A new Boundary Element Approach for the 3D Compressible Fluid Flow Around Obstacles

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Abstract: - The paper presents a boundary element approach for the problem of the finite span airfoil in a subsonic compressible fluid flow. The singular boundary integral equation equivalent with the mathematical model characterized by partial differential equation is solved in this paper by the use of constant and linear isoparametric boundary elements of Lagrangean type. A special attention is given to the treatment of the singular integrals because their evaluation major influences the numerical solutions accuracy. Some aspects about how to treat the integrals of singular kernels in case of solving 3D problems are presented. An efficient method that is applied consists in using suitable geometrical transformations of coordinates to eliminate the singularities. A computer code based on this method is made and numerical results are obtained. The computer code is tested by making an analytical checking and the results show the efficiency of the method.

Key-Words: - boundary element method, linear boundary element, singular integral, compressible fluid flow

1 Introduction

The Boundary Element Method (BEM) is a powerful numeric technique used to solve many kinds of problems of continuum mechanics such as in electrostatics, heat transfer, fluid flow, acoustics, electromagnetism, etc.

Two big steps have to be made when applying this method as described in many books of BEM, for example [1, 2, 3].

At the first step a boundary integral formulation equivalent with the mathematical model of the problem must be obtained and then this boundary integral, that is usually a singular one, must be solved by using a discretization technique.

The problem to solve is so reduced to a linear system of equations, the unknowns being the nodal values of the functions to be find.

The calculation of the matrix coefficients requires several evaluations of integrals with singular and non-singular kernels. An efficient and accurate method of computing the non-singular integrals is to employ Gaussian integration schemes. But how to deal with the singular ones.

An integral whose integrand reaches an infinite value at one or more points in the domain of integration is named singular integral. In general, singular integrals can be defined by eliminating a small space including the singularity, and then taking the limit as this small space disappears. So sometimes integrals can converge, and in this case, they are said to exist. A singular integral can be understand in the sense of Cauchy Principal Value or in Hadamard sense [13].

Integrals of singular kernels evaluation is one of the most important and difficult step in solving problems with BEM and has a big influence on the numerical solutions accuracy.

Sometimes certain integrals, specially over triangles can be integrated exactly. This is happening for example in bi-dimensional problems when using constant and even linear boundary elements. Some methods used for the singularities treatment in case of bi-dimensional problem are presented in papers [11,12].

The analytical calculus is preferred because it doesn't introduce errors but it is not always possible. Semi-analytical approaches are useful too but sometimes numerical integration represents the only possibility that exists. In general, a numerical quadrature technique must be used.

The effectiveness of the BEM is clearly dependent on the implementation of efficient and accurate integration procedures to evaluate boundary and volume integrals of the singular kernels [9,10], and also depends on the types of boundary elements used for the boundary discretization. For two-dimensional problems, each integration is one-dimensional, but for three-dimensional problems the integration is bi-dimensional. Thus, the collocation method requires the evaluation of a large number of single integrals. The integration schemes to use must be designed to obtain accurate approximations to the integrals in an efficient manner, in order to reduce computational effort involved. Studies about the convergence and the errors that appear when singularities are presented, are made for example in paper [4,5].

For the treatment of the singularities there can be used various techniques. Using their rigorous definition, as limit of an usual integral (defined by eliminating a neighborhood of the singularity, and then taking the limit as it disappears) and doing the analytical integration is the first to be considered but it is not easy to apply. Special quadrature rules are very often used and adaptive schemes too. The singularity can also be eliminated if suitable coordinate and variable transformation methods are applied.

For weakening the singularities the integration by parts offers good results.

Other methods based on: series expansion – Taylor expansions around the singularity or expansions in general orthogonal systems- and subtraction, the choice of fictive nodes, not situated on the real boundary, or other regularization techniques can be successfully and easily applied .

In this paper an efficient method that consists in using polar coordinates to eliminate the singularities is described and applied.

2 The Boundary Integral Equation

We first make a short presentation of the problem we want to solve.

We consider a 3D uniform, steady, potential motion of an ideal compressible fluid, of subsonic velocity $U_\infty \bar{i}$, pressure p_∞ and density ρ_∞ perturbed by the presence of a fixed obstacle of a known boundary, noted Σ, assumed to be smooth and closed, which equation is: $F(X,Y,Z)=0$. We want to find out the perturbation, and the fluid action on the body.

The problem was studied by many authors, with different numerical techniques, and even when BEM was applied the boundary integral formulations were obtained in terms of potential functions, or stream function, not in terms of velocity field.

Using dimensionless variables, we have, for the perturbed motion velocity and pressure fields, the following relations and equations:

$$\overline{V_1} = U_\infty\left(\bar{i} + \overline{V}\right),\ p_1 = p_\infty + \rho_\infty U_\infty^2 P \qquad (1)$$

$$\begin{cases} M^2 \dfrac{\partial P}{\partial X} + Div\overline{V} = 0 \\ \dfrac{\partial \overline{V}}{\partial X} + GradP = 0 \end{cases} \qquad (2)$$

From the last equation we deduce that $P = -U$, and further we deduce new equations to solve for finding the perturbation:

$$\begin{cases} \beta^2 \dfrac{\partial U}{\partial X} + \dfrac{\partial V}{\partial Y} + \dfrac{\partial W}{\partial Z} = 0 \\ \dfrac{\partial V}{\partial X} - \dfrac{\partial U}{\partial Y} = 0 \\ \dfrac{\partial W}{\partial X} - \dfrac{\partial U}{\partial Z} = 0 \end{cases} \qquad (3)$$

where *U*, *V, W* are the scalar components of velocity field.

On Σ the following condition must be satisfied:

$$(1+U)N_X + VN_Y + WN_z = 0, \qquad (4)$$

N_X, N_Y, N_z being the components of the inward normal to Σ: $\overline{N} = \dfrac{GradF}{|GradF|}$.

After the following change of coordinates:

$$x = X,\ y = \beta Y, z = \beta Z, \quad u = \beta U,\ v = V, w = W \qquad (5)$$

the mathematical model is obtained in a simple form:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0 \\ \dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z} = 0 \end{cases} \qquad (6)$$

Tthe new boundary condition, deduced from (4), is:

$$un_x + \beta^2\left(vn_y + wn_z\right) = -\beta n_x \ on\ \Sigma, \qquad (7)$$

where $\overline{n} = \frac{grad'F}{\|grad'F\|}$,

$$\|grad'F\| = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \beta^2\left(\left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right)} \quad (8)$$

It is also required that the perturbation velocity vanishes at infinity: $\lim_{\infty}(u, v, w) = 0$.

The first equation ensures the existence of the potential function $\varphi(x, y, z)$, so as:

$$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}, \; w = \frac{\partial \varphi}{\partial z}, \quad \text{and} \quad \Delta\varphi = 0.$$

As it is known (see for example[6]), the fundamental solution of this equation is :

$$\varphi(\overline{x}) = -\frac{1}{4\pi}\frac{1}{\|\overline{x} - \overline{\xi}\|}, \quad (9)$$

where $\| \; \|$ is the usual Euclidean norme in R^3,

$$\|\overline{x} - \overline{\xi}\| = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2},$$

and $\varphi(x, y, z)$ represents the potential of the motion produced by an unitary source situated in point $\overline{\xi}$ (position vector).

The velocity field is given by:

$$\overline{v} = grad\varphi(\overline{x}) = \frac{1}{4\pi}\frac{\overline{x} - \overline{\xi}}{|\overline{x} - \overline{\xi}|^3}. \quad (10)$$

Assimilating the body with a continuous distribution of sources on the boundary, so on Σ, having an unknown intensity $m(\overline{\xi})$ (presumed to satisfy hölder condition on Σ), we have for the perturbation velocity, $\overline{v}$, the integral representation:

$$\overline{v}(\overline{x}) = -\frac{1}{4\pi}\iint_{\Sigma} m(\overline{\xi})\frac{\overline{\xi} - \overline{x}}{|\overline{x} - \overline{\xi}|^3}da. \quad (11)$$

For $\overline{x} \to \overline{x}_0 \in \Sigma$, and replacing $\overline{\xi}$ by $\overline{x}$ we get the perturbation velocity in any point of the boundary:

$$\overline{v}(\overline{x}_0) = -\frac{1}{2}m(\overline{x}_0)\overline{n}_0 - \frac{1}{4\pi}{}' \iint_{\Sigma} m(\overline{x})\frac{\overline{x} - \overline{x}_0}{|\overline{x} - \overline{x}_0|^3}da, \quad (12)$$

where the sign "′" denotes the Cauchy Principal Value sense of the integral, and $\overline{n}_0 = \overline{n}(\overline{x}_0)$.

Using the boundary condition (7) a singular integral equation for the unknown m is obtained in [7]:

$$\left\{(n_x^0)^2 + \beta^2\left[(n_y^0)^2 + (n_z^0)^2\right]\right\}m(\overline{x}_0) + \frac{1}{2\pi}{}' \iint_{\Sigma} m(\overline{x})\frac{(x - x_0)n_x^0 + \beta^2\left[(y - y_0)n_y^0 + (z - z_0)n_z^0\right]}{|\overline{x} - \overline{x}_0|^3}da = 2\beta n_x^0 \quad (13)$$

For $\beta = 1$ we obtain the boundary integral equation for the incompressible fluid flow.

Denoting by $C(\overline{x}^0) = (n_x^0)^2 + \beta^2\left[(n_y^0)^2 + (n_z^0)^2\right]$ and by

$$K(\overline{x}, \overline{x}_0) = \frac{(x - x_0)n_x^0 + \beta^2\left[(y - y_0)n_y^0 + (z - z_0)n_z^0\right]}{\|\overline{x} - \overline{x}_0\|^3} \quad (14)$$

we can write equation (13) as follows:

$$C(\overline{x}_0)m(\overline{x}_0) + \frac{1}{2\pi}{}' \iint_{\Sigma} m(\overline{x})K(\overline{x}, \overline{x}_0)da = 2\beta n_x^0. \quad (15)$$

For simplifying the writing we shall not use the prim sign to specify that an integral must be understand in this sense.

3 Solving the Boundary Integral Equation with constant boundary elements

For the 3D case the simplest mode to discretize a surface is to use triangular plane elements (with straight lines) with vertices on the given surface. Such a boundary element is well known through its vertices coordinates.

In this approach the body surface, Σ, is divided into M triangles, noted T_j, $j = 1, M$, the vertices of the triangles, noted $\overline{x}_i, i = \overline{1, N}$, being situated on Σ. We get the following integral equation on the approximated boundary:

$$C(\bar{x}_0)m(\bar{x}_0)+\frac{1}{2\pi}\sum_{j=1}^{M}{}' \iint_{T_j} m(\bar{x})K(\bar{x},\bar{x}_0)da = 2\beta n_x^0 \tag{16}$$

Next step in applying BEM consists in implementing the local behaviour of the unknowns into the model.

When using constant boundary elements, for the local approximation of the unknown we consider that this constant is equal with the value taken in $\bar{x}_j^0$ the weight centre of the triangle.

So in this case $m(\bar{x}) = m(\bar{x}_j^0) \underset{not}{=} m_j$, $\bar{x} \in T_j$.
Introducing this approximation in (16) we get:

$$C(\bar{x}_0)m(\bar{x}_0)+\frac{1}{2\pi}\sum_{j=1_1}^{M} m_j \iint_{T_j} K(\bar{x},\bar{x}_0)da = 2\beta n_x^0 \tag{17}$$

If we take in the above equation $\bar{x}_0 = \bar{x}_i^0$, $i = \overline{1,M}$, so if we use a collocaion method we obtain a system of equations for m_j, $j = \overline{1,M}$, of the following form:

$$C_i m_i + \sum_j A_{ij} m_j = B_i \ , \tag{18}$$

where

$$C_i = C(\bar{x}_i^0),\ B_i = 2\beta n_x(\bar{x}_i^0)$$
$$A_{ij} = n_x(\bar{x}_i^0)X_{ij} + \beta^2\left(n_y(\bar{x}_i^0)Y_{ij} + n_z(\bar{x}_i^0)Z_{ij}\right), \tag{19}$$

and X_{ij}, Y_{ij}, Z_{ij} are the scalar components of:

$$\bar{X}_{ij} = \frac{1}{2\pi}\iint_{T_j} \frac{\bar{x} - \bar{x}_i^0}{\left\|\bar{x} - \bar{x}_i^0\right\|^3} da \tag{20}$$

Collocation methods and genetic algorithms can be applied to study other boundary value problems as in [14].

The geometry of a usual triangle is described using the coordinates of its vertices, but since the triangles of the mesh are in arbitrary positions in space, for simplicity the local geometry is described using transformations which bring the triangle into a standard configuration. To achieve this goal shape functions and new variables can be used.

For a plane triangular boundary element, T_j, with three nodes, coinciding with its vertices, using a parallel system of notations where $\bar{x}_j^k, k = \overline{1,3}$ represent these geometric nodes, we have:

$$\bar{x} \in T_j,\ \bar{x}(\bar{\lambda}) = \sum_{k=1}^{3} N_k(\bar{\lambda})\bar{x}_j^k,\ \bar{\lambda} \in \Delta_j, \tag{21}$$

where $N_k(\bar{\lambda})$ depends on the new variables, and satisfy relations:

$$N_l(\bar{\lambda}_j^k) = \delta_{lk},\ \sum_{k=1}^{3} N_k(\bar{\lambda}) = 1, (\forall)\bar{\lambda} \in \Delta_j,\ j = \overline{1,M} \tag{22}$$

and $\bar{\lambda}_j^k$ corresponds to $\bar{x}_j^k$.

We have to evaluate singular integrals if $i = j$ and non-singular integrals in $i \neq j$. For evaluating non-singular integrals an analytical calculus can be done.

For evaluating the singular integrals, Taylor formula can be used, and for applying it, polar coordinates, considered to have the origin in node of the base triangle corresponding to the collocation point, noted $\bar{\eta}^j$, can be used, as in [1].
We consider so that

$$\bar{\lambda} = \bar{\eta}^j + \rho\cos\theta\bar{i}' + \rho\sin\theta\bar{j}', \tag{23}$$

where $\bar{i}', \bar{j}'$ are the axes of the new local system of coordinates.
The following relations hold:

$$\lambda_2 = \eta_2^j + \rho\cos\theta,\ \lambda_3 = \eta_3^j + \rho\sin\theta. \tag{24}$$

Using series expansions and truncations we try to simplify then the factors causing the singularities.
Using Taylor's formula we deduce that

$$N_k(\bar{\lambda}) - N_k(\bar{\eta}^j) = (\lambda_2 - \eta_2)N_{k,1}^1(\lambda_2 - \eta_1^j) + \\ + (\lambda_3 - \eta_3)N_{k,2}^1(\lambda_3 - \eta_2^j) \tag{25}$$

$$N_k(\bar{\lambda}) - N_k(\bar{\eta}^j) = \rho\cos\theta N_{k,1}^1(\lambda_2 - \eta_2^j) + \\ + \rho\sin\theta N_{k,2}^1(\lambda_3 - \eta_3^j) = \rho\widehat{N}_k(\rho,\theta,\bar{\eta}^j) \tag{26}$$

where

$$\widehat{N}_k = N_{k,1}^1(\lambda_2 - \eta_2^j)\cos\theta + N_{k,2}^1(\lambda_3 - \eta_3^j)\sin\theta \tag{27}$$

are the new, or modified shape functions.

In this paper the system of coordinates used to shift from the current boundary element to the basic one is the intrinsic system of coordinates. These coordinates are noted with $\lambda_1, \lambda_2, \lambda_3$ and if P is a point of T_j then we have relations:

$$\lambda_1 = \frac{S_{23}}{S}, \lambda_2 = \frac{S_{13}}{S}, \lambda_3 = \frac{S_{12}}{S} \quad (28)$$

where S is the are of T_j, and S_{23}, S_{13}, S_{12} are the areas of sub triangles obtained with P and vertices of T_j.

Obviously $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and satisfy relations: $0 \le \lambda_i \le 1, i = 1,2,3$

If $\overline{x}_j^k, k = \overline{1,3}$ are the geometric nodes of T_j, for $\overline{x} \in T_j$ we can write:

$\overline{x} = N_1(\lambda_2, \lambda_3)\overline{x}_j^1 + N_2(\lambda_2, \lambda_3)\overline{x}_j^2 + N(\lambda_2, \lambda_3)\overline{x}_j^3$,

with $\lambda_2, \lambda_3 \ge 0, 1 - \lambda_2 - \lambda_3 \ge 0$,

$$N_1(\lambda_2, \lambda_3) = (1 - \lambda_2 - \lambda_3),$$
$$N_2(\lambda_2, \lambda_3) = \lambda_2,\ N_3(\lambda_2, \lambda_3) = \lambda_3 \quad (29)$$

In case of using the above shape functions we obtain the following new shape functions:

$$\widehat{N}_1(\rho, \theta, \overline{\eta}^j) = -(\cos\theta + \sin\theta),$$
$$\widehat{N}_2(\rho, \theta, \overline{\eta}^j) = \cos\theta,\ \widehat{N}_3(\rho, \theta, \overline{\eta}^c) = \sin\theta. \quad (30)$$

The distance between $\overline{x}, \overline{x}_j^0$ on T_j can be evaluate with the new shape functions too:

$$\overline{x} - \overline{x}_j^0 = \sum_{k=1}^{3}\left[N_k(\overline{\xi}) - N_k(\overline{\eta}^j)\right]\overline{x}_j^k = \sum_{k=1}^{3}\rho\widehat{N}_k(\rho, \theta, \overline{\eta}^j)\overline{x}_j^k \quad (31)$$

and further can be deduced that

$$\sum_{k=1}^{3}\widehat{N}_k\overline{x}_j^k = \rho\cos\theta\sum_{k=1}^{3}N_{k,1}^1(\overline{\eta}^j)\overline{x}_j^k + \rho\sin\theta\sum_{k=1}^{3}N_{k,2}^1(\overline{\eta}^j)\overline{x}_j^k + O(\rho^2) \quad (32)$$

Noting by

$$\sum_{k=1}^{3}N_{k,1}^1(\overline{\eta}^j)\overline{x}_j^k = \overline{a}_j^1(\overline{\eta}^j) \text{ and by}$$
$$\sum_{k=1}^{3}N_{k,2}^1(\overline{\eta}^j)\overline{x}_j^k = \overline{a}_j^2(\overline{\eta}^j) \quad (33)$$

we have:

$$\sum_{k=1}^{3}\widehat{N}_k\overline{x}_j^k = \rho\left(\overline{a}_j^1(\overline{\eta}^j)\cos\theta + \overline{a}_j^2(\overline{\eta}^j)\sin\theta + O(\rho)\right) = \rho\widehat{r}(\rho, \theta, \overline{\eta}^j) \quad (34)$$

$$\left\|\overline{x} - \overline{x}_0^j\right\| = \rho\widehat{r}(\rho, \theta, \overline{\eta}^j) = \rho\left\|\sum_{k=1}^{3}\widehat{N}_k\overline{x}_j^k\right\|.$$

Using this method we can eliminate the singularities that arise.

Returning to the singular integral $(i = j)$ in (20) we obtain

$$\overline{X}_{ij} = \frac{S_j}{\pi}\sum_{k=1}^{3}\overline{x}_j^k\iint_{T_j'}\frac{\widehat{N}_k}{\rho\hat{r}_j^3}d\rho d\theta.$$

4 Triangular Linear Boundary Elements and Semi-Analytical Approach

In this paragraph, for solving the integral equation (15) we use linear triangular isoparametric boundary elements of Lagrangean type, and for minimizing the errors that appear due to singular integrals evaluations we make a partial analytical calculus.

The body surface, Σ, is divided as before, into M triangles, noted T_j, $j = 1, M$, the vertices of the triangle, noted $\overline{x}_i, i = \overline{1,N}$, situated on Σ.

Considering $\overline{x}_0 = \overline{x}_i$, $i \in \{1,2,...,N\}$ we have to calculate two types of integrals on T_j, with and without singularities, depending on (if $\overline{x}_i$ is one of the triangle T_j vertices). Thus, we have, for a fixed i, the integral equation:

$$C(\overline{x}_i)m(\overline{x}_i) + \frac{1}{2\pi}\sum_{j\in A_1}\iint_{T_j}m(\overline{x})K(\overline{x}, \overline{x}_i)da + \frac{1}{2\pi}\sum_{j\in A_2}{}^{/}\iint_{T_j}m(\overline{x})K(\overline{x}, \overline{x}_i)da = 2\beta n_x^i \quad (35)$$

where A_1 and A_2 represent the sets of triangles that don't have, respective have an extreme in $\overline{x}_i$.

Next step consists in implementing the local behaviour of the unknowns into the model.

In case of using linear isoparametric boundary elements same functions are used for the local approximation of the geometry and unknowns.

Again we have to evaluate singular and non-singular integrals. For evaluating these integrals we use again the intrinsic system of coordinates as before. This changing of coordinates brings the advantage that integrals will be evaluated on the same basic triangle, of vertices ($(0,0,0)$, $(1,0,0)$ and $(0,1,0)$ represented in Fig 1.

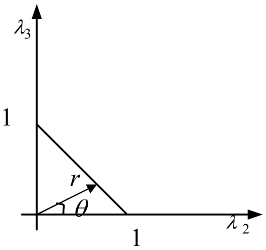

Fig. 1. New domain of integration

First we consider that T_j has all vertices different from $\bar{x}_i$. Naming by $\bar{x}_1^j, \bar{x}_2^j, \bar{x}_3^j$ the vertices of the panel T_j, and by m_j^1, m_j^2, m_j^3 the values of the unknown function in these nodes, and using the above considerations for the geometric approximation, and the local unknown approximation the following relations hold

$$\bar{x} = \bar{x}_j^1 + \left(\bar{x}_j^2 - \bar{x}_j^1\right) r\cos\theta + \left(\bar{x}_j^3 - \bar{x}_j^1\right) r\sin\theta$$

$$m = m_j^1 + \left(m_j^2 - m_j^1\right) r\cos\theta + \left(m_j^3 - m_j^1\right) r\sin\theta \tag{36}$$

where $\theta \in \left[0, \frac{\pi}{2}\right], r \in [0, \rho]$,

with ρ and θ satisfying relation:

$$\rho(\cos\theta + \sin\theta) = 1 \tag{37}$$

Evaluating the Jacobian of the transformation and noting with S_j the area of the initial triangle, T_j we have: $da = 2S_j r dr d\theta$.

We can write that:

$$\frac{1}{2\pi}\iint_{T_j} m(\bar{x}) K(\bar{x}, \bar{x}_i) da = \frac{S_j}{\pi}\left[m_j^1 A_{ij} + m_j^2 B_{ij}^1 + m_j^3 C_{ij}^1\right] \tag{38}$$

where $A_{ij} = A_{ij}^1 - B_{ij}^1 - C_{ij}^1$,

$$A_{ij}^1 = n_x^i\left[\left(x_j^1 - x_i\right)\int_0^{\frac{\pi}{2}} I_1(\theta)d\theta + \int_0^{\frac{\pi}{2}} e_j^1\{\theta\} I_2(\theta)d\theta\right] +$$

$$+\beta^2 n_y^i\left[\left(y_j^1 - y_i\right)\int_0^{\frac{\pi}{2}} I_1(\theta)d\theta + \int_0^{\frac{\pi}{2}} e_j^2\{\theta\} I_2(\theta)d\theta\right] +$$

$$+\beta^2 n_z^i\left[\left(z_j^1 - z_i\right)\int_0^{\frac{\pi}{2}} I_1(\theta)d\theta + \int_0^{\frac{\pi}{2}} e_j^3\{\theta\} I_2(\theta)d\theta\right],$$

$$B_{ij}^1 = n_x^i\left[\left(x_j^1 - x_i\right)\int_0^{\frac{\pi}{2}} \cos\theta I_2(\theta)d\theta + \int_0^{\frac{\pi}{2}} \cos\theta e_j^1\{\theta\} I_3(\theta)d\theta\right] +$$

$$+\beta^2\left\{n_y^i\left[\left(y_j^1 - y_i\right)\int_0^{\frac{\pi}{2}} \cos\theta I_2(\theta)d\theta + \int_0^{\frac{\pi}{2}} \cos\theta e_j^2\{\theta\} I_3(\theta)d\theta\right] +\right.$$

$$\left.+ n_z^i\left[\left(z_j^1 - z_i\right)\int_0^{\frac{\pi}{2}} \cos\theta I_2(\theta)d\theta + \int_0^{\frac{\pi}{2}} \cos\theta e_j^3\{\theta\} I_3(\theta)d\theta\right]\right\},$$

$$C_{ij}^1 = n_x^i\left[\left(x_j^1 - x_i\right)\int_0^{\frac{\pi}{2}} \sin\theta I_2(\theta)d\theta + \int_0^{\frac{\pi}{2}} \sin\theta e_j^1\{\theta\} I_3(\theta)d\theta\right] +$$

$$+\beta^2\left\{n_y^i\left[\left(y_j^1 - y_i\right)\int_0^{\frac{\pi}{2}} \sin\theta I_2(\theta)d\theta + \int_0^{\frac{\pi}{2}} \sin\theta e_j^2\{\theta\} I_3(\theta)d\theta\right] +\right.$$

$$\left.+ n_z^i\left[\left(z_j^1 - z_i\right)\int_0^{\frac{\pi}{2}} \sin\theta I_2(\theta)d\theta + \int_0^{\frac{\pi}{2}} \sin\theta e_j^3\{\theta\} I_3(\theta)d\theta\right]\right\},$$

$$\bar{e}_j(\theta) = \left(\bar{x}_j^2 - \bar{x}_j^1\right)\cos\theta + \left(\bar{x}_j^3 - \bar{x}_j^1\right)\sin\theta,$$

$$\bar{e}_j(\theta) = \left(e_j^1(\theta), e_j^2(\theta), e_j^3(\theta)\right),$$

$$I_n(\theta) = \int_0^{\rho} \frac{r^n}{\left(ar^2 + 2br + c\right)^{\frac{3}{2}}} dr, n = 1,2,3$$

$$a = \left\|\bar{e}_j(\theta)\right\|^2, \; c = \left\|\bar{x}_j^1 - \bar{x}_i\right\|^2,$$

$$b = \left(\bar{x}_j^1 - \bar{x}_i\right)\cdot \bar{e}_j(\theta)$$

(b-dot product between $\bar{x}_j^1 - \bar{x}_i$ and $\bar{e}_j(\theta)$). (39)

Integrals $I_n(\theta)$ from the above relation are the same as in case of an incompressible fluid and they have the following analytical expressions:

$$I_1(\theta)=\frac{\sqrt{c}}{\Delta}-\frac{b\rho+c}{\Delta\sqrt{a\rho^2+2b\rho+c}},$$

$$I_2(\theta)=\frac{(b^2-\Delta)\rho+bc}{a\Delta\sqrt{a\rho^2+2b\rho+c}}-\frac{b\sqrt{c}}{a\Delta}+$$

$$+\frac{1}{a^{3/2}}\ln\frac{\sqrt{a}\sqrt{a\rho^2+2b\rho+c}+a\rho+b}{b+\sqrt{ac}},$$

$$I_3(\theta)=\frac{\sqrt{a\rho^2+2b\rho+c}}{a^2}-$$

$$-\frac{3b}{a^{5/2}}\ln\frac{\left|a\rho+b+\sqrt{a}\sqrt{a\rho^2+2b\rho+c}\right|}{b+\sqrt{ac}}+$$

$$+\frac{\sqrt{c}(b^2-2\Delta)}{a^2\Delta}+\frac{c(\Delta-b^2)+\rho b(3\Delta-b^2)}{a^2\Delta\sqrt{a\rho^2+2b\rho+c}}. \tag{40}$$

Considering now that the triangle, noted T_j, has a vertex in $\overline{x}_i$ we calculate the singular integrals occuring in (35) using the following relations:

$$\overline{x}=\overline{x}_i+\left(\overline{x}_j^2-\overline{x}_i\right)r\cos\theta+\left(\overline{x}_j^3-\overline{x}_i\right)r\sin\theta$$
$$m=m_i+\left(m_j^2-m_i\right)r\cos\theta+\left(m_j^3-m_i\right)r\sin\theta \tag{41}$$

where $\overline{x}_j^2,\overline{x}_j^3$ are the other two nodes of T_j, and m_j^2,m_j^3 are the values of the unknown function, m, in these nodes.

We have:

$$\frac{1}{2\pi}\iint_{T_j} m(\overline{x})\frac{(x-x_i)n_x^i+\beta^2\left((y-y_i)n_y^i+(z-z_i)n_z^i\right)}{\left\|\overline{x}-\overline{x}_i\right\|^3}da=$$

$$=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\left[m_i+\left(m_j^2-m_i\right)r\cos\theta\right]\frac{r\langle\overline{e}_j,\overline{n}\rangle}{r^3\left\|\overline{e}_j(\theta)\right\|^3}2S_j\,rdrd\theta+$$

$$+\frac{\overline{n}_i}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\left[\left(m_j^3-m_i\right)r\sin\theta\right]\frac{r\langle\overline{e}_j,\overline{n}\rangle}{r^3\left\|\overline{e}_j(\theta)\right\|^3}2S_j\,rdrd\theta \tag{42}$$

where

$$\left\langle\langle\overline{e}_j,\overline{n}\rangle\right\rangle\underset{not}{=}e_j^1(\theta)n_x^i+\beta^2\left(e_j^2(\theta)n_y^i+e_j^3(\theta)n_z^i\right),$$

e_j^1,e_j^2,e_j^3 being components of

$$\overline{e}_j(\theta)=\left(\overline{x}_j^2-\overline{x}_i\right)\cos\theta+\left(\overline{x}_j^3-\overline{x}_i\right)\sin\theta \tag{43}$$

When evaluating the singular integrals we use the same parametric representation and the notion of finite part of integral $\int_0^{\rho}\frac{1}{r}dr$ **:**

$$FP\int_0^{\rho}\frac{1}{r}dr=\ln(\rho) \tag{44}$$

and we obtain:

$$\frac{1}{2\pi}\,'\!\iint_{T_j} m(\overline{x})\frac{(x-x_i)n_x^i+\beta^2\left((y-y_i)n_y^i+(z-z_i)n_z^i\right)}{\left\|\overline{x}-\overline{x}_i\right\|^3}da=$$
$$=\frac{S_j}{\pi}\left[m_iA_{ij}^2+\left(m_j^2-m_i\right)B_{ij}^2+\left(m_j^3-m_i\right)C_{ij}^2\right] \tag{45}$$

where

$$A_{ij}^2=\int_0^{\frac{\pi}{2}}\frac{e_j^1(\theta)n_x^i+\beta^2\left(e_j^2(\theta)n_y^i+e_j^3(\theta)n_z^i\right)}{\left\|\overline{e}_j(\theta)\right\|^3}\ln\rho(\theta)d\theta$$

$$B_{ij}^2=\int_0^{\frac{\pi}{2}}\frac{\left[e_j^1(\theta)n_x^i+\beta^2\left(e_j^2(\theta)n_y^i+e_j^3(\theta)n_z^i\right)\right]\cos\theta}{\left\|\overline{e}_j(\theta)\right\|^3}\rho(\theta)d\theta$$

$$C_{ij}^2=\int_0^{\frac{\pi}{2}}\frac{\left[e_j^1(\theta)n_x^i+\beta^2\left(e_j^2(\theta)n_y^i+e_j^3(\theta)n_z^i\right)\right]\sin\theta}{\left\|\overline{e}_j(\theta)\right\|^3}\rho(\theta)d\theta. \tag{46}$$

Denoting by $A'_{ij}=A_{ij}^2-B_{ij}^2-C_{ij}^2$, we further get:

$$\frac{1}{2\pi}^{/}\iint_{T_j} m(\bar{x})\frac{(x-x_i)n_x^i+\beta^2\left((y-y_i)n_y^i+(z-z_i)n_z^i\right)}{\|\bar{x}-\bar{x}_i\|^3}da = $$
$$= \frac{S_j}{\pi}\left[m_i A_{ij}^{/} + m_j^2 B_{ij}^2 + m_j^3 C_{ij}^2\right] \quad (47)$$

and finally equation (9) has the form:

$$m_i A_i + \sum_{j\in A_1}\frac{S_j}{\pi}\left(m_j^1 A_{ij} + m_j^2 B_{ij}^1 + m_j^3 C_{ij}^1\right) +$$
$$+ \sum_{j\in A_2}\frac{S_j}{\pi}\left(m_j^2 \overline{B}_{ij}^2 + m_j^3 \overline{C}_{ij}^2\right) = 2\beta n_i^x \quad (48)$$

where

$$A_i = \left(n_x^i\right)^2 + \beta^2\left(\left(n_y^i\right)^2 + \left(n_x^i\right)^2\right) + \frac{1}{\pi}\left(\sum_{j\in A_2} S_j A'_{ij}\right). \quad (49)$$

Returning to the global system of notation the problem is reduced to the following system of equations:

$$\sum_{j=1}^{N} \widetilde{A}_{ij} f_j = 2\beta n_i^x \quad (50)$$

After solving system (50) we may compute the velocity for the N nodes choosen for the boundary discretization.

5 Numerical results

In order to test the method we shall consider the uniform motion in the presence of a sphere of radius R, centered in the origin of the system of coordinates, and an incompressible fluid. In this case the integral equation (15) can be solved analytically. A solution of this equation can be found in [8].

Using the spherical coordinates for the nodal points $\bar{x} = R(\sin q_1 \cos q_2 \bar{i} + \sin q_1 \sin q_2 \bar{j} + \cos q_1 \bar{k})$ and the method of successive approximations to integrate equation (15) we obtain the exact solution which has the expression:

$$m(q_1, q_2) = \frac{3}{2} U_\infty \cos q_1$$

Comparisons between the analytical values of the intensity m, on the sphere, and the values calculated by means of the boundary element method (with a computer code in MATHCAD) are performed in Fig.2, and Fig.3. The boundary mesh is represented by 24 planar triangles and has 14 control points.

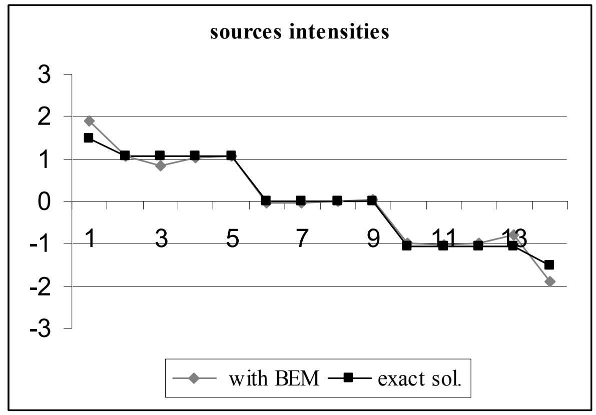

Fig.2. Sources intensities for the 14 control points

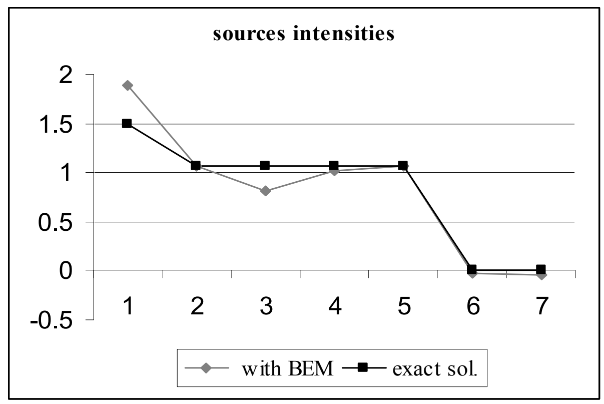

Fig.3. The sources intensities for the first 7 control points

We can observe that the calculated and analytical values of the intensity are very close.

Another simple method that can be applied for the treatment of the singularities when applying BEM with linear boundary elements of plane triangles type consists in using a more refined mesh near the singularity and disposing the element with the singularity.

For example, if we have to evaluate a singular integral, noted $I = \iint_T K da$ over triangle T represented, whose nodes are noted A_1, A_2, A_3, with a singularity in one of it is nodes, for example in node 1, (like the situation that exist in case of using linear boundary element and a collocation method with collocations points in the corners of the triangles), we may apply the following procedure.

We consider the midpoints of the triangle sides and with them we construct sub triangles T_1, T_2, T_3, T_4 as in Fig 4, and we then do the same with triangle T_1 instead of T. We obtain a new mesh of the initial triangle as in Fig.5.

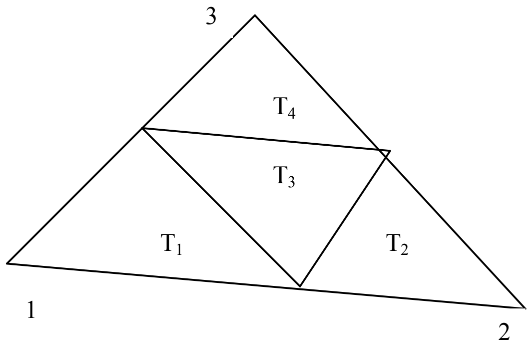

Fig.4. The mesh after the first step

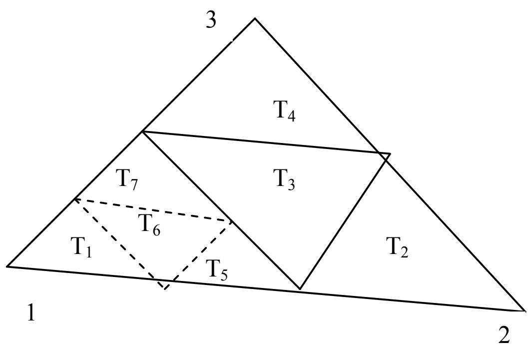

Fig.5. The mesh after the second step

So we have:

$$I = \iint_T Kda = \sum_{j=1}^{4} \iint_{T_j} Kda = \sum_{j=1}^{7} \iint_{T_j} Kda = ...$$

We continue this algorithm in the same manner, refining at each step the single triangle with a corner point in node 1, until the mesh size is small enough near this node and the integral over T will not be too much affected if we will eliminate the small triangle near the singularity.

For evaluating the integral left we can use usual quadrature schemes.

2 Conclusions

This paper briefs out some aspects about how to apply the BEM to solve 3D problems of fluid flow, through a concrete case: the problem of the subsonic compressible fluid flow around obstacles. Techniques that can be used for the treatment of singularities, when using triangular boundary elements for the boundary discretization have been discussed too.

Taking into account that the boundary integral equation itself is a statement of the exact solution to the problem posed, we can say that errors arise in principal due to discretization and numerical approximations, due to our inability to carry out the required integrations in closed form.

The effectiveness of the BEM is clearly dependent on the integrals of the singular kernels evaluation. If the numerical integration procedure is made sufficiently sophisticated (by using for example curved boundary elements and continuously varying distributions of functions over the boundary) than the errors so introduced can be very small indeed. Numerical integration is, of course, always a much more stable and precise process than numerical differentiation and neither the direct nor the indirect BEM require any differentiation of numerical quantities whatsoever, and this recommends the BEM as an efficient numerical technique for solving boundary values problems of continuum mechanics, described by partial differential equations with fundamental solutions.

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