}, x_{10}, x_{20}, p_0)$ is (23) to which we add:

$$x_0(n) = x_{20} + l_0 z(n) + \bar{l}_0 \bar{z}(n) + \frac{1}{2} w_{020} z(n)^2 + w_{011} z(n) \bar{z}(n) + \frac{1}{2} w_{002} \bar{z}(n)^2. \quad (41)$$

The Lyapunov coefficient associated to the normal form is given by (25) with the coefficients from (35):

For the numerical simulation, we consider the values:

$\alpha_1 = 0.6$, $\alpha_2 = 0.5$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $c = 5$, $k_1 = 0.3$.

Using a program in Maple 12, we obtain: $k = 0.102$, $k_{20} = 0.5607$, $L_0 = 98.43$. Because $L_0 > 0$ we have an unstable orbit.

Fig. 5 represents a visualisation of the orbit $(n, p(n))$, for $k_{20} = 0.001$, $\alpha = 0.01$. Fig.6 and Fig. 7 represent visualisations of the orbit $(x_1, p(n))$, and $(x_2, p(n))$ respectively.

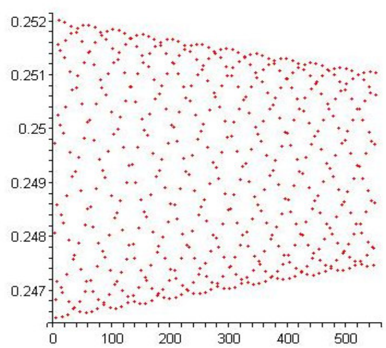

Fig.5 The orbit $(n, x_3(n))$

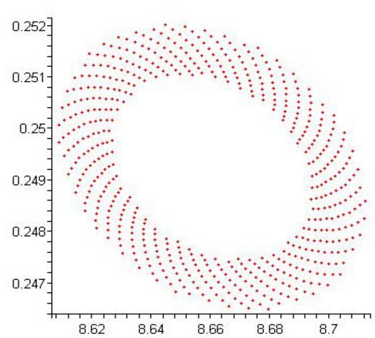

Fig.6 The orbit $(x_1(n), p(n))$

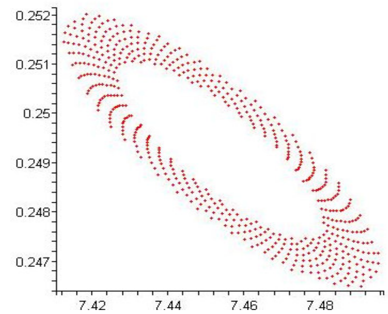

Fig.7 The orbit $(x_2(n), p(n))$

If $q_1 = 1$, $q_2 = 0$, the model (4) is given by:

$$\begin{aligned} x_0(n+1) &= x_1(n) \\ x_1(n+1) &= x_1(n) + k_1 x_0(n)\left(\frac{\alpha_1}{x_1(n)} - \beta_1 x_1(n) p(n)\right) \\ x_2(n+1) &= x_2(n) + k_2 x_2(n)\left(\frac{\alpha_2}{x_2(n)} - \beta_2 x_2(n) p(n)\right) \\ p(n+1) &= x_3(n) + kp(n)(x_0(n) + x_2(n) - c) \end{aligned} \quad (42)$$

If we liniarize the system (42) in the neighbourhood of the equilibrium point (6) and obtain:

$$\begin{aligned} y_0(n+1) &= y_1(n) \\ y_1(n+1) &= c_{11} y_1(n) + c_{13} y_3(n) \\ y_2(n+1) &= c_{22} y_2(n) + c_{23} y_3(n) \\ y_3(n+1) &= c_{30} y_0(n) + c_{32} y_1(n) + c_{33} y_3(n) \end{aligned} \quad (43)$$

where

$$\begin{aligned} & c_{11} = 1 - 2k_1 \beta_1 x_{10} p_0,\ c_{13} = -k_1 \beta_1 x_{10}^2, \\ & c_{22} = 1 - 2k_2 \beta_2 x_{20} p_0,\ c_{23} = -k_2 p_2 x_{20}^2, \\ & c_{30} = kp_0,\ c_{32} = kp_0, c_{33} = 1. \end{aligned} \quad (44)$$

The characteristic equation of the system (43) is given by:

$$\lambda^4 - C_1 \lambda^3 + C_2 \lambda^2 - C_3 \lambda - C_4 = 0. \quad (45)$$

where

$$\begin{aligned} C_1 &= c_{11} + c_{22} + c_{33} \\ C_2 &= c_{11}(c_{22} + c_{33}) + c_{22} c_{33} - c_{23} c_{32} \\ C_3 &= c_{11}(c_{22} c_{33} - c_{23} c_{32}) + c_{13} c_{30} \\ C_4 &= -c_{30} c_{22} c_{13} \end{aligned} \quad (46)$$

For fixed k_1, k_2, let k_0 given by

$$k_0 = \frac{c_{11} c_{22}}{p_0 (c_{11} c_{23} - c_{13})} \quad (47)$$

Using the Schur criteria, we obtain:

Proposition 6:

a) If k_1, k_2 satisfy the relations:

$|C_4|<1, |C_1|<2((1+C_4), (C_2-1+C_4)(1+C_4)^2<C_1^2C_4,$ where C_1, C_2, C_4 are given by (46) and $k=k_0$ given by (47), then the equilibrium of the system (42) is locally asymptotically stable.

b) If k_{10} is the positive root of the equation $(C_2-1+C_4)(1+C_4)^2-C_1^2C_4=0,$ and k_2 satisfies the relations $|C_4|<1, |C_1|<2(1+C_4)$ then k_{10} is a Neimark-Sacker bifurcation, namely there exists a $\beta>0$ sufficiently small so that, for $k_1=k_{10}-\beta$, the equation (28) has the roots in modulus less than 1 and for $k_1=k_{10}+\beta$, it has the roots in modulus greater than 1.

The normal form of the system (42) on the central manifold corresponding to the value k_{10} of the bifurcation parameter k_1 is obtained in a similar manner to that of the system (25)

4 Flip bifurcation for the system (4)

Let the model:

$$\begin{aligned} x_1(n+1) &= x_1(n)+k_1x_1(n-q)(\frac{\alpha_1}{x_1(n)}-\beta_1x_1(n)p(n)) \\ x_2(n+1) &= x_2(n)+k_2x_2(n)(\frac{\alpha_2}{x_2(n)}-\beta_2x_2(n)p(n)) \\ p(n+1) &= x_3(n)+kp(n)(x_1(n-q)+x_2(n)-c) \end{aligned} \tag{48}$$

where

$q\in N$ and

$$k_1=h_1-a,\ k_2=h_2-\sqrt{\frac{\alpha_1\beta_1}{\alpha_2\beta_2}}a \tag{49}$$

with

$$\begin{aligned} h_1 &= \frac{2c\beta_1}{\sqrt{\alpha_1\beta_2}(\sqrt{\alpha_1\beta_2}+\sqrt{\alpha_2\beta_1})} \\ h_2 &= \frac{2c\beta_2}{\sqrt{\alpha_2\beta_1}(\sqrt{\alpha_1\beta_2}+\sqrt{\alpha_2\beta_1})} \end{aligned} \tag{50}$$

and a is a real parameter.

The liniarized system of the system (48) is given by:

$$\begin{aligned} y_1(n+1) &= d_{11}y_1(n)+d_{13}y_3(n) \\ y_2(n+1) &= d_{22}y_2(n)+d_{23}y_3(n) \\ y_3(n+1) &= d_{30}y_1(n-q)+d_{32}y_2(n)+d_{33}y_3(n) \end{aligned} \tag{51}$$

where

$$\begin{aligned} & d_{11}=1-2k_1\beta_1x_{10}p_0,\ d_{13}=-k_1\beta_1x_{10}^2, \\ & d_{22}=1-2k_2\beta_2x_{20}p_0,\ d_{23}=-k_2\beta_2x_{20}^2, \\ & d_{30}=kp_0,\ d_{32}=kp_0, d_{33}=1. \end{aligned} \tag{52}$$

and x_{10}, x_{20}, p_0 is given by (6).

From (52), (49), it follows that:

$$\begin{aligned} & d_{11}=d_{22}=-1+2a\beta_1x_{10}p_0, \\ & d_{13}=-(h_1-a)\beta_1x_{10}^2, \\ & d_{23}=-\left(h_2-a\sqrt{\frac{\alpha_1\beta_1}{\alpha_2\beta_2}}\right)\beta_2x_{20}^2, \\ & d_{30}=kp_0,\ d_{32}=kp_0,\ d_{33}=1. \end{aligned} \tag{53}$$

Proposition 7:

The characteristic equation of the system (51) for $d_{11}, d_{22}, d_{13}, d_{23}, d_{30}, d_{32}, d_{33}$ given by (53) is:

$$(\lambda+1-2aB)(\lambda^{q+2}-(1-kh_1\alpha_1)\lambda^q+kh_2\alpha_2- a\left((2B+k\alpha_1)\lambda^q+k\sqrt{\frac{\alpha_1\beta_1}{\alpha_2\beta_2}}\alpha_2\right)=0 \tag{54}$$

where

$B=\beta_1x_{10}p_0$.

From (54), it follows that:

Proposition 8:

a) If $q=0$ and k satisfies the inequality:

$$0<k<\frac{1}{h_1\alpha_1+h_2\alpha_2} \tag{55}$$

then $a=0$ is flip bifurcation.

b) If $q=1$ and k satisfies the inequality

$$0<k<\frac{1}{h_2\alpha_2} \tag{56}$$

then $a=0$ is a flip bifurcation.

c) If there exists a $k>0$ so that the equation:

$$\lambda^{q+2}-(1-kh_1\alpha_1)\lambda^q+kh_2\alpha_2=0 \tag{57}$$

may have the roots in modulus less than 1, then $a=0$ is a flip bifurcation.

For the parameter values $\alpha_1=0.6$, $\alpha_2=0.9$, $\beta_1=0.5$, $\beta_2=0.2$, $c=2$ and $q=0$, the solutions of the system (48) in relation to the bifurcation parameter $a=\alpha$ are displayed in the following figures: Fig. 8 The orbit $(\alpha, x_1(n))$, Fig. 9 The orbit $(\alpha, x_2(n))$, Fig. 10 The orbit $(\alpha, p(n))$.

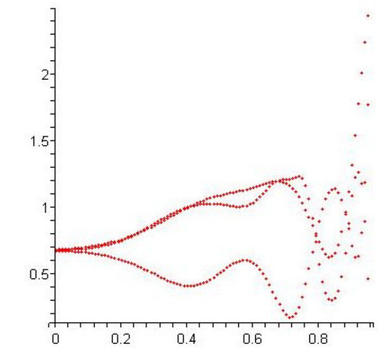

Fig.8 The orbit $(\alpha, x_1(n))$

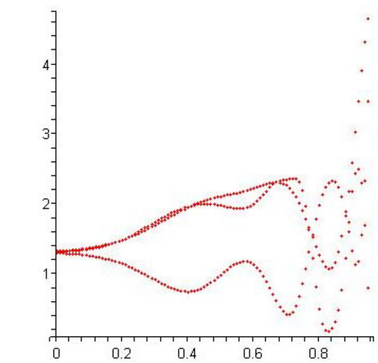

Fig.9 The orbit $(\alpha, x_2(n))$

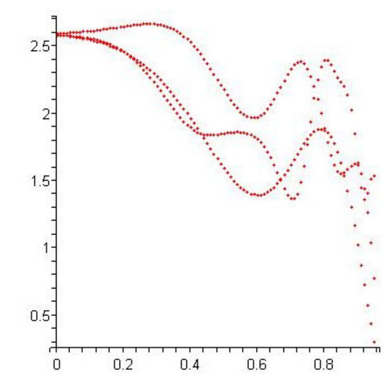

Fig.10 The orbit $(\alpha, p(n))$

For the parameter values $\alpha_1 = 0.6$, $\alpha_2 = 0.9$, $\beta_1 = 0.5$, $\beta_2 = 0.2$, $c = 2$ and $q = 3$, the solutions of the system (48) in relation to the bifurcation parameter $a = \alpha$ are displayed in the following figures: Fig. 11 The orbit $(\alpha, x_1(n))$, Fig. 12 The orbit $(\alpha, x_2(n))$, Fig. 13 The orbit $(\alpha, p(n))$.

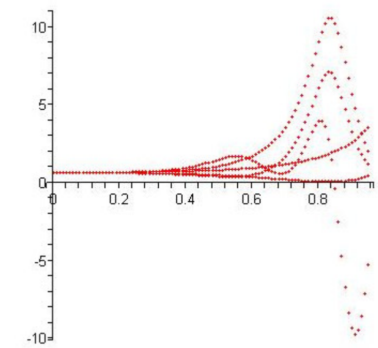

Fig.11 The orbit $(\alpha, x_1(n))$

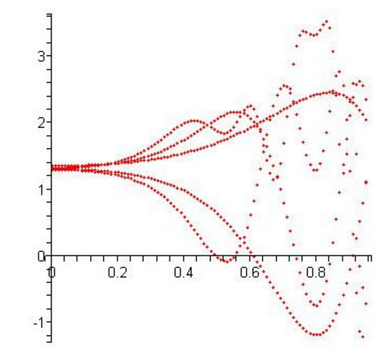

Fig.12 The orbit $(\alpha, x_2(n))$

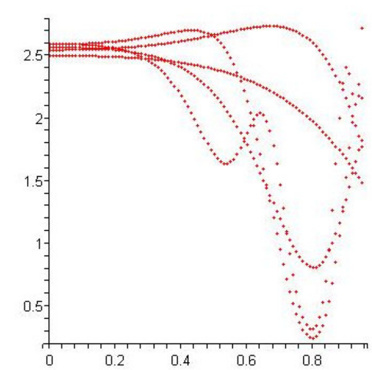

Fig.13 The orbit $(\alpha, p(n))$

These graphics justify the behavior of the model's solutions as obtained in the theoretical section.

5 Conclusion

In this paper, an Internet congestion control, discrete model with one link and two sources has been studied. We have considered the parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$ as being positive, subunitary, positive numbers. The convenient choice of parameter k has determined the conditions for the existence of a Neimark-Sacker bifurcation. The analysis has been carried out for $q_1 = 0, q_2 = 0, \quad q_1 = 1, q_2 = 0$ and $q_1 = 0, q_2 = 1.$

For these cases we have determined the normal forms and the Lyapunov coefficients. We have performed numerical simulations for the cases when the parameters k_1 and k_2 depend on $\alpha_1, \beta_1, \alpha_2, \beta_2$ and the real parameter a. We have determined the conditions for the existence of a flip bifurcation. The numerical simulations confirm the theoretical results

The analysis can be carried out in a similar manner for dynamic systems with discrete time and delay, with more sources.

Acknowledgement: The research was supported by the Grant No. 1085/2008, The National University Research Council from the Ministry of Education and Research of Romania.

References:

[1] Chunguang L., Guanrong C., Xiaofeng L., Juebang Y., - *Hopf bifurcation in an Internet congestion control model* Haos, Solitons and Fractals 19, pp. 853-862, 2004;

[2] Kelly FP., Maulloo A., Tan DKH. - *Rate control in communication networks: shadow prices, proportional fairness, and stability* J Oper. Res. Soc. 49:237–52, 1998;

[3] Kuznetsov Y.A., - *Elements of Applied Bifurcations Theory*, Springer Verlag, New_York, 1995;

[4] Mircea G., Neamțu M., Opriş D., - *Hopf bifurcations for dynamical systems with time delay and application,* Mirton Publishing House, Timişoara, 2004;

[5] Mircea G., Opris D.- *Internet model with n access link and feedback delay* – Economy Informatics, Vol. III, Number 1/2003, ISSN 1582-7941, Published by Economy Informatics Department and INFOREC Association, with the support of Education and Research Ministry, Economic Publishing House, Bucharest, pp.78-83, 2003;

[6] Mircea G. - *Internet congestion control model,* Proceedings of the 9th WSEAS International Conference on Mathematics and Computers in Business and Economics, Bucharest, Romania, pp.158-162, 2008;

[7] Mircea G., Muntean M.- *Numerical simulation for the Hopf bifurcation in TCP systems with continuous distributed network delays* – WSEAS International Conference on European Computing Conference, Athens, Greece, Lecture Notes in Electrical Engineering, vol 27, Springer-Verlag, pp.609-616, 2009.

[8] Mircea G., Opris D. - *Neimark-Saker bifurcation in a discrete time dynamic system for Internet Congestion* -Proceedings of the 10th WSEAS International Conference on Mathematics and Computers in Business and Economics, Prague, Czech Republic, pp.186-191, 2009;

[9] Neamtu M., Mircea G., Opris D. - *A discrete IS-LM model with tax ravenues,* Proceedings of the 10th WSEAS International Conference on Mathematics and Computers in Business and Economics, Prague, Czech Republic, pp.171-176, 2009;

[10] Planas-Garcia M. - *Internet congestion control model,* WSEAS Transaction on Mathematics, Issue 1, vol. 7, January 2008;

[11] Ugail H., Raja N., Satnoianu R., - *Method of Numerical Simulation of Stable Structure,* WSEAS Transaction on Mathematics, Issue 9, vol. 5, Sept. 2006;

[12] Yang H.Y., Tian Y.P., - *Hopf bifurcation in REM algorithm with communication delay* - Haos, Solitons and Fractals 25, pp. 1093-1105, 2005;

[13] Zhihong M., Xinghuo Y. - *Modelling and Stability Analysis of Nonlinear TCP Dynamics with RED*- WSEAS Transaction on Mathematics, Issue 8, vol. 6, August 2007;