The study of some discrete IS-LM models with tax revenues and time delay

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Abstract: In this paper, we study some discrete IS-LM models with tax revenues and time delay. Considering its parameters as variables, we analyze the existence of the Neimark-Sacker bifurcation. We find the normal form and its Lyapunov coefficient. For each considered model, using programs in Maple 11 we make some numerical simulations that verify the theoretical results.

Key–Words: IS-LM model, tax revenues, Neimark-Sacker bifurcation.

1 Introduction

Recently, there has been great interest in dynamical characteristics of economic, biologic, informatics models with time delay, [6], [8], [9], [10], [12], [14], [15]. In [2], the authors consider an IS-LM model with delayed taxation revenues, which is augmented by a government budget constraint in the tradition of the well known Schinasi paper [13]. Varying the length of lag and applying the "stability switch criteria", they proved that the equilibrium may lose or gain its local stability and that the existence of limit cycles generated by subcritical and supercritical Hopf bifurcation is obtained. In [11] an IS-LM model with the same lag in the tax revenues and the capital accumulation equation is presented. The authors analyzed the quantitative behavior of the model via the Hopf bifurcation of stability switch criteria. In [4] an IS-LM model with distributed tax collection lag is considered offering an explanation of the multiperiodicity and irregularity in business cycles.

Almost all real economic processes have the state variables defined at different moments, thus the discrete models are important in obtaining the practical results. In the present paper, we will use the discretization method from [3].

After this introduction, in Section 2 we study the discrete IS-LM model with tax revenues. This model is obtained by the discretization of the continuous IS-LM model with tax revenues. The functions that describe the model are the investment function I and the liquidity function L. Section 3, respectively Section 4 presents the qualitative analysis of a IS-LM discrete model with time delay and respectively with exponential density distribution. Also, some numerical simulations are performed in sections 2, 3, and 4, using programs in Maple 11. Concluding comments are presented in Section 5.

We start from the following dynamical system [2]:

$$\begin{aligned}\dot{Y}(t) &= a[I(Y(t), R(t)) - S(Y^D(t)) - T(t) + g]\\ \dot{R}(t) &= b(L(Y(t), R(t)) - M(t))\\ \dot{M}(t) &= c(g - T(t)),\end{aligned} \tag{1}$$

with: the national income $Y(t)$, the interest rate R, the real money supply M (prices are fixed at unity). The tax revenues $T(t)$ are given by:

$$T(t) = (1-\varepsilon)qY(t) + \varepsilon q \int_0^t k(s)Y(t-s)ds,$$

the disposable income given by:

$$Y^D(t) = Y(t) - T(t),$$

the tax rate q and the saving function $S(Y^D(t))$.

Further on, we assume that investment I is a function of national income Y and the interest rate R, i.e. $I(Y, R)$, the liquidity preference function L is a function of national income Y and the interest rate, i.e. $L(Y, R)$, g represents the government expenditure, a, b, c represent respectively the speed of adjustment in the goods market, depreciation rate of the interest rate, the speed of adjustment in the money market.

The following is assumed about the derivatives:

$$I_Y = \frac{\partial I(Y,R)}{\partial Y} > 0, I_R = \frac{\partial I(Y,R)}{\partial R} > 0,$$
$$L_Y = \frac{\partial L(Y,R)}{\partial Y} > 0, L_R = \frac{\partial L(Y,R)}{\partial R} > 0,$$
$$S_Y = \frac{\partial S}{\partial Y} = s, 0 < s < 1, q \in (0,1).$$

Let $k : \mathbb{R}_+ \to \mathbb{R}_+$, be the density of distribution that verifies the following properties:

$$k(s) \geq 0, s \in \mathbb{R}_+, \int_0^\infty k(s)ds = 1, \int_0^\infty sk(s)ds < \infty$$

and it is called kernel.

If k is the Dirac density of distribution, then:

$$\int_0^t k(s)Y(t-s)ds = Y(t-\tau)$$

where $\tau \geq 0$ is the delay.

If k is the exponential density of distribution, then:

$$k(s) = \alpha e^{-\alpha s}, \alpha > 0$$

and the variable $z(t) = \int_0^\infty k(s)Y(t-s)ds$ satisfies the differential equation:

$$\dot{z}(t) = \alpha(Y(t) - z(t)).$$

If k is the Erlang density of distribution:

$$k(s) = \alpha^2 s e^{-\alpha s}, \alpha > 0$$

then $z(t) = \int_0^\infty k(s)Y(t-s)ds$ satisfies the following relations:

$$\dot{z}(t) = \alpha(u(t) - z(t))$$
$$\dot{u}(t) = \alpha(Y(t) - u(t))$$
$$u(t) = \alpha \int_0^\infty e^{-\alpha s}Y(t-s)ds$$

With these assumptions and with the function $S(Y^D(t)) = s(Y(t) - T(t))$, $s \in (0,1)$, dynamic system (1) is given by:

$$\begin{aligned}
&\dot{Y}(t) = a[I(Y(t),R(t)) - s(1-q)Y(t) - qY(t) + g]\\
&\dot{R}(t) = b(L(Y(t),R(t)) - M(t))\\
&\dot{M}(t) = c(g - qY(t)),
\end{aligned} \tag{2}$$

the dynamic model with continuous time;

$$\begin{aligned}
&\dot{Y}(t) = a(I(Y(t),R(t)) - (s+(1-\varepsilon)(1-s)q)Y(t) -\\
&-(1-s)\varepsilon qY(t-\tau) + g)\\
&\dot{R}(t) = b(L(Y(t),R(t)) - M(t))\\
&\dot{M}(t) = c(g - (1-\varepsilon)qY(t) - \varepsilon qY(t-\tau)),
\end{aligned} \tag{3}$$

the dynamic model with continuous time and delay;

$$\begin{aligned}
&\dot{Y}(t) = a[I(Y(t),R(t)) - (s+(1-\varepsilon)(1-s)q)Y(t) -\\
&-(1-s)\varepsilon qz(t) + g]\\
&\dot{R}(t) = b(L(Y(t),R(t)) - M(t))\\
&\dot{M}(t) = c(g - (1-\varepsilon)qY(t) - \varepsilon qz(t))\\
&\dot{z}(t) = \alpha((1-\varepsilon)qY(t) - (1-\varepsilon q)z(t))
\end{aligned} \tag{4}$$

the dynamic model with continuous time and exponential density of distribution;

$$\begin{aligned}
&\dot{Y}(t) = a[I(Y(t),R(t)) - (s+(1-\varepsilon)(1-s)q)Y(t) -\\
&-(1-s)\varepsilon qz(t) + g]\\
&\dot{R}(t) = b(L(Y(t),R(t)) - M(t))\\
&\dot{M}(t) = c(g - (1-\varepsilon)qY(t) - \varepsilon qz(t))\\
&\dot{z}(t) = \alpha(u(t) - z(t))\\
&\dot{u}(t) = \alpha((1-\varepsilon)qY(t) + \varepsilon qz(t) - u(t))
\end{aligned} \tag{5}$$

the dynamic model with continuous time and Erlang density of distribution.

The model (3) with the functions:

$$I(Y,R) = a_1 Y^{a_2} R^{a_3}, L(Y,R) = b_1 Y + b_2 Y^{b_3}(R - b_4)^{-b_5}$$

was analyzed in [12].

The discrete models are obtained by the discretization of the models (2), (3), (4), (5) with the adjustment coefficient different from the ones above. For simplicity, we will employ the same coefficients:

$$\begin{aligned}
&Y(n+1) = Y(n) + a(I(Y(n),R(n)) -\\
&-s(1-q)Y(n) - qY(n) + g)\\
&R(n+1) = R(n) + b(L(Y(n),R(n)) - M(n))\\
&M(n+1) = M(n) + c(g - qY(n)), n \in N
\end{aligned} \tag{6}$$

the discrete IS-LM model;

$$\begin{aligned}&Y(n+1)=Y(n)+a(I(Y(n),R(n))-(s+(1-\varepsilon)\\&(1-s)q)Y(n)-(1-s)\varepsilon qY(n-p)+g)\\&R(n+1)=R(n)+b(L(Y(n),R(n))-M(n))\\&M(n+1)=M(n)+c(g-(1-\varepsilon)qY(n)-\\&-\varepsilon qY(n-p)),\end{aligned} \tag{7}$$

$n\in N, p\in N, p\geq 1$, the discrete IS-LM model with delay;

$$\begin{aligned}&Y(n+1)=Y(n)+a(I(Y(n),R(n))-(s+(1-\varepsilon)\\&(1-s)q)Y(n)-(1-s)\varepsilon qz(n)+g)1\\&R(n+1)=R(n)+b(L(Y(n),R(n))-M(n))\\&M(n+1)=M(n)+c(g-(1-\varepsilon)qY(n)-\varepsilon qz(n))\\&z(n+1)=z(n)+\alpha((1-\varepsilon)qY(n)-\\&-(1-\varepsilon q)z(n))\end{aligned} \tag{8}$$

the discrete IS-LM model with delay corresponding to (4);

$$\begin{aligned}&Y(n+1)=Y(n)+a[I(Y(n),R(n))-(s+(1-\varepsilon)\\&(1-s)q)Y(n)-(1-s)\varepsilon qz(n)+g]\\&R(n+1)=R(n)+b(L(Y(n),R(n))-M(n))\\&M(n+1)=M(n)+c(g-(1-\varepsilon)qY(n)-\varepsilon qz(n))\\&z(n+1)=z(n)+\alpha(u(n)-z(n))\\&u(n+1)=u(n)+\alpha((1-\varepsilon)qY(n)+\varepsilon qz(n)-u(n))\end{aligned}$$

the discrete IS-LM model with delay corresponding to (5);

2 The analysis of the discrete IS-LM model with tax revenues (6)

We analyze system (6). Because $I_R<0$ we assume that:

$$\lim_{R\to 0^+} I(\frac{g}{q},R)\geq\frac{s(1-q)}{q}g,$$
$$\lim_{R\to\infty} I(\frac{g}{q},R)\geq\frac{s(1-q)}{q}g.$$

Under the previous conditions, the model (6) has the equilibrium state (Y_0, R_0, M_0), so that: $Y_0=\frac{g}{q}$, R_0 is the solution of $I(\frac{g}{q},R)=\frac{s(1-q)}{q}g$ and $M_0=L(Y_0,R_0)$.

The linearization of system (6) in the neighborhood of the equilibrium yields:

$$\begin{aligned}v_1(n+1)&=a_{11}v_1(n)+a_{12}v_2(n)+a_{13}v_3(n)\\v_2(n+1)&=a_{21}v_1(n)+a_{22}v_2(n)+a_{23}v_3(n)\\v_3(n+1)&=a_{31}v_1(n)+a_{32}v_2(n)+a_{33}v_3(n)\end{aligned} \tag{9}$$

where

$$\begin{aligned}&a_{11}=1+a(I_1-s(1-q)-q), a_{12}=aI_2, a_{13}=0,\\&a_{21}=bL_1, a_{22}=1+bL_2, a_{23}=-b,\\&a_{31}=-cq, a_{32}=0, a_{33}=1\end{aligned} \tag{10}$$

and $I_1=I_Y(Y_0,R_0)$, $I_2=I_R(Y_0,R_0)$, $L_1=L_Y(Y_0,R_0)$, $L_2=L_R(Y_0,R_0)$.

The characteristic equation of system (9) is given by:

$$\lambda^3-A_1\lambda^2+A_2\lambda-A_3=0 \tag{11}$$

where

$$\begin{aligned}&A_1=1+a_{11}+a_{22},\\&A_2=a_{11}a_{22}-a_{12}a_{21}+a_{11}+a_{22},\\&A_3=a_{11}a_{22}-a_{12}a_{21}+a_{12}a_{23}a_{31};\end{aligned} \tag{12}$$

From (10) and (12), we get:

$$\begin{aligned}&A_1=3+(I_1-s(1-q)-q)a+L_2b,\\&A_2=3+2(I_1-s(1-q)-q)a+2L_2b-abs(1-q)L_2,\\&A_3=1+(I_1-s(1-q)-q)a-L_2b-\\&-abs(1-q)L_2+abcqI_2.\end{aligned} \tag{13}$$

According to the Schur criterion [7], equation (11) has the roots in modulus less than 1, if and only if:

$$|A_3|<1, |A_1-A_3|<2, 1-A_2+A_3(A_1-A_3)>0. \tag{14}$$

Then, if the parameters of the model satisfy the relations (14), the equilibrium point (Y_0,R_0,M_0) is asymptotically stable.

We analyze the roots of equation (11) considering the parameters a and b as fixed and the parameter c as variable. We denote by:

$$d_1=a_{11}a_{22}-a_{12}a_{21}, d_2=qa_{12}a_{23}. \tag{15}$$

From (13) and (15), we have:

$$A_3=d_1-cd_2.$$

The following proposition holds:

Proposition 1 *(i) If the parameters a, b, c satisfy the relations:*

$$|d_1 - cd_2| < 0, |A_1 - d_1 + cd_2| < 2, \quad (16)$$

$$1 - A_2 + d_1(A_1 - d_1) + d_2(2d_1 - A_1)c - d_2^2c^2 = 0, \quad (17)$$

then equation (11) has one root in modulus less than 1 and two complex roots in modulus equal to 1.

(ii) If c_0 is one solution of (17) then for $c = c_0+\alpha$, there is $\alpha > 0$ sufficiently small so that the roots of equation (11) are in modulus less than 1; for $c = c_0 - \alpha$ equation (11) has one root in modulus less than 1 and the complex conjugate roots are in modulus equal to 1.

(iii) The value c_0 is a Neimark-Sacker bifurcation for equation (11).

The proof of the proposition results from (11) and from the definition of the Neimark-Sacker bifurcation [7].

In what follows, using the method from [7], [12], we find the normal form of system (6) for the Neimark-Sacker bifurcation given by c_0.

We consider A the matrix of the linear system (9) with the coefficients (10) and $c = c_0 + \alpha$, where c_0 satisfies (17) and $|\alpha|$ is sufficiently small.

We have:

Proposition 2 *(i) The eigenvector corresponding to the eigenvalue μ is the nontrivial solution of the system $Al = \mu l$ and has the components:*

$$l_1 = -a_{12}(\mu - a_{33}), l_2 = -(\mu - a_{11})(\mu - a_{33}),$$
$$l_3 = -a_{12}a_{31}; \quad (18)$$

(ii) The eigenvector corresponding to the eigenvalue $\bar{\mu}$ is the nontrivial solution of the system $A^Tm = \bar{\mu}m$ and has the components:

$$m_1 = \frac{\bar{\mu} - a_{22}}{a_{12}V}, m_2 = \frac{1}{V}, m_3 = \frac{a_{23}}{(\bar{\mu} - a_{33})V}, \quad (19)$$

where $V = \frac{\bar{\mu} - a_{22}}{a_{12}}\bar{l}_1 + \bar{l}_2 + \frac{a_{23}}{\bar{\mu} - a_{33}}\bar{l}_3$.

Moreover, the relation $l_1\bar{m}_1 + l_2\bar{m}_2 + l_3\bar{m}_3 = 1$ holds.

We denote by:

$$\begin{aligned}&I_{20} = I_{YY}(Y_0, R_0), I_{11} = I_{YR}(Y_0, R_0),\\ &I_{02} = I_{RR}(Y_0, R_0),\\ &I_{30} = I_{YYY}(Y_0, R_0), I_{21} = I_{YYR}(Y_0, R_0),\\ &I_{12} = I_{YRR}(Y_0, R_0), I_{03} = I_{RRR}(Y_0, R_0),\end{aligned} \quad (20)$$

$$\begin{aligned}&L_{20} = L_{YY}(Y_0, R_0), L_{11} = L_{YR}(Y_0, R_0),\\ &L_{02} = L_{RR}(Y_0, R_0),\\ &L_{30} = L_{YYY}(Y_0, R_0), L_{21} = L_{YYR}(Y_0, R_0),\\ &L_{12} = L_{YRR}(Y_0, R_0), L_{03} = L_{RRR}(Y_0, R_0),\end{aligned} \quad (21)$$

and

$$\begin{aligned}&B_1 = \begin{pmatrix} I_{20} & I_{11} \\ I_{11} & I_{02}\end{pmatrix}, B_2 = \begin{pmatrix} L_{20} & L_{11} \\ L_{11} & L_{02}\end{pmatrix}\\ &C_1 = \begin{pmatrix} I_{30} & I_{21} \\ I_{21} & I_{12}\end{pmatrix}, D_1 = \begin{pmatrix} I_{21} & I_{12} \\ I_{12} & I_{03}\end{pmatrix}\\ &C_2 = \begin{pmatrix} L_{30} & L_{21} \\ L_{21} & L_{12}\end{pmatrix}, D_2 = \begin{pmatrix} L_{21} & L_{12} \\ L_{12} & L_{03}\end{pmatrix}.\end{aligned} \quad (22)$$

We consider $l = (l_1, l_2)^T$, $m = (m_1, m_2)^T$, where l_1, l_2, m_1, m_2 are given by (18) and (19) and

$$\begin{aligned}&B^i(l, l) = l^TB_il, B^i(l, \bar{l}) = l^TB_i\bar{l},\\ &B^i(\bar{l}, \bar{l}) = \bar{l}^TB_i\bar{l}, i = 1, 2\\ &C^i(l, l, \bar{l}) = l^T(l_iC_i + l_2D_i)\bar{l}, i = 1, 2,\end{aligned}$$

$$\begin{aligned}&g_{20} = (B^1(l, l), B^2(l, l))m, g_{11} = (B^1(l, \bar{l}), B^2(l, \bar{l}))m,\\ &g_{02} = (B^1(\bar{l}, \bar{l}), B^2(\bar{l}, \bar{l}))m,\end{aligned} \quad (23)$$

$$\begin{aligned}&h_{i20} = B^i(l, l) - ((B^1(l, l), B^2(l, l))m)l_i-\\ &- ((B^1(l, l), B^2(l, l))\bar{m})\bar{l}_i, i = 1, 2\\ &h_{320} = -((B^1(l, l), B^2(l, l))m)l_3-\\ &- ((B^1(l, l), B^2(l, l))\bar{m})\bar{l}_3,\end{aligned}$$

$$\begin{aligned}&h_{i11} = B^i(l, \bar{l}) - ((B^1(l, \bar{l}), B^2(l, \bar{l}))m)l_i-\\ &- ((B^1(l, \bar{l}), B^2(l, \bar{l}))\bar{m})\bar{l}_i, i = 1, 2\\ &h_{311} = -((B^1(l, \bar{l}), B^2(l, \bar{l}))m)l_3-\\ &- ((B^1(l, \bar{l}), B^2(l, \bar{l}))\bar{m})\bar{l}_3,\end{aligned}$$

$$\begin{aligned}&h_{i02} = B^i(\bar{l}, \bar{l}) - ((B^1(\bar{l}, \bar{l}), B^2(\bar{l}, \bar{l}))m)l_i-\\ &- ((B^1(\bar{l}, \bar{l}), B^2(\bar{l}, \bar{l}))\bar{m})\bar{l}_i, i = 1, 2\\ &h_{302} = -((B^1(\bar{l}, \bar{l}), B^2(\bar{l}, \bar{l}))m)l_3-\\ &- ((B^1(\bar{l}, \bar{l}), B^2(\bar{l}, \bar{l}))\bar{m})\bar{l}_3.\end{aligned}$$

We denote by:

$$A(\mu^2) = \begin{pmatrix} \mu^2 - a_{11} & -a_{12} & 0 \\ -a_{21} & \mu^2 - a_{22} & -a_{23} \\ -a_{31} & 0 & \mu^2 - a_{33} \end{pmatrix},$$

$$A(1) = \begin{pmatrix} 1 - a_{11} & -a_{12} & 0 \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & 0 & 1 - a_{33} \end{pmatrix}$$

$$A(\bar{\mu}^2) = \begin{pmatrix} \bar{\mu}^2 - a_{11} & -a_{12} & 0 \\ -a_{21} & \bar{\mu}^2 - a_{22} & -a_{23} \\ -a_{31} & 0 & \bar{\mu}^2 - a_{33} \end{pmatrix}$$

and

$$\begin{aligned} w_{20} &= A(\mu^2)^{-1}h_{20}, w_{11} = A(1)^{-1}h_{11}, \\ w_{02} &= A(\bar{\mu}^2)^{-1}h_{02}, \end{aligned} \tag{24}$$

where

$$h_{20} = (h_{120}, h_{220}, h_{320})^T,$$
$$h_{11} = (h_{111}, h_{211}, h_{311})^T, h_{02} = (h_{102}, h_{202}, h_{302})^T$$

and

$$\begin{aligned} g_{21} &= (B^1(\bar{l}, w_{20}), B^2(\bar{l}, w_{20}))m + 2(B^1(l, w_{11}), \\ & B^2(l, w_{11}))m + (C^1(l, l, \bar{l}), C^2(l, l, \bar{l}))l. \end{aligned} \tag{25}$$

Using the method from [7], [12], for the determination of the normal form, we obtain:

Proposition 3 *(i) The normal form of system (6) is:*

$$\begin{aligned} z(n+1) &= \mu z(n) + \frac{1}{2}g_{20}z(n)^2 + g_{11}z(n)\bar{z}(n) + \\ &+ \frac{1}{2}g_{02}\bar{z}(n)^2 + \frac{1}{2}g_{21}z(n)^2\bar{z}(n), \end{aligned} \tag{26}$$

where $z(n) \in C$ and the coefficients are given by (23) and (25);

(ii) System (6) in the neighborhood of the state equilibrium (Y_0, R_0, M_0) is:

$$\begin{aligned} Y(n) &= Y_0 + l_1 z(n) + \bar{l}_1\bar{z}(n) + \frac{1}{2}w_{120}z(n)^2 + \\ &+ w_{111}z(n)\bar{z}(n) + + \frac{1}{2}w_{102}\bar{z}(n)^2 \\ R(n) &= R_0 + l_2 z(n) + \bar{l}_2\bar{z}(n) + \frac{1}{2}w_{220}z(n)^2 + \\ &+ w_{211}z(n)\bar{z}(n) + \frac{1}{2}w_{202}\bar{z}(n)^2 \\ M(n) &= M_0 + l_3 z(n) + \bar{l}_3\bar{z}(n) + \frac{1}{2}w_{320}z(n)^2 + \\ &+ w_{311}z(n)\bar{z}(n) + \frac{1}{2}w_{302}\bar{z}(n)^2 \end{aligned} \tag{27}$$

where $z(n)$ is one solution of (26) and the coefficients are given by (24);

(iii) The Lyapunov coefficient associated to the normal form (26) is given by:

$$\begin{aligned} C_1(\alpha) &= \frac{g_{20}(\alpha)g_{11}(\alpha)(\bar{\mu}(\alpha) - 3 - 2\mu(\alpha))}{2(\mu(\alpha) - \bar{\mu}(\alpha))(\bar{\mu}(\alpha) - 1)} + \\ &+ \frac{|g_{11}(\alpha)|^2}{1 - \bar{\mu}(\alpha)} + \frac{|g_{02}(\alpha)|^2}{2(\mu^2(\alpha) - \bar{\mu}(\alpha))} + \frac{g_{21}(\alpha)}{2}; \end{aligned} \tag{28}$$

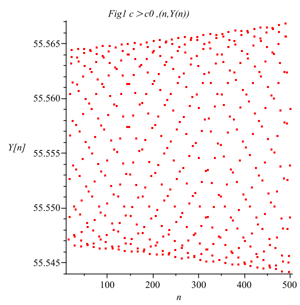
(iv) If $\theta_0 = arg(\mu(0))$, $L_0 = Re(e^{-i\theta}C_1(0))$ and $L_0 < 0 (> 0)$ in the neighborhood of the equilibrium state $(Y_0,\ R_0,\ M_0)$, then there is a stable (unstable) limit cycle.

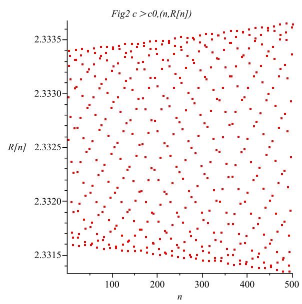
The numerical simulation was made using a program in Maple 11. For:

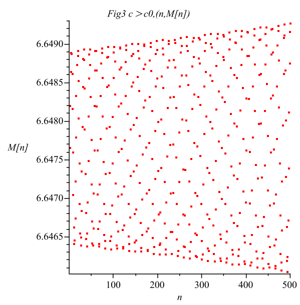
$$I = a_1 Y^{a_2} R^{-a_3}, L = b_1 Y + b_2(R - b_3)^{-b_4}$$

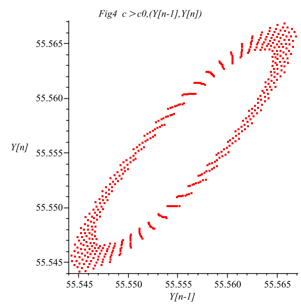
where $a_1 = 0.38$, $a_2 = 1.05$, $a_3 = 0.83$, $b_1 = 0.07$, $b_2 = 1$, $b_3 = 0.003$, $b_4 = -1.2$, $s = 0.5$, $q = 0.18$, $g = 10$, $a = 0.96$, $b = 0.8$, we obtain the following results: $Y_0 = 5.55$, $R_0 = 2.33$, $M_0 = 6.64$, $c_0 = 0.3560$. The Lyapunov coefficient is $L_0 = 0.337$ and in the neighborhood of the equilibrium state there is an unstable limit cycle.

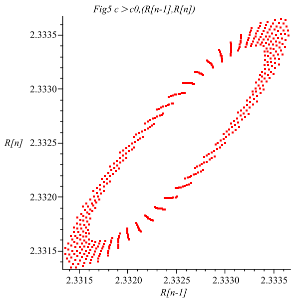
For $c = c_0 + \beta$, $\beta = 0.001$, the following trajectories are displayed: $(n, Y(n))$, $(n, R(n))$, $(n, M(n))$ in Fig 1, Fig 2, Fig 3. In the figures Fig 4, Fig 5, Fig 6 are displayed: $(Y(n-1), Y(n))$, $(R(n-1), R(n))$, $(M(n-1), M(n))$,

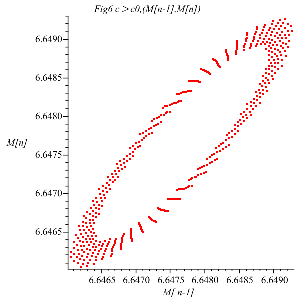




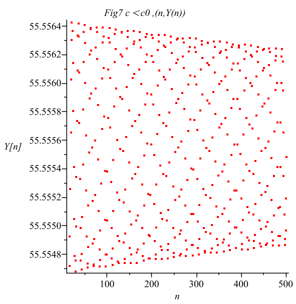


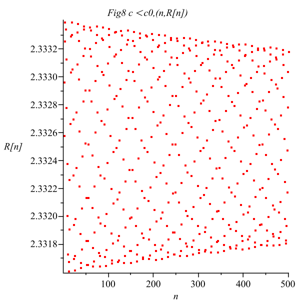


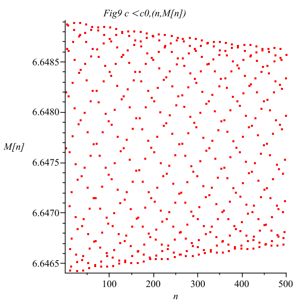


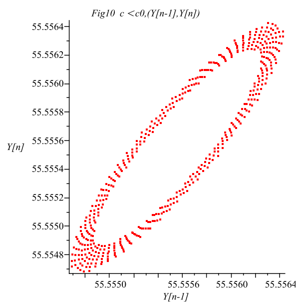


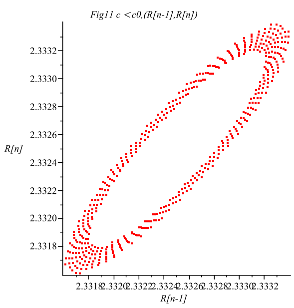
For $c = c_0 - \beta$, $\beta = 0.001$ in Fig 7, Fig 8, Fig 9 the following trajectories: $(n, Y(n))$, $(n, R(n))$, $(n, M(n))$ are displayed. In the figures Fig 10, Fig 11, Fig 12 are displayed: $(Y(n-1), Y(n))$, $(R(n-1), R(n))$, $(M(n-1), M(n))$,

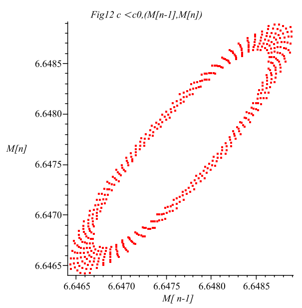












These graphics justify the behavior of the model's solutions as obtained in the theoretical section.

3 The analysis of the discrete IS-LM dynamic model with time delay (7)

We analyze system (7) with $p = 1$. System (7) become:

$$\begin{aligned}
&u(n+1) = Y(n)\\
&Y(n+1) = Y(n) + a[I(Y(n), R(n)) + g-\\
&-(s + (1-\varepsilon)(1-s)q)Y(n) - (1-s)\varepsilon q u(n)]\\
&R(n+1) = R(n) + b(L(Y(n), R(n)) - M(n))\\
&M(n+1) = M(n) + c(g - (1-\varepsilon)qY(n)-\\
&-\varepsilon q u(n)), n \in N.
\end{aligned} \tag{29}$$

In the hypothesis from Section 2, system (29) has the equilibrium state $(u_0,\ Y_0,\ R_0,\ M_0)$, where $u_0 = Y_0$, $Y_0 = \frac{g}{q}$, R_0 is the solution of $I(\frac{g}{q}, R) = \frac{(1-q)s}{q} g$ and $M_0 = L(Y_0, R_0)$.

The linearized system of (29) in the neighborhood of the equilibrium $(u_0,\ Y_0,\ R_0,\ M_0)$ is:

$$\begin{aligned}
v_0(n+1) &= a_{00}v_0(n) + a_{01}v_1(n) + a_{02}v_2(n) + a_{03}v_3(n)\\
v_1(n+1) &= a_{10}v_0(n) + a_{11}v_1(n) + a_{12}v_2(n) + a_{13}v_3(n)\\
v_2(n+1) &= a_{20}v_0(n) + a_{21}v_1(n) + a_{22}v_2(n) + a_{23}v_3(n)\\
v_3(n+1) &= a_{30}v_0(n) + a_{31}v_1(n) + a_{32}v_2(n) + a_{33}v_3(n)
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
&a_{00} = 0, a_{01} = 1, a_{02} = 0, a_{03} = 0,\\
&a_{10} = -a(1-s)\varepsilon q,\\
&a_{11} = 1 + a(I_1 - (s + (1-\varepsilon)(1-s)q)),\\
&a_{12} = aI_2, a_{13} = 0, a_{20} = 0,\\
&a_{21} = bL_1, a_{22} = 1 + bL_2, a_{23} = -1,\\
&a_{30} = -c\varepsilon q, a_{31} = -c(1-\varepsilon)q, a_{32} = 0, a_{33} = 1.
\end{aligned} \tag{31}$$

The characteristic equation of system (30) is given by:

$$\lambda^4 - B_1\lambda^3 + B_2\lambda^2 - B_3\lambda - B_4 = 0 \tag{32}$$

where

$$\begin{aligned}
&B_1 = 1 + a_{11} + a_{22},\\
&B_2 = a_{11} + a_{22} + a_{11}a_{22} - a_{12}a_{21} - a_{10},\\
&B_3 = a_{11}a_{22} + a_{12}a_{23}a_{31} - a_{12}a_{21} - 1 - a_{22},\\
&B_4 = a_{10}a_{22} - a_{12}a_{23}a_{30}.
\end{aligned} \tag{33}$$

We analyze the model (29), choosing the adjustment coefficient c as function of the adjustment coefficients a, b so that:

$$B_3 = a_{11}a_{22} - a_{12}a_{23}c(1-\varepsilon)q - a_{12}a_{21} - 1 - a_{22} = 0. \quad (34)$$

From (34) we obtain:

$$c = \frac{a_{11}a_{22} - a_{12}a_{21} - 1 - a_{22}}{a_{12}a_{23}(1-\varepsilon)q}. \quad (35)$$

From (33) and (35) we have:

$$\begin{aligned} B_1 &= 1 + a_{11} + a_{22}, \\ B_2 &= a_{11} + a_{22} + a_{11}a_{22} - a_{12}a_{21} - a_{10}, \\ B_4 &= a_{10}a_{22} - \frac{\varepsilon}{1-\varepsilon}(a_{11}a_{22} - a_{12}a_{21} - 1 - a_{22}); \end{aligned} \quad (36)$$

and characteristic equation (32) becomes:

$$\lambda^4 - B_1\lambda^3 + B_2\lambda^2 - B_4 = 0 \quad (37)$$

According to the Schur criterion, equation (37) has the roots in modulus less than 1, if and only if:

$$\begin{aligned} &|B_4| < 1, |B_1| < 2(1 + B_4), \\ &(B_2 - 1 + B_4)(1 + B_4)^2 < B_1^2 B_4. \end{aligned} \quad (38)$$

Then, if the parameters of the model satisfy the relations (38) then the equilibrium point (u_0, Y_0, R_0, M_0) is asymptotically stable.

We analyze the roots of equation (37) considering the parameter a as fixed and the parameter b as variable. We denote by:

$$\begin{aligned} d_3 &= a_{10} - \frac{\varepsilon}{1-\varepsilon}(a_{11} - 2), \\ d_4 &= (a_{10} - \frac{\varepsilon}{1-\varepsilon}(a_{11} - 1))L_2 + \frac{\varepsilon}{1-\varepsilon}a_{12}L_1. \end{aligned} \quad (39)$$

From (36) and (39) we have:

$$B_4 = d_3 + bd_4.$$

The following statements hold:

Proposition 4 *(i) If the parameter b satisfies the relations:*

$$\begin{aligned} &|d_3 + bd_4| < 1, |2 + a_{11} + bL_2| < 2(1 + d_3 + bd_4), \\ &(2a_{11} + d_3 + ((1 + a_{11})L_2 - a_{12}L_1 + d_4)b)\cdot \\ &\cdot (1 + d_3 + d_4 b)^2 = (2 + a_{11} + L_2 b)^2(d_3 + d_{11}b) \end{aligned} \quad (40)$$

then equation (37) has one root in modulus less then 1 and two complex roots in modulus equal to 1.

(ii) If b_0 is one solution of (40) then for $b = b_0 + \alpha$, there is $\alpha > 0$ sufficiently small so that the roots of equation (37) are in modulus less than 1; for $b = b_0 - \alpha$ equation (37) has one root in modulus less than 1 and the complex conjugate roots are in modulus equal to 1.

(iii) The value b_0 is a Neimark-Sacker bifurcation for equation (37).

The proof of the proposition results from (38) and from the definition of the Neimark-Sacker bifurcation [7].

In what follows, using the method from [7], [12] we find the normal form of system (29) for the Neimark-Sacker bifurcation given by b_0.

We consider B the matrix of the linear system (30) with the coefficients (31) and $b = b_0 + \alpha$, where b_0 satisfies (40) and $|\alpha|$ is sufficiently small and with c given by (35).

Let $\mu = \mu(\alpha)$ one root of the characteristic equation (37).

We have:

Proposition 5 *(i) The eigenvector corresponding to the eigenvalue μ is the nontrivial solution of the system $Bl = \mu l$ and has the components:*

$$\begin{aligned} &l_0 = \frac{a_{12}a_{23}}{\mu}, l_1 = a_{12}a_{23}, l_2 = a_{23}(\mu - a_{11} - \frac{a_{10}}{\mu}), \\ &l_3 = (\mu - a_{22})(\mu - a_{11} - \frac{a_{10}}{\mu}) - a_{12}a_{21}; \end{aligned} \quad (41)$$

(ii) The eigenvector corresponding to the eigenvalue $\bar{\mu}$ is the nontrivial solution of the system $B^T m = \bar{\mu} m$ and has the components:

$$\begin{aligned} m_0 &= \frac{a_{10}(\bar{\mu} - a_{22})(\bar{\mu} - 1) + a_{12}a_{30}a_{23}}{a_{12}\bar{\mu}(\bar{\mu} - 1)V} \\ m_1 &= \frac{\bar{\mu} - a_{22}}{a_{12}V}, m_2 = \frac{1}{V}, m_3 = \frac{a_{23}}{(\bar{\mu} - 1)V} \end{aligned} \quad (42)$$

where

$$\begin{aligned} V = &\frac{(\bar{\mu} - a_{22})}{a_{12}}\bar{l}_1 + \bar{l}_2 + \frac{a_{23}}{\bar{\mu} - 1}\bar{l}_3 + \\ &+ \frac{(a_{10}(\bar{\mu} - a_{22})(\bar{\mu} - 1)) + a_{12}a_{30}a_{23}\bar{l}_0}{a_{12}\bar{\mu}(\bar{\mu} - 1)}. \end{aligned} \quad (43)$$

Moreover, the relation $l_0\bar{m}_0 + l_1\bar{m}_1 + l_2\bar{m}_2 + l_3\bar{m}_3 = 1$ holds.

We consider $l = (l_1, l_2)^T$, $m = (m_1, m_2)^T$ given by (41), (42), the formulas (22), (23), (24) and

$$\begin{aligned} h_{020} &= -((B^1(l,l), B^2(l,l))m)l_0 - \\ &- ((B^1(l,l), B^2(l,l))\bar{m})\bar{l}_0, \\ h_{011} &= -((B^1(l,\bar{l}), B^2(l,\bar{l}))m)l_0 - \\ &- ((B^1(l,\bar{l}), B^2(l,\bar{l}))\bar{m})\bar{l}_0, \\ h_{002} &= -((B^1(\bar{l},\bar{l}), B^2(\bar{l},\bar{l}))m)l_0 - \\ &- ((B^1(\bar{l},\bar{l}), B^2(\bar{l},\bar{l}))\bar{m})\bar{l}_0, \end{aligned} \quad (44)$$

Let be the matrices:

$$B(\mu^2) = \begin{pmatrix} \mu^2 & -1 & 0 & 0 \\ -a_{10} & \mu^2 - a_{11} & -a_{12} & 0 \\ 0 & -a_{21} & \mu^2 - a_{22} & -a_{23} \\ -a_{30} & -a_{31} & 0 & \mu^2 - a_{33} \end{pmatrix},$$

$$B(1) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -a_{10} & 1 - a_{11} & -a_{12} & 0 \\ 0 & -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{30} & -a_{31} & 0 & 1 - a_{33} \end{pmatrix}$$

$$B(\bar{\mu}^2) = \begin{pmatrix} \bar{\mu}^2 & -1 & 0 & 0 \\ -a_{10} & \bar{\mu}^2 - a_{11} & -a_{12} & 0 \\ 0 & -a_{21} & \bar{\mu}^2 - a_{22} & -a_{23} \\ -a_{30} & -a_{31} & 0 & \bar{\mu}^2 - a_{33} \end{pmatrix}. \quad (45)$$

and $w_{20} = B(\mu^2)^{-1}h_{20}$, $w_{11} = B(1)^{-1}h_{11}$, $w_{02} = B(\bar{\mu}^2)^{-1}h_{02}$ where

$$\begin{aligned} h_{20} &= (h_{020}, h_{120}, h_{220}, h_{320})^T, \\ h_{11} &= (h_{011}, h_{111}, h_{211}, h_{311})^T, \\ h_{02} &= (h_{002}, h_{102}, h_{202}, h_{302})^T. \end{aligned} \quad (46)$$

and

$$\begin{aligned} g_{21} &= (B^1(\bar{l}, w_{20}), B^2(\bar{l}, w_{20}))m + \\ &+ 2(B^1(l, w_{11}), B^2(l, w_{11}))m + \\ &+ (C^1(l,l,\bar{l}), C^2(l,l,\bar{l}))l. \end{aligned} \quad (47)$$

Using the method from [7], [12] for the determination the normal form we obtain:

Proposition 6 *(i) The normal form of system (7) is given by (26) and the coefficients are given by (23) and (47)*

(ii) System (7) in the neighborhood of the state equilibrium (u_0, Y_0, R_0, M_0) is:

$$\begin{aligned} u(n) &= u_0 + l_0 z(n) + \bar{l}_1 \bar{z}(n) + \frac{1}{2} w_{020} z(n)^2 + \\ &+ w_{011} z(n)\bar{z}(n) + \frac{1}{2} w_{002} \bar{z}(n)^2 \\ Y(n) &= Y_0 + l_1 z(n) + \bar{l}_1 \bar{z}(n) + \frac{1}{2} w_{120} z(n)^2 + \\ &+ w_{111} z(n)\bar{z}(n) + \frac{1}{2} w_{102} \bar{z}(n)^2 \\ R(n) &= R_0 + l_2 z(n) + \bar{l}_2 \bar{z}(n) + \frac{1}{2} w_{220} z(n)^2 + \\ &+ w_{211} z(n)\bar{z}(n) + \frac{1}{2} w_{202} \bar{z}(n)^2 \\ M(n) &= M_0 + l_3 z(n) + \bar{l}_3 \bar{z}(n) + \frac{1}{2} w_{320} z(n)^2 + \\ &+ w_{311} z(n)\bar{z}(n) + \frac{1}{2} w_{302} \bar{z}(n)^2 \end{aligned} \quad (48)$$

where $z(n)$ is one solution of the normal form for the system (7).

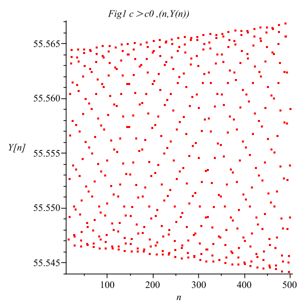
(iii) The Lyapunov coefficient associated to the normal form is given by (28).

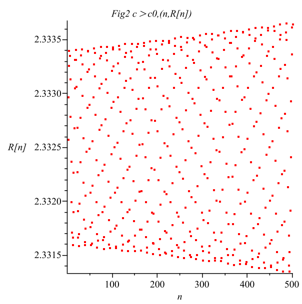
The numerical simulation was made using a program in Maple 11. For:

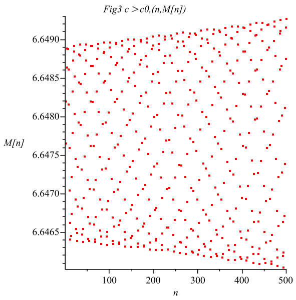
$$I = a_1 Y^{a_2} R^{-a_3}, L = b_1 Y + b_2 (R - b_3)^{-b_4}$$

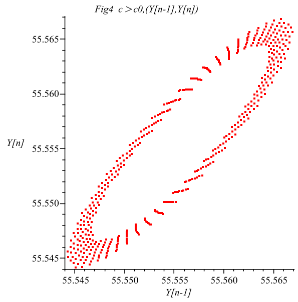
where $a_1 = 0.38$, $a_2 = 1.05$, $a_3 = 0.83$, $b_1 = 0.07$, $b_2 = 1$, $b_3 = 0.003$, $b_4 = -1.2$, $s = 0.08$, $q = 0.18$, $g = 1$, $a = 0.96$, $b = 0.8$, we obtain the following results: $Y_0 = 5.55$, $R_0 = 1.876043$, $M_0 = 2.512414$, $c_0 = -70.377794$. The Lyapunov coefficient is $L_0 = 2.590623$ and in the neighborhood of the equilibrium state there is an unstable limit cycle.

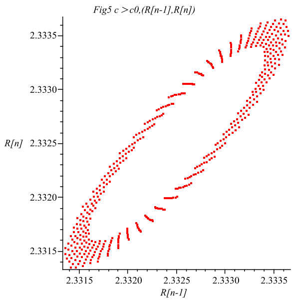
For $c = c_0 + \beta$, $\beta = 0.001$, the following trajectories are displayed: $(n, Y(n))$, $(n, R(n))$, $(n, M(n))$ in Fig 1, Fig 2, Fig 3. In the figures Fig 4, Fig 5, Fig 6 are displayed: $(Y(n-1), Y(n))$, $(R(n-1), R(n))$, $(M(n-1), M(n))$,

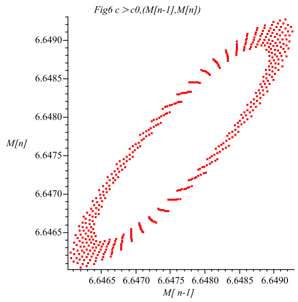




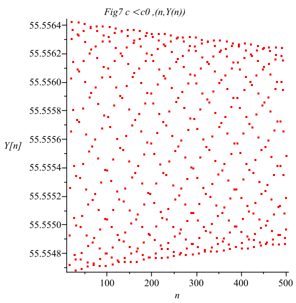


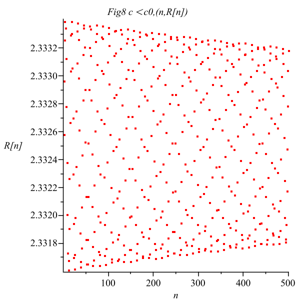


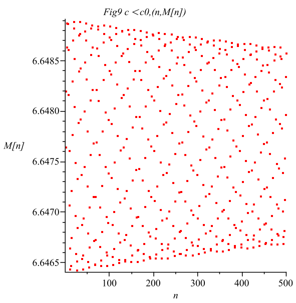


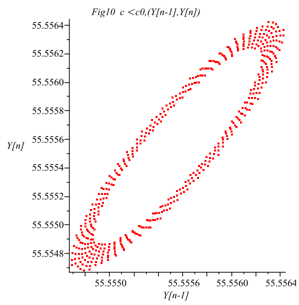


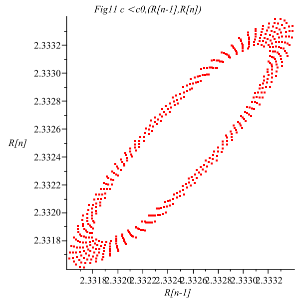
For $c = c_0 - \beta$, $\beta = 0.001$ in Fig 7, Fig 8, Fig 9 the following trajectories: $(n, Y(n))$, $(n, R(n))$, $(n, M(n))$ are displayed. In the figures Fig 10, Fig 11, Fig 12 are displayed: $(Y(n-1), Y(n))$, $(R(n-1), R(n))$, $(M(n-1), M(n))$,

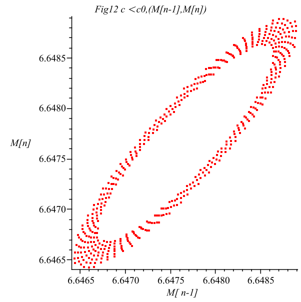












These graphics justify the behavior of the model's solutions as obtained in the theoretical section.

4 The analysis of system (8)

In the hypothesis from Section 2, system (8) has the equilibrium state (Y_0, R_0, M_0, z_0), where $Y_0 = \dfrac{(1-\varepsilon q)g}{(1-\varepsilon)q}$, R_0 is the solution of $I(Y_0, R) = (1+(1-s)(1-\varepsilon)q)Y_0 - \varepsilon qg$, $M_0 = L(Y_0, R_0)$ and $z_0 = g$.

The linearized system of (8) in the neighborhood of the equilibrium (Y_0, R_0, M_0, z_0) is:

$$v_i(n+1) = \sum_{j=1}^{4} a_{ij} v_j(n) \tag{49}$$

where

$$\begin{aligned} & a_{11} = 1 + a(I_1 - (s + (1-s)(1-\varepsilon)\varepsilon)q), \\ & a_{12} = aI_2, a_{13} = 0, a_{14} = -a(1-s)\varepsilon q, \\ & a_{22} = 1 + bL_2, a_{21} = bL_1, a_{23} = -b, a_{24} = 0, \\ & a_{31} = -c(1-\varepsilon)q, a_{32} = 0, a_{33} = 1, a_{34} = -c\varepsilon q, \\ & a_{41} = \alpha(1-\varepsilon)q, a_{42} = 0, a_{43} = 0, \\ & a_{44} = 1 - \alpha(1-\varepsilon q). \end{aligned} \tag{50}$$

The characteristic equation of system (49) is given by:

$$\lambda^4 - D_1\lambda^3 + D_2\lambda^2 - D_3\lambda - D_4 = 0 \tag{51}$$

where

$$\begin{aligned} & D_1 = 1 + a_{11} + a_{22} + a_{44}, \\ & D_2 = a_{11} + a_{22} + a_{11}a_{22} - a_{12}a_{21} - a_{41}a_{14} - \\ & - a_{11}(1 + a_{11} + a_{22}), \\ & D_3 = a_{41}a_{12}a_{23} - a_{14}(a_{22} + a_{33} + a_{11}a_{22} - \\ & - a_{12}a_{21} + a_{12}a_{31}a_{23}) + \\ & + a_{44}(a_{11} + a_{22} + a_{11}a_{22} - a_{12}a_{21}) \\ & D_4 = a_{12}a_{23}a_{41}a_{42} + a_{14}a_{22}a_{33} - \\ & - a_{44}(a_{11}a_{22} - a_{12}a_{21} + a_{12}a_{31}a_{23}). \end{aligned} \tag{52}$$

We analyze the model (8), choosing the adjustment coefficient c as function of the adjustment coefficients a, b and the coefficient α so that $D_3 = 0$. From (50) and (52) we have:

$$c = \frac{a_{11}a_{22} + (1-\alpha(1-\varepsilon q))}{(1-\varepsilon)qa_{12}a_{23}} \cdot \frac{(a_{11}+a_{22}+a_{11}a_{22}-a_{12}a_{21})-}{(1-\varepsilon)qa_{12}a_{23}} - \frac{\alpha(1-\varepsilon)qa_{14}(1+a_{22}-a_{12}a_{21})}{(1-\varepsilon)qa_{12}a_{23}} \quad (53)$$
$$D_4 = ad_5 + bd_6,$$

where

$$\begin{aligned} d_5 &= a_{14} - a_{44}((1-\alpha(1-\varepsilon q))(2a_{11}+1) - 2\alpha(1-\varepsilon)qa_{14}) \\ d_6 &= -a_{12}a_{41}a_{42} + a_{14}L_2 - a_{44}(-L_1a_{12} + (1-\alpha(1-\varepsilon q))((1+a_{11})L_2 - a_{12}L_1) - \alpha(1-\varepsilon)qa_{14}L_2 - L_1a_{12}). \end{aligned} \quad (54)$$

We have:

Proposition 7 *(i) If the parameter b satisfies the relations:*

$$\begin{aligned} &|d_5+bd_6|<1, |2+a_{11}+a_{44}+bL_2|<2(1+d_5+bd_6), \\ &(-a_{41}a_{14} - a_{11}^2 + 1 + d_5 + (L_2 - a_{12}L_1 + d_6)b)\cdot \\ &\cdot(1+d_5+d_6b)^2 = (2+a_{11}+L_2b)^2(d_5+d_6b) \end{aligned} \quad (55)$$

then the equation

$$\lambda^4 - D_1\lambda^3 + D_2\lambda^2 - D_4 = 0 \quad (56)$$

has two roots in modulus equal to 1 and two complex roots in modulus less to 1.

(ii) If b_0 is one solution of (55) then for $b = b_0+\alpha$, there is $\beta > 0$ sufficiently small so that the roots of equation (56) are in modulus less than 1; for $b = b_0 - \alpha$ equation (56) has two roots in modulus less than 1.

(iii) The value b_0 is a Neimark-Sacker bifurcation for equation (55).

In what follows, using the method from [7], [12], we find the normal form of system (8) for the Neimark-Sacker bifurcation given by b_0.

We consider C the matrix of the linear system (49) with the coefficients (50), $b = b_0 + \beta$, where b_0 satisfies (55) and $|\beta|$ is sufficiently small and with c given by (53).

Let $\mu = \mu(\beta)$ one root of the characteristic equation (56).

We have:

Proposition 8 *(i) The eigenvector corresponding to the eigenvalue μ is the nontrivial solution of the system $Cl = \mu l$ and has the components:*

$$\begin{aligned} &l_1 = -a_{12}(\mu - a_{33}), l_2 = -(\lambda - a_{33})(\mu - a_{11} - \frac{a_{14}a_{41}}{\mu - a_{44}}), \\ &l_3 = -a_{12}(a_{31} + \frac{a_{41}a_{34}}{\mu - a_{44}}), l_4 = -\frac{a_{41}a_{12}(\mu - a_{33})}{\mu - a_{44}}; \end{aligned} \quad (57)$$

(ii) The eigenvector corresponding to the eigenvalue $\bar{\mu}$ is the nontrivial solution of the system $C^T m = \bar{\mu} m$ and has the components:

$$\begin{aligned} &m_1 = \frac{\bar{\mu} - a_{22}}{a_{12}V}, m_2 = \frac{1}{V}, m_3 = \frac{a_{22}}{(\bar{\mu} - a_{33})V}, \\ &m_4 = \frac{a_{14}(\bar{\mu} - a_{22})}{a_{12}(\bar{\mu} - a_{44})(\bar{\mu} - a_{33})V} \cdot \frac{(\bar{\mu} - a_{33}) + a_{12}a_{22}a_{34}}{a_{12}(\bar{\mu} - a_{44})(\bar{\mu} - a_{33})V}, \end{aligned} \quad (58)$$

where

$$V = \frac{(\bar{\mu} - a_{22})}{a_{12}}\bar{l}_1 + \bar{l}_2 + \frac{a_{23}}{\bar{\mu} - a_{33}}\bar{l}_3 + \frac{a_{14}(\bar{\mu} - a_{22})(\bar{\mu} - a_{33}) + a_{12}a_{22}a_{34}}{a_{12}(\bar{\mu} - a_{44})(\bar{\mu} - a_{33})}l_4. \quad (59)$$

Moreover, the relation $l_1\bar{m}_1 + l_2\bar{m}_2 + l_3\bar{m}_3 + l_4\bar{m}_4 = 1$ holds.

We consider $l = (l_1, l_2)^T$, $m = (m_1, m_2)^T$ given by (57), (58), the formulas (22), (23), (24) and

$$\begin{aligned} h_{420} &= -((B^1(l,l), B^2(l,l))m)l_4 - ((B^1(l,l), B^2(l,l))\bar{m})\bar{l}_4, \\ h_{411} &= -((B^1(l,\bar{l}), B^2(l,\bar{l}))m)l_4 - ((B^1(l,\bar{l}), B^2(l,\bar{l}))\bar{m})\bar{l}_4, \\ h_{402} &= -((B^1(\bar{l},\bar{l}), B^2(\bar{l},\bar{l}))m)l_4 - ((B^1(\bar{l},\bar{l}), B^2(\bar{l},\bar{l}))\bar{m})\bar{l}_4, \end{aligned} \quad (60)$$

Let be the matrices:

$$C(\mu^2) = \begin{pmatrix} \mu^2 - a_{11} & -a_{12} & 0 & -a_{14} \\ -a_{21} & \mu^2 - a_{22} & -a_{23} & 0 \\ -a_{31} & 0 & \mu^2 - a_{33} & -a_{34} \\ -a_{41} & 0 & 0 & \mu^2 - a_{44} \end{pmatrix}, \quad (61)$$

$C(\bar{\mu}^2) = \overline{C(\mu^2)}$, $C(1)$ and $w_{20} = C(\mu^2)^{-1}h_{20}$, $w_{11} = C(1)^{-1}h_{11}$, $w_{02} = C(\bar{\mu}^2)^{-1}h_{02}$ where

where

$$h_{20} = (h_{120}, h_{220}, h_{320}, h_{420})^T,$$
$$h_{11} = (h_{111}, h_{211}, h_{311}, h_{411})^T, \quad (62)$$
$$h_{02} = (h_{102}, h_{202}, h_{302}, h_{402})^T.$$

Using the method from [7], [12] for the determination the normal form we obtain:

Proposition 9 *(i) The normal form of system (8) is given by (26) where the coefficients are obtained for* $l = (l_1, l_2)^T$ *and* $m = (m_1, m_2)^T$ *given by (57) and (58);*

(ii) System (8) in the neighborhood of the state equilibrium (Y_0, R_0, M_0, g) *is:*

$$\begin{aligned} Y(n) &= Y_0 + l_1 z(n) + \bar{l}_1 \bar{z}(n) + \frac{1}{2} w_{120} z(n)^2 + \\ &+ w_{111} z(n)\bar{z}(n) + \frac{1}{2} w_{102} \bar{z}(n)^2 \\ R(n) &= R_0 + l_2 z(n) + \bar{l}_2 \bar{z}(n) + \frac{1}{2} w_{220} z(n)^2 + \\ &+ w_{211} z(n)\bar{z}(n) + \frac{1}{2} w_{202} \bar{z}(n)^2 \\ M(n) &= M_0 + l_3 z(n) + \bar{l}_3 \bar{z}(n) + \frac{1}{2} w_{320} z(n)^2 + \\ &+ w_{311} z(n)\bar{z}(n) + \frac{1}{2} w_{302} \bar{z}(n)^2 \\ v(n) &= g + l_4 z(n) + \bar{l}_4 \bar{z}(n) + \frac{1}{2} w_{420} z(n)^2 + \\ &+ w_{411} z(n)\bar{z}(n) + \frac{1}{2} w_{402} \bar{z}(n)^2 \end{aligned} \quad (63)$$

where $z(n)$ *is one solution of the normal form for the system (7).*

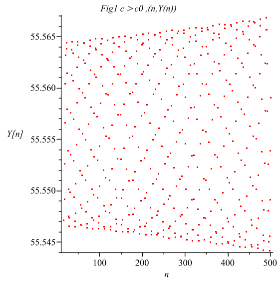
(iii) The Lyapunov coefficient associated to the normal form is given by (28).

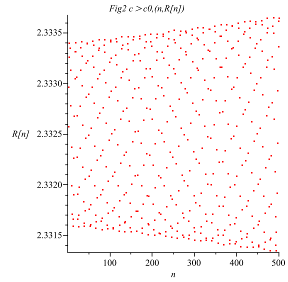
The numerical simulation was made using a program in Maple 11. For:

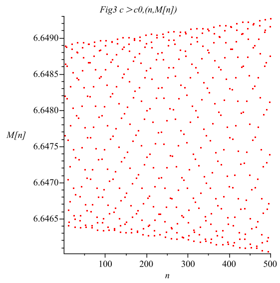
$$I = a_1 Y^{a_2} R^{-a_3}, L = b_1 Y + b_2 (R - b_3)^{-b_4}$$

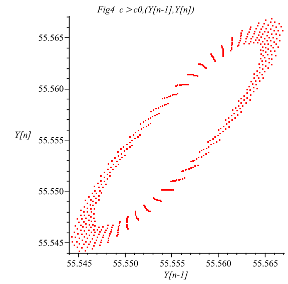
where $a_1 = 0.38$, $a_2 = 1.05$, $a_3 = 0.83$, $b_1 = 0.07$, $b_2 = 1$, $b_3 = 0.003$, $b_4 = -1.2$, $s = 0.5$, $q = 0.18$, $g = 1$, $a = 0.96$, $b = 0.8$, $\varepsilon = 0.02$ we obtain the following results: $Y_0 = 5.648526$, $R_0 = 0.656110$, $M_0 = 0.995165$, $c_0 = -3.506258$. The Lyapunov coefficient is $L_0 = -177.547698$ and in the neighborhood of the equilibrium state there is a stable limit cycle.

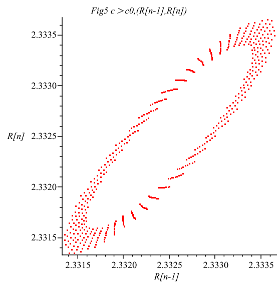
For $c = c_0 + \beta$, $\beta = 0.001$, the following trajectories are displayed: $(n, Y(n))$, $(n, R(n))$, $(n, M(n))$ in Fig 1, Fig 2, Fig 3. In the figures Fig 4, Fig 5, Fig 6 are displayed: $(Y(n-1), Y(n))$, $(R(n-1), R(n))$, $(M(n-1), M(n))$,

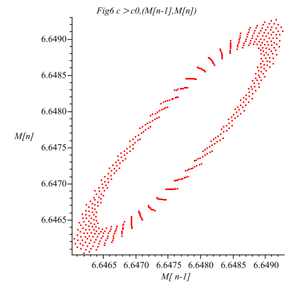




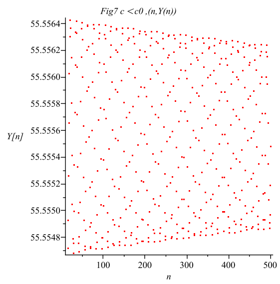


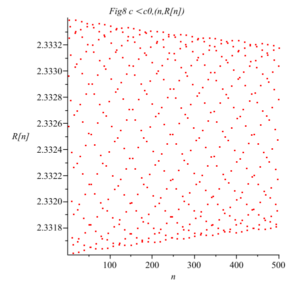


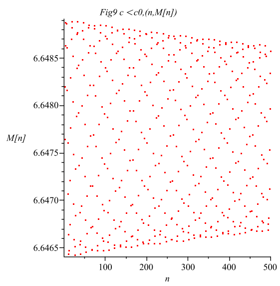


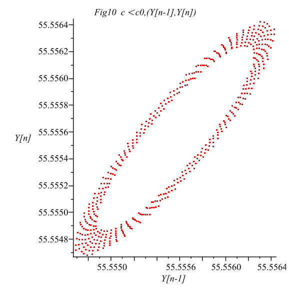


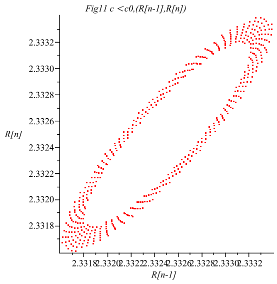
For $c = c_0 - \beta$, $\beta = 0.001$ in Fig 7, Fig 8, Fig 9 the following trajectories: $(n, Y(n))$, $(n, R(n))$, $(n, M(n))$ are displayed. In the figures Fig 10, Fig 11, Fig 12 are displayed: $(Y(n-1), Y(n))$, $(R(n-1), R(n))$, $(M(n-1), M(n))$,

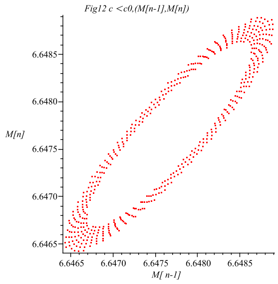












These graphics justify the behavior of the model's solutions as obtained in the theoretical section.

5 Conclusion

The discrete time IS-LM model with tax revenues and time delay is a complex model with many parameters. The model allows us to study the real process using the temporal numeric series of the income, the interest rate, the money stock, the liquidity.

The analysis of the model leads to different scenarios by considering the adjustment coefficient of the equation which describes the dynamics of the money stock as variable.

In the present paper we have analyzed the discrete IS-LM models with the parameter c as variable and we have shown the existence of the Neimark-Sacker bifurcation. The normal form of the model has also been presented. We will carry out the same analysis for the discrete model with delay. In a future paper the scenario for which the model has a chaotic behavior will be analyzed.

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