More on the Green Solow Model with Logistic Population Change

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Abstract: This paper generalizes the model introduced by Ferrara and Guerrini [13], where two different research lines have been joined together: the one studying the effects of incorporating technological progress in pollution abatement in the Solow-Swan model (Brock and Taylor [6]), and that analyzing the role of a logistic population growth rate within the Solow-Swan model (Ferrara and Guerrini [13]). In this framework, the economy is described by a three dimensional dynamical system, whose solution can be explicitly determined. We note that physical capital can be expressed in closed-form via Hypergeometric functions. As well, we prove the model's solution to be convergent in the log-run. We characterize the economy balanced growth path equilibrium, and find that sustainable growth occurs if technological progress in abatement is faster than technological progress in production. An environmental Kuznets curve may result along the transition to the balanced growth path. If there is no technological progress in abatement, then there is no EKC. Furthermore, the economy has a unique equilibrium (a node), which is locally asymptotically stable.

Key-Words: Green Solow, Logistic population, Pollution, Sustainable growth, Environmental Kuznets curve.

1 Introduction

One of the most important models elaborated to explain economic growth is the growth model originating from the work of Solow [22] and Swan [23]. The resulting model has become famously known as the Solow-Swan model, or simply the neoclassical growth model. This model is an extension of the Harrod-Domar model, which was independently developed by Harrod [16] and Domar [9]. Their theories related an economy's rate of growth to its capital stock, and emphasized how investment spending also increased an economy's productive capacity (a supply-side effect). Their assumptions involved an exogenous rate of labour force growth, a given technology exhibiting fixed factor proportions (constant capital-labour ratio) and a fixed capital-output ratio.

The Harrod-Domar model became tremendously influential in the development economics literature during the third quarter of the twentieth century, and it was a key component within the framework of economic planning.

However, the assumption of zero substitutability between capital and labour, i.e. a fixed factor proportions production function, appeared to be an inappropriate assumption for a model concerned with long-run growth. In fact, it gave knife-edge equilibria, with the implausible implication that any deviation at all from equilibrium would cause the model to diverge further and further away from equilibrium.

Solow and Swan claimed that the capital-output ratio of the Harrod-Domar model should not be regarded as exogenous. They proposed a growth model where the capital-output ratio was precisely the adjusting variable that would lead a system back to its steady-state growth path.

The key assumptions of the Solow-Swan model are: the economy consists of one-sector producing one type of commodity that can be used for either investment or consumption purposes; the economy is closed to international transactions and the government sector is ignored; all output that is saved is invested; the assumptions of full price flexibility and monetary neutrality apply and the economy is always producing its potential (natural) level of total output; an abandon of the Harrod-Domar assumptions of a fixed capital-output ratio and fixed capital-labour ratio, and the assumption of a production function consisting of constant returns to scale, Inada conditions, and diminishing returns on all inputs and some degree of substitution among them; the rate of technological progress, population growth and the depreciation rate of the capital stock are all determined exogenously.

On the basis of these assumptions, an economy, regardless of its starting point, converges to a balanced growth path, where long-run growth of output and capital are determined solely by the rate of labor-augmenting technological progress and the rate of population growth (see, for example, Barro and Sala-i-Martin [4]).

In the neoclassical model of economic growth, it is usually assumed that labor (population) force grows at a positive constant rate (Malthusian model [17]). The main problem of this assumption is that population exponentially grows without limits. In addition, the Malthusian model considers homogeneous populations, i.e. it supposes that all the individuals of such a population are physiologically identical, as well as that the population lives isolated in an invariable habitat and with limitless resources, so that the population depends, respectively, on constant fertility and mortality rates.

Although one assumes that variations do not take place in the external habitat, the population itself causes changes in life conditions due to competition for the survival resources. Consequently, one could admit that the fertility and mortality rates depend on the total size of the population, replacing the linear model of Malthus by a non-linear model.

The first model of this type was proposed by Verhulst [24]. His model, known as logistic model, corrected the most significant objections against the Malthusian model by imposing a maximum size for the total population size (carrying capacity). The logistic model could be interpreted as a Malthusian model with constant fertility rate, and a mortality rate proportional to the relative size of the total population with respect to the carrying capacity.

The result of Verhulst's work was a demonstration that any population growth rate would essentially follow a bell-shaped curve, starting from zero, steadily increasing to a maximum, and declining once again to zero in a fashion symmetrical to the positive growth phase. The population stock then evolves according to the elongated S-curve, which has a point of inflection at the maximal value of the growth rate, and then levels off at a new but higher plateau, at which point the growth rate declines to zero.

On the other hand, it is a well known fact that the population growth rate is decreasing since the 1950s, and it is projected to go down to zero during the next six decades. The cause of this decrease in the growth rate is mainly due to the population aging, i.e. to the dramatic increase in the number of deaths. As well, from 2030 to 2050, the world population would grow more slowly than ever before in its history (see Day [8]). Consequently, as observed by Maynard Smith [18], a more realistic approach would be to consider a logistic law for the population growth rate.

Several attempts to analyze how the logistic population growth hypothesis might affect the dynamics of some growth models have been recently done in many and different directions (see, for example, [2],[3],[10],[11],[12],[13][13],[15],[19],[20]).

Recently, Brock and Taylor [6] has extended the Solow-Swan model by incorporating technological progress in abatement. The resulting model, which they called the Green Solow model, generates an environmental Kuznets curve (hereafter EKC) relationship between the flow of pollution emissions and income per capita, and the stock of environmental quality and income per capita (for the EKC, see, for example, Grossman and Krueger [14]). Moreover, there can be sustainable development, here meaning decreasing pollution along with increasing per capita income, if the rate of progress in the development of environmental technology is sufficiently high.

Ferrara and Guerrini [13] has explored the implications of combining, within the same framework, these two different research lines, that have been analyzed separately in the recent past. This was done by including in an augmented Solow-Swan model emissions, abatement and a stock of pollution, and by assuming the population to grow according to the logistic model. The resulting model was then a three dimensional non-autonomous dynamical system, which the authors proved to be solvable in closed-form. As well, for sustainable growth to be possible, technological progress in abatement has to be faster than technological progress in production. Finally, an EKC may result along the transition to the balanced growth path.

The main objective of our paper is to extend the results obtained in [13] by assuming that the exogenous technological progress in abatement is not necessarily lowering the units of pollution generated as a joint product of output. The corresponding model is then described by a three dimensional dynamical system, whose solution can be determined recursively. In addition, one is able to show that the transitional path of capital stock can be written in terms of Gaussian Hypergeometric functions in their Euler integral representation. As well, we characterize the economy balanced growth path equilibrium, and find out that there is sustainable growth if technological progress in abatement is faster than technological progress in production. In this case, an EKC may result along the transition to the balanced growth path. If there is no technological progress in abatement, then the economy is described by an autonomous dynamical system, having a unique equilibrium, which is a stable node. Moreover, no EKC can occur.

2 The basic model

We consider a one-sector production technology in which output Y_t is a homogeneous good that can be consumed, C_t, or invested, I_t, in a process described by a Cobb-Douglas production function

$$Y_t = K_t^{\alpha}(B_t L_t)^{1-\alpha},\ 0 < \alpha < 1, \qquad (1)$$

where K_t denotes physical capital, L_t is labor, and B_t is the level of technology.

We assume that B_t increases over time at the exogenous and constant rate $g > 0$, i.e.

$$\frac{\dot{B}_t}{B_t} = g, \qquad (2)$$

where a dot over a variable denotes differentiation with respect to time.

Gross investment I_t has two components: net investment, defined as the variation in the stock of capital, and the loss by depreciation. Assuming that capital is a homogeneous good that depreciates at the constant rate $\delta > 0$, this means

$$I_t = \dot{K}_t - \delta K_t. \qquad (3)$$

Each worker has a unit of time available each period that is supplied inelastically in the labor market. Thus, we can identify the number of workers and the supply of labor each period. As well, there is full employment in the economy, so that employment and labor supply coincide. The economy is supposed to be closed, i.e. households cannot buy foreign goods or assets and cannot sell home goods or assets abroad, and there are no government purchases of good or services. This implies that aggregate savings and investment are equal to each other every period, i.e.

$$S_t = I_t.$$

Additionally, we assume savings to evolve over time as a constant fraction s of output, i.e.

$$S_t = sY_t.$$

The growth of population is modelled according to Ferrara and Guerrini [12], i.e. we have

$$\frac{\dot{L}_t}{L_t} = a - bL_t,\ a > b > 0 \ \text{ (logistic model)}, \qquad (4)$$

where, for simplicity, the initial population has been normalized to one, $L_0 = 1$. The fundamental prediction of the constant growth rate population model,

$$\frac{\dot{L}_t}{L_t} = n > 0 \ \text{ (Malthusian model)},$$

is that the population exponentially grows without limits. Although this model may accurately reflect experiments in the initial stages, we realize that no population will grow exponentially indefinitely. Since unbounded growth is unrealistic, more sophisticated models take into account factors such as limited resources for reproduction. The first model of this type was proposed by Verhulst [24], and it corrected the most significant objections against the Malthusian model. Equation (4) is also known as the logistic equation.

We now assume that the production process generates emissions. Similarly to Copeland and Taylor [7], we assume that every unit of economic activity Y_t generates Ω_t units of pollution as a joint product of output. Pollution emitted E_t is equal to pollution created minus pollution abated. We assume abatement is a constant returns to scale activity. Abatement of pollution A_t takes as inputs the flow of pollution, which is proportional to the gross flow of output Y_t, and abatement inputs, denoted by Y_t^A. If abatement at level A_t removes the $\Omega_t A_t$ units of pollution from the total created, then we can write pollution emitted as

$$E_t = \Omega_t Y_t - \Omega_t A(Y_t, Y_t^A) = \phi(\theta)\Omega_t Y_t, \qquad (5)$$

where

$$\theta = \frac{Y_t^A}{Y_t}$$

is the fraction of economic activity dedicated to abatement, and

$$\phi(\theta) = 1 - A(1, \theta).$$

Following Brook and Taylor [6], we assume fixed abatement intensity, and we require the economy to employ a fixed fraction of its inputs, both capital and effective labor, in abatement. This means that the fraction of total output allocated to abatement θ is fixed much like the familiar fixed saving rate assumption. As a result, output available for consumption or investment becomes $(1-\theta)Y_t$. Since emissions are measured by physical unit per time unit, E_t is a flow variable, while total pollution X_t can be interpreted as a stock variable which accumulates through permanent emissions, but has also a natural decay. Therefore, pollution is accumulated in the ambient environment according to

$$\dot{X}_t = E_t - \eta X_t, \qquad (6)$$

with $\eta > 0$ as the natural decay rate which reflects the ability of the ecosystem to absorb pollution.

Based on the feature of constant returns to scale, we can specify the economy's output in terms of effective labor as follows

$$y_t = k_t^{\alpha},$$

where

$$y_t = \frac{Y_t}{B_t L_t}, \quad k_t = \frac{K_t}{B_t L_t}$$

denote output per unit of effective labor, and capital per unit of effective labor, respectively. Taking derivatives with respect to time in the definition of k_t yields

$$\dot{k}_t = \frac{d[K_t/(B_t L_t)]}{dt} = \frac{\dot{K}_t}{B_t L_t} - \left(\frac{\dot{B}_t}{B_t} + \frac{\dot{L}_t}{L_t} \right) k_t. \quad (7)$$

From (2) and (4), we know that $\dot{L}_t/L_t$ and $\dot{B}_t/B_t$ are $a - bL_t$ and g, respectively. $\dot{K}_t$ is given by (3). Substituting these facts into (7), we get

$$\dot{k}_t = Mk_t^\alpha - [\delta + g + n(L_t)] k_t,$$

with

$$n(L_t) = a - bL_t, \quad M = (1-\theta)s.$$

Similarly, defining pollution in efficiency units as

$$x_t = \frac{X_t}{B_t L_t},$$

the pollution accumulation equation (6) becomes

$$\dot{x}_t = \phi(\theta)\Omega_t k_t^\alpha - [\eta + g + n(L_t)]x_t.$$

In conclusion, we have that the economy of our modified Solow-Swan model is described by the following set of differential equations

$$\begin{aligned} \dot{k}_t &= Mk_t^\alpha - [\delta + g + n(L_t)]k_t, & (8) \\ \dot{x}_t &= \phi(\theta)\Omega_t k_t^\alpha - [\eta + g + n(L_t)]x_t, & (9) \\ \dot{L}_t &= n(L_t)L_t. & (10) \end{aligned}$$

Given $k_0 > 0$, $x_0 > 0$, this Cauchy problem has a unique solution (k_t, x_t, L_t), defined on $[0, \infty)$ (see Birkhoff and Rota [5]).

3 The model solution

We will now work out an explicit solution to the differential equations appearing in $(8) - (10)$.

Lemma 1. *For all t, the transitional path of labor is*

$$L_t = \frac{ae^{at}}{a - b + be^{at}}, \quad \lim_{t\to\infty} L_t = \frac{a}{b}. \quad (11)$$

Proof: An explicit solution to (10) can be obtained since this equation is separable. We have

$$\frac{dL_t}{L_t(a - bL_t)} = dt.$$

The method of partial fractions will be successful in integrating this equation. Since

$$\frac{1}{L_t(a-bL_t)} = \frac{1}{aL_t} + \frac{b}{a(a-bL_t)},$$

integrating between 0 and t yields

$$\frac{1}{a} \ln \left[\frac{(a-b)L_t}{a - bL_t} \right] = t. \quad (12)$$

The statement is obtained multiplying (12) by a, and then exponentiating. Note that (10) can also be solved as a Bernoulli differential equation by making the substitution $z_t = L_t^{-1}$. □

Remark 2. *The graph of L_t looks like an S-shaped curve (sigmoid), which lies between the two equilibrium solutions, $L_t = 0$ and $L_t = a/b$. Since one has $\dot{L}_t > 0$, this curve has positive slope throughout the region. In addition,population is limited to stay below $L_t = a/b$. In fact, by computing the limit as t gets large, we found that the population predicted is a/b.*

Remark 3. *Set $L_\infty = \lim_{t\to\infty} L_t$. The function L_t increases monotonically from $L_0 = 1$ to $L_\infty = a/b$. Moreover, $n(L_\infty) = 0$, i.e. L_∞ is a constant solution of (10).*

Proposition 4. *Set*

$$\varphi_t = \frac{ae^{(\delta+g+a)t}}{a - b + be^{at}}.$$

For all t, the time path of the capital stock measured in intensive units is

$$k_t = \varphi_t^{-1} \left[k_0^{1-\alpha} + (1-\alpha)M \int_0^t \varphi_t^{1-\alpha} dt \right]^{\frac{1}{1-\alpha}} \quad (13)$$

$$\lim_{t\to\infty} k_t = \left(\frac{M}{\delta + g} \right)^{\frac{1}{1-\alpha}}.$$

Proof: See Ferrara and Guerrini [13]. □

Remark 5. *Set $k_\infty = \lim_{t\to\infty} k_t$. Then k_∞ is a constant solution of (8).*

Substituting (11) in (8) yields

$$\dot{k}_t = M k_t^{\alpha} - (\delta + g + n_t)\, k_t,$$

where

$$n_t = \frac{a(a-b)}{a-b+be^{at}}.$$

Note that the function n_t is monotone decreasing from $n_0 = a-b$ to $n_\infty = 0$. In particular, $0 \le n_t \le a-b$. Therefore, applying theorem 7 of Guerrini [15], with

$$n_* = a-b,\ k_1^* = \left(\frac{M}{\delta+g+a-b}\right)^{\frac{1}{1-\alpha}},\ k_2^* = k_\infty,$$

we get the following result.

Proposition 6.

(a) *If $k_0 \le k_1^*$, then $\dot{k}_t \ge 0$, for all t.*

(b) *If $k_1^* < k_0 \le k_2^*$, there exists $\tau > 0$ such that $\dot{k}_t \le 0$ for $t \in (0,\tau]$ and $\dot{k}_t \ge 0$, for $t \in [\tau, \infty)$.*

(c) *If $k_2^* < k_0$, $\dot{k}_t \le 0$, for all t.*

Following Guerrini [15], the solution (13) can be written in closed-form through the Hypergeometric function ${}_2F_1$ (see appendix).

Proposition 7. *Let*

$$\gamma_1 = (\delta+g+a)\gamma_3,\ \gamma_2 = a,\ \gamma_3 = 1-\alpha,\ B = \frac{b}{b-a}.$$

Then

$$k_t = \varphi_t^{-1}\left\{k_0^{\gamma_3} + \frac{M\gamma_3\,(1-B)^{\gamma_3}}{\gamma_1}\cdot \right.$$
$$\cdot\left[e^{\gamma_1 t}{}_2F_1\left(\frac{\gamma_1}{\gamma_2},\gamma_3,\frac{\gamma_1}{\gamma_2}+1;Be^{\gamma_2 t}\right)\right.$$
$$\left.\left. -{}_2F_1\left(\frac{\gamma_1}{\gamma_2},\gamma_3,\frac{\gamma_1}{\gamma_2}+1;B\right)\right]\right\}^{\frac{1}{\gamma_3}}.$$

Proof: The statement follows by writing the integral

$$\int_0^t \varphi_t^{1-\alpha} dt = \int_0^t e^{(1-\alpha)(\delta+g+a)t} L_t^{1-\alpha}$$

in terms of Hypergeometric functions. This can be derived from Guerrini [15], theorem 13, with $\beta = 0$, and the term δ replaced by $\delta + g$. □

Proposition 8. *Let $\lim_{t\to\infty} \Omega_t = \Omega_\infty < \infty$. Set*

$$\psi_t = \frac{ae^{(\eta+g+a)t}}{a-b+be^{at}}.$$

For all t, the time path of the stock of pollution measured in intensive units is given by

$$x_t = \psi_t^{-1}\left(x_0 + \phi(\theta)\int_0^t \Omega_t k_t^{\alpha}\psi_t dt\right), \qquad (14)$$

$$\lim_{t\to\infty} x_t = \frac{\phi(\theta)\Omega_\infty k_\infty^{\alpha}}{\eta+g}.$$

Proof: Equation (9) is a first order linear differential equation, whose solution is known to be given by the formula

$$x_t = e^{-\int_0^t [\eta+g+n(L_t)]dt}\cdot$$
$$\cdot\left\{x_0 + \int_0^t \phi(\theta)\Omega_t k_t^{\alpha} e^{\int_0^t [\eta+g+n(L_t)]dt} dt\right\}.$$

Evaluating the integral

$$\int_0^t [\eta+g+n(L_t)]dt = (\eta+g)t + \ln L_t,$$

we have that the solution becomes

$$x_t = e^{-(\eta+g)t} L_t^{-1}\cdot$$
$$\cdot\left[x_0 + \phi(\theta)\int_0^t e^{(\eta+g)t}\Omega_t k_t^{\alpha} L_t dt\right].$$

The statement now follows using (11). Next, rewrite the above formula as

$$x_t = \frac{x_0 + \phi(\theta)\int_0^t e^{(\eta+g)t}\Omega_t k_t^{\alpha} L_t dt}{e^{(\eta+g)t} L_t}. \qquad (15)$$

Since $\Omega_t \to \Omega_\infty$, $k_t \to k_\infty$, $L_t \to L_\infty$, we derive that both numerator and denominator of (15) go to infinity in the long-run. L'Hopital's rule is a general method for evaluating the indeterminate form ∞/∞. All we need to do is to differentiate the numerator and denominator of our expression, and then take the limit. This yields

$$\begin{aligned}\lim_{t\to\infty} x_t &= \lim_{t\to\infty}\frac{\phi(\theta)\Omega_t k_t^{\alpha} L_t}{(\eta+g)L_t + \dot{L}_t}\\ &= \lim_{t\to\infty}\frac{\phi(\theta)\Omega_t k_t^{\alpha}}{\eta+g+n(L)} = \frac{\phi(\theta)\Omega_\infty k_\infty^{\alpha}}{\eta+g}.\end{aligned}$$

□

Corollary 9. *Given $k_0 > 0$, $x_0 > 0$, the unique solution of the dynamical system described by $(8)-(10)$ satisfies*

$$\lim_{t\to\infty}(k_t, x_t, L_t) = \left(k_\infty, \frac{\phi(\theta)\Omega_\infty k_\infty^{\alpha}}{\eta+g}, L_\infty\right).$$

4 Balanced growth path

In this section we focus on a balanced growth path (BGP) equilibrium, i.e. a trajectory along which all the relevant variables either stay constant or grow at a constant rate. The next result will be useful to characterize the economy's balanced growth path.

Lemma 10.

$$\gamma_{C_t} = \gamma_{Y_t} = \alpha\gamma_{K_t} + (1-\alpha)(g + \gamma_{L_t}),$$

$$\gamma_{E_t} = \gamma_{\Omega_t} + \gamma_{Y_t},$$

where γ_z denotes the growth rate of the variable z.

Proof: First, consumption and output are proportional, so that taking logs and time derivative of these yields $\gamma_{C_t} = \gamma_{Y_t}$. Second, consider equations (1), (5), and take their logs. This yields

$$\ln Y_t = (1-\alpha)\left(\ln B_t + \ln L_t\right) + \alpha \ln K_t,$$

$$\ln E_t = \ln \phi(\theta) + \ln \Omega_t + \ln Y_t.$$

Differentiating with respect to time implies

$$\frac{\dot{Y}_t}{Y_t} = \alpha\frac{\dot{K}_t}{K_t} + (1-\alpha)\left(g + \frac{\dot{L}_t}{L_t}\right),$$

$$\frac{\dot{E}_t}{E_t} = \frac{\dot{\Omega}_t}{\Omega_t} + \frac{\dot{Y}_t}{Y_t},$$

i.e. we have the statement. □

Lemma 11. *Along a balanced growth path*

$$\gamma_{L_t} = 0,\ \gamma_{C_t} = \gamma_{Y_t} = \gamma_{K_t} = g,$$

$$\gamma_{X_t} = \gamma_{E_t} = g + \gamma_{\Omega_t}.$$

Proof: By definition γ_{L_t} is constant along a BGP. Thus, $L_t = a/b$, and so, from (10), we get

$$\gamma_{L_t} = 0. \tag{16}$$

Next, rewrite equations (3) and (6) as

$$\gamma_{K_t} = s\frac{Y_t}{K_t} - \delta,\ \ \gamma_{X_t} = \frac{E_t}{X_t} - \eta. \tag{17}$$

Since γ_{K_t} and γ_{X_t} are constant along a BGP, we derive from (17) that Y_t/K_t and E_t/X_t must also be constant. Taking logs and time derivatives of these yields

$$\gamma_{Y_t} = \gamma_{K_t},\ \ \gamma_{X_t} = \gamma_{E_t}. \tag{18}$$

The statement now follows using (16) and (18) in Lemma 10. □

We define sustainable growth as a balanced growth path with increasing environmental quality and ongoing growth in income per capita.

Lemma 11 implies that, along a balanced growth path, the growth rate of emissions γ_{E_t} may be positive, negative or zero.

Proposition 12. *If $g > -\gamma_{\Omega_t}$, then there exists sustainable growth.*

5 Absence of technological progress in abatement

In this section, we assume that Ω_t is taken as constant over time and by choice of units set to one. Consequently, equations $(8) - (10)$ become

$$\dot{k}_t = Mk_t^{\alpha} - \left[\delta + g + n(L_t)\right]k_t, \tag{19}$$

$$\dot{x}_t = \phi(\theta)k_t^{\alpha} - [\eta + g + n(L_t)]x_t, \tag{20}$$

$$\dot{L}_t = n(L_t)L_t. \tag{21}$$

The economy is thus described by an autonomous system of differential equations. Before starting the study of the local dynamics of this system, it is better to recall the definition of steady state equilibrium. A point (k_*, x_*, L_*) is said to be a steady state equilibrium of the equations $(19) - (21)$ if it solves

$$\dot{k}_t = \dot{x}_t = \dot{L}_t = 0.$$

We confine the analysis to interior steady states only, i.e. we will exclude the economically meaningless null solutions.

Proposition 13. *There exists a unique steady state equilibrium (k_*, x_*, L_*), with*

$$k_* = \left(\frac{M}{\delta+g}\right)^{\frac{1}{1-\alpha}},\ x_* = \frac{\phi(\theta)k_*^{\alpha}}{\eta+g},\ L_* = \frac{a}{b}. \tag{22}$$

Proof: On applying the definition given above, in the steady-state equilibrium we must have

$$Mk_t^{\alpha-1} - (\delta+g) = 0, \quad n(L_t) = 0.$$

$$\phi(\theta)k_t^{\alpha} - (\eta+g)x_t = 0.$$

Solving this three-equations system leads to the identities (22). □

Outside the steady state the growth rate of the economy is not constant but, rather, it behaves according to $(19) - (21)$. In order to determine what the equilibrium path of the economy looks like, we need to study the transitional dynamics of the dynamical system. We already know from Corollary 9 that the economy will tend to the steady state. Of special interest is the answer to the question of how it will behave along the transition path.

Proposition 14. *The steady state equilibrium described by equations $(19) - (21)$ is a stable node.*

Proof: From the theory of linear approximation, we know that in a neighborhood of the steady state the dynamic behavior of a non-linear system is characterized by the behavior of the linearized system around the steady state. In our case this means

$$\begin{bmatrix} \dot{k}_t \\ \dot{x}_t \\ \dot{L}_t \end{bmatrix} = J^* \begin{bmatrix} k_t - k_* \\ x_t - x_* \\ L_t - L_* \end{bmatrix}, \tag{23}$$

where

$$J^* = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix}$$

denotes the Jacobian matrix evaluated at the steady state (k_*, x_*, L_*). Set $P_* = (k_*, x_*, L_*)$. By definition

$$J_{11}^* = \frac{\partial \dot{k}_t}{\partial k_t}|_{P_*},\ J_{12}^* = \frac{\partial \dot{k}_t}{\partial x_t}|_{P_*},\ J_{13}^* = \frac{\partial \dot{k}_t}{\partial L_t}|_{P_*}.$$

Similarly for the other J_{ij}^* entries. Computing these elements yields

$$J_{11}^* = -(1-\alpha)(\delta+g), \quad J_{12}^* = 0, \quad J_{13}^* = bk_*,$$

$$J_{21}^* = \alpha\phi(\theta)k_*^{\alpha-1}, \quad J_{22}^* = -(\eta+g), \quad J_{23}^* = bx_*,$$

$$J_{31}^* = 0, \quad J_{32}^* = 0, \quad J_{33}^* = -a.$$

Therefore, J^* is immediately seen to be given by

$$\begin{bmatrix} -(1-\alpha)(\delta+g) & 0 & bk_* \\ \alpha\phi(\theta)k_*^{\alpha-1} & -(\eta+g) & bx_* \\ 0 & 0 & -a \end{bmatrix}.$$

One eigenvalue of this matrix is $-a$. The other two eigenvalues are those of the submatrix

$$D = \begin{bmatrix} -(1-\alpha)(\delta+g) & 0 \\ \alpha\phi(\theta)k_*^{\alpha-1} & -(\eta+g) \end{bmatrix}.$$

We recall that the determinant (resp. trace) of a matrix is also equal to the product (resp. sum) of its eigenvalues. Since

$$Det(D) = (1-\alpha)(\delta+g)(\eta+g) > 0,$$

$$Trace(D) = -[(1-\alpha)(\delta+g)+\eta+g] < 0,$$

we derive that these two roots are both real and negative. Since the matrix J^* has three real negative (stable) roots, we can conclude that the equilibrium is a stable node, where the term node refers to the characteristic shape of the ensemble of orbits around the equilibrium (see Simon and Blume [21]). □

Remark 15. *The point (k_*, x_*, L_*) is locally asymptotically stable. All solutions which start near it remain near the steady state for all time, and, furthermore, they tend towards (k_*, x_*, L_*) as t grows to infinity.*

Remark 16. *$\gamma_{\Omega_t} = 0$ implies $\gamma_{E_t} = g > 0$ along a balanced growth path. Thus, there is no sustainable growth.*

6 Presence of technological progress in abatement

In this section, we assume that the emissions per unit of output are not constant. In particular, we assume they fall at the exogenous rate g_A, i.e.

$$\dot{\Omega}_t/\Omega_t = -g_A, \tag{24}$$

with $g_A > 0$. From (24) we get $\Omega_t = \Omega_0 e^{-g_A t}$. Therefore, the economy of this model becomes described by a non-autonomous system of differential equations

$$\begin{aligned} \dot{k}_t &= Mk_t^\alpha - [\delta+g+n(L_t)]\,k_t, \\ \dot{x}_t &= \phi(\theta)\Omega_0 e^{-g_A t}k_t^\alpha - [\eta+g+n(L_t)]x_t, \\ \dot{L}_t &= n(L_t)L_t. \end{aligned}$$

From sections 3 and 4, we derive the next results.

Corollary 17. *Starting from any $k_0 > 0$, $x_0 > 0$, the long-run behavior of the model's solution is as follows:*

$$\lim_{t\to\infty}(k_t, x_t, L_t) = (k_\infty, 0, L_\infty).$$

Proposition 18. *There exists sustainable growth if*

$$g_A > g.$$

Technological progress in abatement must exceed growth in aggregate output in order for pollution to fall and the environment to improve.

Remark 19. *Brock and Taylor* [6] *showed that*

$$g_A > g + n$$

is the condition for sustainable growth in case of constant population growth rate n.

Let us assume that growth is sustainable. Contrary to the previous section, where there was no technological progress in abatement, we will be able to show that environmental quality deteriorates initially, and improves with economic development in later stage as the economy converges on its balanced growth path. This implies that the model produces a transition path for income and environmental quality, which traces out an environmental Kuznets curve, i.e. an inverted-U shaped relationship between emissions and income.

Lemma 20.

$$\gamma_{E_t} = g - g_A + \gamma_{L_t} + \alpha\gamma_{k_t}. \quad (25)$$

Proof: Take logs of (5), and, then, replace (24), and (1), written as $Y_t = B_t L_t k_t^{\alpha}$, to get

$$\ln E_t = \ln\phi(\theta) + \ln\Omega_t + \ln B_t + \ln L_t + \alpha\ln k_t.$$

Differentiation of this identity with respect to time yields

$$\frac{\dot{E}_t}{E_t} = \frac{\dot{\Omega}_t}{\Omega_t} + \frac{\dot{B}_t}{B_t} + \frac{\dot{L}_t}{L_t} + \alpha\frac{\dot{k}_t}{k_t},$$

i.e. the statement. □

Lemma 21. *Set*

$$N = g - g_A - \alpha(\delta + g) < 0.$$

Then the function

$$\begin{aligned}\gamma_{E_t} &= \alpha M k_t^{\alpha-1} + (1-\alpha)(a - bL_t) + N \\ &\equiv \gamma_{E_t}(k_t, L_t)\end{aligned}$$

is monotone decreasing in k_t and L_t.

Proof: Equations (8) and (10) replaced in (25) give

$$\gamma_{E_t}(k_t, L_t) = \alpha M k_t^{\alpha-1} + (1-\alpha)(a - bL_t) + N.$$

Consequently,

$$\begin{aligned}\frac{\partial\gamma_{E_t}(k_t, L_t)}{\partial k_t} &= -(1-\alpha)\alpha M k_t^{\alpha-2} < 0, \\ \frac{\partial\gamma_{E_t}(k_t, L_t)}{\partial L_t} &= -(1-\alpha)b < 0.\end{aligned}$$

□

$\gamma_{E_t} = \gamma_{E_t}(k_t, L_t)$ is a surface in the three dimensional space (k_t, L_t, γ_{E_t}), whose zero locus is the planar curve $\gamma_{E_t}(k_t, L_t) = 0$. Let us consider the intersection of this surface with the plane of equation $L_t = a/b$.

Lemma 22. *Set $\gamma_{E_t}^{a/b} = \gamma_{E_t}(k_t, a/b) \equiv \gamma_{E_t}(k_t)$. Then*

$$\gamma_{E_t}^{a/b} = 0 \;\; \textit{if} \;\; k_t = \left(-\frac{\alpha M}{N}\right)^{\frac{1}{1-\alpha}}.$$

Proof: $\gamma_{E_t}^{a/b} = 0$ implies $\alpha M k_t^{\alpha-1} + N = 0$. Solving this equation gives the statement. □

Remark 23. *Say T the time at which $\gamma_{E_t}^{a/b} = 0$, i.e.*

$$k_T = \left(-\frac{\alpha M}{N}\right)^{\frac{1}{1-\alpha}}. \quad (26)$$

Setting (13) equal to k_T yields an implicit equation for the time T defined by

$$k_T = \varphi_T^{-1}\left[k_0^{1-\alpha} + (1-\alpha)M\int_0^T \varphi_t^{1-\alpha}dt\right]^{\frac{1}{1-\alpha}}.$$

Proposition 24. *If growth is sustainable and $k_T > k_0$, then the growth rate of emissions is at first positive but turns negative in finite time. If growth is sustainable and $k_T < k_0$, then the growth rate of emissions is negative for all t. If growth is unsustainable, then emissions growth declines with time, but remains positive for all t.*

Proof: From Lemmas 21 and 22, we know that $\gamma_{E_t}^{a/b}$ is decreasing as a function of k_t, and $\gamma_{E_t}^{a/b} = 0$ at k_T. Therefore, we derive that $\gamma_{E_t}^{a/b} > 0$ if $k_0 < k_T$, and $\gamma_{E_t}^{a/b} < 0$ if $k_0 > k_T$. If growth is sustainable, then, from (26) and the fact $k_\infty = [M/(\delta+g)]^{\frac{1}{1-\alpha}}$, we get $k_T < k_\infty$. Equation (13) implies that k_T is reached in finite time from $k_0 < k_T$. If growth is not sustainable, then we have $k_T > k_\infty$. Since k_t converges to k_∞ as time goes to infinity, then $k_T > k_\infty$ always, and by definition of k_T emissions growth remains positive. □

7 Conclusion

In this paper, we have considered a modified version of the Solow-Swan growth model, obtained by the assumption of a logistic-type population growth law and the introduction of environmental pollution. This set up has led the model to be described by a three dimensional dynamical system, whose solution, determined recursively, converges in long-run. In addition, physical capital stock has a closed-form expression via Hypergeometric functions. Investigating sustainable growth, we prove that technological progress in abatement must exceed growth in aggregate output for pollution to fall and the environment to improve. In this case, there is an EKC relationship between the stock of environmental quality and income. Finally, in case there is no technological progress in abatement, the economy is shown to have a unique equilibrium, which is a stable node, and no EKC occurs.

8 Appendix

Let recall some facts about Hypergeometric functions (see Abramowitz and Stegun [1] for details). The Gauss Hypergeometric function ${}_2F_1(c_1, c_2, c_3; z)$, with complex arguments c_1, c_2, c_3, and z, is given by the series

$$
\begin{aligned}
&{}_2F_1(c_1, c_2, c_3; z) \\
&= \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_2)} \sum_{m=1}^{\infty} \frac{\Gamma(c_1+m)\Gamma(c_2+m)}{\Gamma(c_3+m)} \frac{z^m}{m!},
\end{aligned}
$$

where $\Gamma(\cdot)$ is the special function Gamma. The above series is convergent for any c_1, c_2 and c_3 if $|z| < 1$, and for any c_1, c_2 and c_3 such that $Re(c_1+c_2-c_3) < 0$ if $|z| = 1$. Fortunately, there are many continuation formulas of the Gamma Hypergeometric function outside the unit circle. The most practical continuation formulas consist in the integral representations of the Gamma Hypergeometric function. We shall use the following formula

$$
\begin{aligned}
{}_2F_1(c_1, c_2, c_3; z) &= \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_3-c_1)} \cdot \\
&\cdot \int_0^1 t^{c_1-1}(1-t)^{c_3-c_1-1}(1-zt)^{-c_2} dt,
\end{aligned}
$$

where $Re(c_1) > 0$, $Re(c_3 - c_1) > 0$, commonly known as the Euler integral representation.

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