Bifurcation diagrams of generic families of singular systems under proportional and derivative feedback

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Abstract:- In this paper we study qualitative properties about nearby singular systems using stratification method. We construct the stratification partitioning the set of singular systems according to the complete set of discrete invariants. We show that it is a constructible stratification and that it is Whitney regular for one input regularizable systems. We give an application to the obtention of bifurcation diagrams for few parameter generic families of singular regularizable systems.

Key-Words:- Singular systems, Feedback and derivative feedback equivalence, stratification, canonical form, orbit, stratum.

1 Introduction

Let M be the differentiable manifold of triples of matrices (E, A, B) where $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, which represent singular time-invariant linear systems in the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

It is well known that a system $E\dot{x} = Ax + Bu$ is called regular if and only if $\det(\alpha E - \beta A) \neq 0$ for some $(\alpha, \beta) \in \mathbb{C}^2$. Remember that the regularity of the system guarantees the existence and uniqueness of classical solutions.

For no regular systems one can ask for whether the close loop system is uniquely solvable for all consistent initial solution, when this is possible the system will be called regularizable by proportional and derivative feedback. That is to say, a system is regularizable if and only if, there exist matrices $F_E, F_A \in M_{m\times n}(\mathbb{C})$ such that the system $(E+BF_E)\dot{x} = (A+BF_A)x+Bu$ is regular. A special subset of regularizable systems is the subset formed for the so called standardizable systems that they are systems (E, A, B), such that there exists a derivative feedback F_E such that $E + BF_E$ is invertible, so after to apply the derivative feedback F_E and premultiplying the equation by $(E+BF_E)^{-1}$, we obtain a standard system.

Notice that, the set M_R consisting in all regularizable triples is an open and dense set in the space of all triples.

The central goal of this paper is to construct a stratification of the space M_R of regularizable systems, that is to say, a partition of M_R in a locally finite (in fact finite) family of differentiable manifolds, called strata. As a starting point, the space of regularizable systems is partitioned into equivalent classes under the proportional and derivative feedback equivalence relation considered. After to observe that the equivalent classes can be seen as orbits by a group Lie action, geometrical techniques can be used and perturbations of a given system are obtained from the local description of this partition, which can be derived from versal deformations. Nevertheless, the partition into orbits is not locally finite. Following the method used by Arnold [1], we consider "strata" formed by the uncountable union of the orbits having the same collection of discrete invariants varying only in the value of eigenvalues. Therefore, the local description of this new partition gives information about the perturbation of the controllability indices, etc. of a system. The key point lies in showing that each stratum is a differentiable manifold, and as a consequence the new partition is in fact a stratification that we will call Kronecker stratification. The proof follows from the local description of the stratification given by the miniversal deformation. The authors Elmroth, Johansson, Kågström [6], [7], Puerta, Helmke [13], study stratifications for the case of standard systems, obviously our case of regularizable systems include the case of standard ones. Pervouchine [12] stratifis the space of matrix pencils associate to standard ssytems. Several other people futhermore study related topics as distance to uncontrollability see [2], [3], [10], [11] for example. Computing the canonical structure of a singular linear system is an ill-posed problem, that is to say small changes in the input data matrices E, A and B may drastically change the computed canonical structure. Besides knowing the canonical structure it is also important to identify nearby canonical structures in order to explain the behavior and to study the robustness of a singular system under small perturbations.

Finally, given a parametrized differentiable family of singular regularizable systems, the stratification induces a partition in the space of parameters, known as the bifurcation diagram of the family. The bifurcation diagram of a family of singular reguarizable systems gives precise information about the qualitative properties of the systems arising in the family and about the effects of local perturbations of the parameters.

2 Preliminaries

For every integers p, q, we will denote by $M_{p\times q}(\mathbb{C})$ the space of p-rows and q-columns complex matrices, and if $p = q$ we will write only $M_p(\mathbb{C})$, and by $Gl(n;\mathbb{C})$ the linear group formed by the invertible matrices of $M_p(\mathbb{C})$. In all the paper, M denote the space of triples of matrices (E, A, B) with $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$ and M_R denote the open and dense space of regularizable systems.

In order to classify systems preserving regularizability character, we consider the following equivalence relation.

Definition 1 *The triples (E, A, B) and (E', A', B') in M, are said to be equivalent if and only if*

$$(E', A', B') = (QEP + QBF_E, QAP + QBF_A, QBR) \tag{2}$$

for some $Q, P \in Gl(n;\mathbb{C})$, $R \in Gl(m;\mathbb{C})$, $F_E, F_A \in M_{m\times n}(\mathbb{C})$. In a matrix form:

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_E & F_A & R \end{pmatrix}.$$

As a consequence and after to observe that the set M_R is closed under equivalence relation considered, a regularizable system can be reduced as follows.

Proposition 1 *Let (E, A, B) be a n-dimensional m-input regularizable system. Then, it can be reduced to $\left(\begin{pmatrix} I_r & \\ & N_1 \end{pmatrix}, \begin{pmatrix} A_1 & \\ & I_{n-r} \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}\right)$ where (A_1, B_1) is a pair in its Kronecker canonical form and N_1 is a nilpotent matrix in its canonical reduced form. Concretely, $(A_1, B_1) = \left(\begin{pmatrix} N_2 & 0 \\ 0 & J \end{pmatrix}, \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}\right)$ where $N_2 = \operatorname{diag}(N_{21}, \ldots, N_{2p})$, N_{2i} k_i-nilpotent matrices $\begin{pmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{pmatrix}$, $B' = \operatorname{diag}(e_1, \ldots, e_p)$, $e_i = (0, \ldots, 0, 1)^t \in M_{k_i\times 1}(\mathbb{C})$, J is a Jordan matrix having $\lambda_1, \ldots, \lambda_t$ distinct eigenvalues and σ_i as Segre characteristic for each eigenvalue.*

A complete system of invariants permitting to obtain the canonical form, can be found in [4].

The equivalence relation may be seen as induced by Lie group action. Let us consider the following Lie group $\mathcal{G} = Gl(n;\mathbb{C}) \times Gl(n;\mathbb{C}) \times Gl(m;\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$ acting on M.

The action $\alpha : \mathcal{G} \times M \longrightarrow M$ is defined as follows:

$$\alpha((P,Q,R,F_E,F_A),(E,A,B)) = (QEP + QBF_E, QAP + QBF_A, QBR). \tag{3}$$

So, the orbits are equivalence classes of triples of matrices under the equivalence relation considered.

$$\mathcal{O}(E,A,B) = \{(E_1,A_1,B_1)$$

with

$$\begin{aligned} E_1 &= QEP + QBF_E, \\ A_1 &= QAP + QBF_A, \\ B_1 &= QBR, \end{aligned}$$

$\forall Q,P \in Gl(n;\mathbb{C}),\ R \in Gl(m;\mathbb{C}),\ F_E,F_A \in M_{m\times n}(\mathbb{C})$.

Proposition 2 *Any orbit is a constructible subset of M.*

It is well known that the orbits are embedded submanifolds of M.

Given a triple $(E,A,B) \in M_R$ in its canonical reduced form, a minitransversal differentiable family to the orbits is (see ([4])):

$(E + X, A + Y, B + Z)$ with $X = \begin{pmatrix} 0 & 0 \\ X_3 & X_4 \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} Z_1 & 0 \end{pmatrix}$. Y_1, Z_1 in such away that $(A_2 + Y_1, B_1 + Z_1)$ being a minimal deformation of the pair (A_1,B_1). Concretely, $Y_1 = \begin{pmatrix} 0 & 0 \\ Y_1^2 & Y_2^2 \end{pmatrix}$, $Z_1 = \begin{pmatrix} Z_1^1 & Z_2^1 \\ 0 & Z_2^2 \end{pmatrix}$ where the block-decomposition correspond to that of (A_1,B_1) and

i) all the entries in Y_1^2 are zero except $y_i^{p+1},\dots,y_i^n$, $\quad i = 1, k_1+1,\dots,k_1+\dots+k_{p-1}+1$,

ii) the matrices Y_2^2 are such that $J + Y_2^2$ is the miniversal deformation of J given by Arnold [1],

iii) all the entries in Z_1^1 are zero except z_i^j, $\quad 2 \le i \le p$, $\quad k_1+\ldots+k_{i-2}+k_i+1 \le j \le k_1+\ldots+k_{i-2}+k_{i-1}-1$ (provided that $k_i \le k_{i-1}+2$,

iv) Z_2^1 is such that $z_{p+1}^i = \ldots = z_m^i = 0$, $\quad i = k_1, k_1+k_2, \dots, k_1+\ldots+k_p$,

v) all the entries in Z_2^2 are arbitrary.

$N_1 + X_4$ is a miniversal deformation of the square matrix N_1 given by Arnold (see [1]), and $X_3 = (X_{ij})$ with

$$X_{ij} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ & & \\ 0 & \dots & 0 \\ x_1 & \dots & x_\ell \end{pmatrix},$$

$$X_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ & & & & \\ 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & x_1 & \dots & x_\ell \end{pmatrix},$$

corresponding to size in the nilpotent submatrices N_1 and N_2.

3 The strata

A stratification of a subset V of a manifold N is a partition $\cup_i X_i$ of V into submanifolds X_i of N (called the strata) which satisfies the local finiteness condition: every point in V has a neighborhood in N which meets only finitely many strata.

Let W, N manifolds, $f : W \longrightarrow N$ a differentiable map, and $V = \cap_i X_i$ a stratification on N. We say that f is transversal to V if it is transversal to each stratum X_i of V.

Let N be a finite-dimensional vector space, X, Y submanifolds of N, and $x \in X \cap \overline{Y}$. We say that Y is Whitney regular over X at x when the following condition holds: let (x_i), (y_i) be sequences in X, Y, respectively, both converging to x, with $x_i \neq y_i$ for all i. Take L_i to be the line spanned by $x_i - y_i$ and T_i to be the tangent space T_{y_i}. If (L_i) converges to L (in the Grassmannian of 1-dimensional subspaces of N) and T_i converges to T (in the Grassmannian of q-dimensional subspaces of N, $q = \dim Y$), then $L \subseteq T$.

It is not difficult to see that this condition is invariant under diffeomorphisms. Hence, we can define in an obvious way the Whitney regular condition when N is a manifold.

Finally, let $\cup_i X_i$ be a stratification of a subset V of a manifold N We say that this stratification is Whitney regular when every stratum X_i is Whitney regular over X_j for all $i \neq j$.

The space M_R of all triples of matrices is formed by the disjoint union of all orbits of the triples and the frontier of each orbit is formed by orbits of strictly lower dimension.

We remark that this partition is not locally finite (as example, in a neighborhood of any triple having continuous invariants, there are infinite orbits that we will call "having the same type", they are orbits having the same collection of discrete invariants varying only in the continuous ones).

In order to obtain a finite partition preserving the orbit structure, we group the orbits having the same type, we call this set stratum in M_R. There are only finitely many strata, each is an orbit or an uncountable union of orbits partitioning M_R.

So, the strata in M_R are determined by the controllability indices k, Segre characteristic σ and ∞-characteristic τ . We denote each strata by $\mathcal{S}(k, \sigma, \tau)$, and (k, σ, τ) will be called symbol of strata. We denote Σ the partition $\cup_{(k,\sigma,\tau)}\mathcal{S}(k, \sigma, \tau)$ of M_R which will be called Kronecker stratification

When we write (k, σ, τ) we do not exclude the possibilities that $(k, \sigma, \tau) = (k)$, $(k, \sigma, \tau) = (k, \sigma)$, $(k, \sigma, \tau) = (k, \tau)$, $(k, \sigma, \tau) = (\sigma, \tau)$,$(k, \sigma, \tau) = (\sigma)$ or $(k, \sigma, \tau) = (\tau)$. First case (k) means that the triple is a standardizable and controllable triple, (k, σ) that the triple is standardizable with a no-controllable part, (k, τ) that the standardizable part of the system is controllable, (σ, τ), (σ), and (τ) that $B = 0$ with standard and non-standard part, standard triple, and no-standard one respectively.

Proposition 3 *Any stratum in M_R is a constructible and connected subset of M_R.*

Proof. Let $\mathcal{S}(k, \sigma, \tau)$ any stratum in M_R corresponding to the controllability indices $k = (k_1, \dots, k_p)$, Segre characteristic $\sigma = (\sigma_1, \dots, \sigma_t)$ and ∞-characteristic $\tau = (\tau_1, \dots, \tau_s)$. Let us consider $\mathbb{C}^{(t)} = \{(\lambda_1, \dots, \lambda_t) \mid \lambda_i \neq \lambda_j, i \neq j\} \subset \mathbb{C}^t$. For each $(\lambda_1, \dots, \lambda_t) \in \mathbb{C}^{(t)}$ we consider (E, A, B) the triple of matrices in its canonical reduced form in the stratum $\mathcal{S}(k, \sigma, \tau)$ having these eigenvalues. Finally we consider the map $\rho : \mathcal{G} \times \mathbb{C}^{(t)} \longrightarrow M_R$ defined by $\rho(g, (\lambda_1, \dots, \lambda_t)) = \alpha(g, (E, A, B))$. Obviously, $\mathcal{G} \times \mathbb{C}^{(t)}$ is a constructible set, ρ is a rational map, and $\rho(\mathcal{G} \times \mathbb{C}^{(t)}) = \mathcal{S}(k, \sigma, \tau)$, so that, according to the Chevalley theorem $\mathcal{S}(k, \sigma, \tau)$ is a constructible set. Moreover, it is connected because ρ is continuous and $\mathcal{G} \times \mathbb{C}^{(t)}$ is connected. $\square$

Lemma 1 *Let $\varphi : \Lambda \longrightarrow M_R$ be a deformation of (E, A, B) minitransversal to the orbit $\mathcal{O}(E, A, B)$ given in §2. Let $V \subset \mathcal{G}$ a subvariety minitransversal to the stabilizer $St(E, A, B)$. Then, the map $\beta : \Lambda \times V \longrightarrow M_R$ defined by $\beta(\lambda, (P, Q, R, F_E, F_A) = Q \begin{pmatrix} E(\lambda) & A(\lambda) & B(\lambda) \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_E & F_A & R \end{pmatrix}$, with $\lambda = (\lambda_1, ..., \lambda_\ell)$ and $(E(\lambda), A(\lambda), B(\lambda)) = \varphi(\lambda)$, is a diffeomorphism at $(0, I)$.*

Remember that $St(E, A, B)$ is the set of $(P, Q, R, F_E, F_A) \in \mathcal{G}$ such that $\alpha((P, Q, R, F_E, F_A), (E, A, B)) = (E, A, B)$.

Proof. The inverse function theorem ensures that β is a local diffeomorphism at $(0, I)$, if and only if $d\beta_{(0,I)}$ is a diffeomorphism.

Taking into account that $\dim(V \times \Lambda) = 2n^2 + mn = \dim M_R$, it suffices to observe that $d\beta$ is surjective. $\square$

Lemma 2 *Let (E, A, B) be a triple in M_R, $\mathcal{O}(E, A, B)$ its orbit, $\mathcal{S}(k, \sigma, \tau)$ its stratum, and Γ the variety transversal to the orbit (we will consider the miniversal minimal deformation explicited in §3). Then, in a* *neighborhood of* (E, A, B)*,* $\mathcal{S}(k, \sigma, \tau)$ *is a regular subvariety at* (E, A, B) *if and only if* $\mathcal{S}(k, \sigma, \tau) \cap \Gamma$ *is.*

Proof. Suppose $\mathcal{S}(k, \sigma, \tau)$ regular at (E, A, B). Taking into account that Γ is transversal to $\mathcal{O}(E, A, B)$, it also is transversal to $\mathcal{S}(k, \sigma, \tau)$. Then, $\mathcal{S}(k, \sigma, \tau) \cap \Gamma$ is regular at (E, A, B).

Conversely, suppose $\mathcal{S}(k, \sigma) \cap \Gamma$ regular at (E, A, B). The local triviality given in lemma 2, we have

$$\mathcal{S}(k, \sigma, \tau) = \beta((\mathcal{S}(k, \sigma, \tau) \cap \Gamma) \times V)$$

locally in (E, A, B). Then $\mathcal{S}(k, \sigma, \tau)$ is regular at (E, A, B). $\square$

Now we analyze $\mathcal{S}(k, \sigma, \tau) \cap \Gamma$ and $\mathcal{S}_i \cap \Gamma$.

Proposition 4 *Let* (E, A, B) *be a triple in* $\mathcal{S}(k, \sigma, \tau)$*, in its canonical reduced form*

i) if $(X, Y, Z) \neq (0, 0, 0)$ *then* $(E, A, B) + (X, Y, Z) \notin \mathcal{O}(E, A, B)$

ii) $(E, A, B) + (X, Y, Z) \in \mathcal{S}(k, \sigma, \tau)$ *if and only if* $X = 0$*,* $Y_1^2 = 0$*,* $Z = 0$ *and* $J + Y_2^2$ *has the same Segre characteristic* σ *than* J*.*

Theorem 1 *The strata are submanifolds of* M_R*.*

Proof.
It is obvious for strata that they are orbits. Let $(E, A, B) \in M_R$, taking into account the homogeneity of its orbits we can assume that the triple is in its canonical reduced form. By lemma 2, it suffices to prove that $\mathcal{S}(k, \sigma, \tau) \cap \Gamma$ is regular at (E, A, B), where Γ is the particular one considered in §2. From proposition 4, it follows that $\mathcal{S}(k, \sigma, \tau) \cap \Gamma$ is formed by triples of the form (E, A, B) with $E = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & N_1 \end{pmatrix}$, $A = \begin{pmatrix} N_2 & 0 & 0 \\ 0 & J{+}Y_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ and $B = \begin{pmatrix} B' & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $I_1, N_2 \in M_{n_1}(\mathbb{C})$, $I_2, J + Y_2 \in M_{n_2}(\mathbb{C})$, $I_3, N_1 \in M_{n_3}(\mathbb{C})$, such that $J + Y_2$ has the Segre characteristic of J, or equivalently, such that $J + Y_2$ belongs to the Segre stratum $\mathcal{S}(J)$ of J in the Segre stratification of square matrices under similarity given by Arnold in [1].

Therefore, the mapping $\phi : M_{n_2} \longrightarrow M_R$ defined by

$$\phi(C) = \left(\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & N_1 \end{pmatrix}, \begin{pmatrix} N_2 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \begin{pmatrix} B' & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

is a diffeomorphism such that $\phi(\mathcal{S}(J) \cap \Gamma_J) = \mathcal{S}(k, \sigma, \tau) \cap \Gamma$ (Γ_J denotes the variety transversal to the orbit given in [1] and $S(J)$ the Segre stratum in the stratification of the space of square matrices under similarity). Gibson in [9] proves that the Segre strata of the stratification of the space of square matrices are regular, so the proof is completed. $\square$

We can compute the dimension of the strata.

Proposition 5 *Let* $(E, A, B) \in M_R$*,* $\mathcal{O}(E, A, B)$ *be its orbit and* $\mathcal{S}(k, \sigma, \tau)$*, its stratum, then*

$$\dim \mathcal{S}(k, \sigma, \tau) = t + \dim \mathcal{O}(E, A, B)$$

where t *is the number of distinct eigenvalues of* (E, A, B)*.*

Proof. It suffices to bear in mind that $\dim \mathcal{S}(k, \sigma, \tau) \cap \Gamma) = t$. $\square$

Finally, and as example we explicit the set S of strata $S(i)$ of two dimensional one input singular systems.

$$S(1) = \mathcal{O}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

$$S(2) = \cup_{a \in \mathbb{C}} \mathcal{O}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ & a \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right),$$

$$S(3) = \cup_{\mu_1 \neq \mu_2} \mathcal{O}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right).$$

$$S(4) = \cup_{\mu} \mathcal{O}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 1 & \mu \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right).$$

$$S(5)=\mathcal{O}\left(\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right).$$

$$S(6)=\cup_{\lambda\neq 0}\mathcal{O}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}\lambda&0\\0&1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right).$$

$$S(7)=\mathcal{O}\left(\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right).$$

$$S(8)=\mathcal{O}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right).$$

$$S(9)=\mathcal{O}\left(\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right).$$

$$S(10)=\mathcal{O}\left(\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right).$$

$$S(11)=\mathcal{O}\left(\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right).$$

$$S(12)=\cup_{\mu}\mathcal{O}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}\mu&0\\0&0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right).$$

$$S(13)=\mathcal{O}\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right).$$

$$S(14)=\mathcal{O}\left(\begin{pmatrix}0&1\\x_2&0\end{pmatrix},\begin{pmatrix}y_1&0\\y_2&1\end{pmatrix},\begin{pmatrix}z_1\\z_2\end{pmatrix}\right).$$

Obviously $S(i)\cap S(j)=\emptyset$, $\forall i\neq j$ and $\cup_i S(i))M$.

4 Regularity of the stratification ($m=1$)

Given a triple $(E,A,B)\in M_R$ and Γ the minitransversal variety considered in §2, let us to consider the stratification $(\cup_{(k,\sigma,\tau)}(\mathcal{S}(k,\sigma,\tau)\cap\Gamma)$ of a sufficiently small neighborhood Γ of (E,A,B) which we denote by $\Sigma\cap\Gamma$. Notice that, it is well defined because Γ is not only transversal to $\mathcal{O}(E,A,B)$ but also to every orbit sufficiently close to (E,A,B) and hence to every stratum sufficiently close to (E,A,B).

Lemma 3 *With the above notations, the stratification Σ is Whitney regular over (E,A,B) at (E,A,B) if and only if $\Sigma\cap\Gamma$ is Whitney regular over $\mathcal{S}(E,A,B)\cap\Gamma$ at (E,A,B).*

The proof is analogous to one in [9] for square matrices under similarity.

First, we proof the regularity for a special kind of strata.

Definition 2 *A triple of matrices $(E,A,B)\in M_R$ is called simple if it has one eigenvalue at most. A stratum $\mathcal{S}(k,\sigma,\tau)$ is called simple if its elements are simple.*

The simple strata verify a particular homogeneity property.

Lemma 4 *If $\mathcal{S}(k,\sigma,\tau)$ is a stratum in M_R, $(E,A,B)\in\mathcal{S}(k,\sigma,\tau)$ and $\lambda\in\mathbb{C}$, then*

1) μ is an eigenvalue of (E,A,B) if and only if $\lambda+\mu$ is an eigenvalue of $(E,A+\lambda E,B)$,

2) $(E,A+\lambda E,B)\in\mathcal{S}(k,\sigma,\tau)$.

Proof. 1) According definition of eigenvalue we have

$$\begin{array}{l}\operatorname{rank}\begin{pmatrix}-\mu E+A & B\end{pmatrix}=\\ \operatorname{rank}\begin{pmatrix}-\mu E+\lambda E-\lambda E+A & B\end{pmatrix}=\\ \operatorname{rank}\begin{pmatrix}-(\lambda+\mu)E+A+\lambda E & B\end{pmatrix}<n.\end{array}$$

2) To see that (E,A,B) and $(E,A+\lambda E,B)$ have the same controllability indices, the same Segre characteristic and the same ∞-characteristic first of all we observe that we can consider the triple in its canonical reduced form

$$\begin{pmatrix}E_c & A_c & B_c\end{pmatrix}=Q\begin{pmatrix}E & A & B\end{pmatrix}\begin{pmatrix}P & & \\ & P & \\ F_E & F_A & R\end{pmatrix},$$

so, for each λ we have

$$\begin{pmatrix} E_c & A_c+\lambda E_c & B_c \end{pmatrix} = \\ Q\begin{pmatrix} E & A+\lambda E & B \end{pmatrix}\begin{pmatrix} P & & \\ & P & \\ F_E & F_A+\lambda F_E & R \end{pmatrix},$$

Taking into account that $(E_c, A_c, B_c) = \left(\begin{pmatrix} I_1 & \\ & N\end{pmatrix}, \begin{pmatrix} A_1 & \\ & I_2\end{pmatrix}, \begin{pmatrix} B_1 \\ 0\end{pmatrix}\right)$ and $(E_c, A_c + \lambda E_c, B_c) = \left(\begin{pmatrix} I_1 & \\ & N\end{pmatrix}, \begin{pmatrix} A_1+\lambda I_1 & \\ & I_2\end{pmatrix}, \begin{pmatrix} B_1 \\ 0\end{pmatrix}\right)$, we observe that the controllability indices and the Segre characteristic depends only on the controllability indices and the Segre characteristic of the pairs (A_1, B_1) and $(A_1 + \lambda I_1, B_1)$, and following [8] they are the same.

Now, we compute the ∞-characteristic for both triples. We observe that we can consider the subsystem $(N, I_2, 0)$.

$$\operatorname{rank}\begin{pmatrix} I_2 & & \dots & & 0 \\ N & I_2 & & & \\ & N & & & \\ & & \ddots & & \\ & & & I_2 & 0 \\ & & & N & N \end{pmatrix} = \operatorname{rank}\begin{pmatrix} I_2 & & \dots & 0 \\ & I_2 & & \\ & & \ddots & \\ & & & N \end{pmatrix},$$

$\vdots$

$$\operatorname{rank}\begin{pmatrix} I_2 & & \dots & N \\ N & I_2 & & \\ & N & & \\ & & \ddots & \\ & & N & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} I_2 & & \dots & 0 \\ & I_2 & & \\ & & \ddots & \\ & & & I_2 & 0 \\ & & & & N^n \end{pmatrix}.$$

$$\operatorname{rank}\begin{pmatrix} I_2+\lambda N & & \dots & & 0 \\ N & I_2+\lambda N & & & 0 \\ & N & & & \\ & & \ddots & & \\ & & & I_2+\lambda N & 0 \\ & & & N & N \end{pmatrix} = \\ \operatorname{rank}\begin{pmatrix} I_2 & & \dots & 0 \\ & I_2 & & \\ & & \ddots & \\ & & & N \end{pmatrix},$$

$\vdots$

$$\operatorname{rank}\begin{pmatrix} I_2+\lambda N & & \dots & N \\ N & I_2+\lambda N & & \\ & N & & \\ & & \ddots & \\ & & N & 0 \end{pmatrix} = \\ \operatorname{rank}\begin{pmatrix} I_2 & & \dots & 0 \\ & I_2 & & \\ & & \ddots & \\ & & & I_2 & 0 \\ & & & & N^n \end{pmatrix}.$$

□

Proposition 6 *Let $\mathcal{S}(k,\sigma,\tau)$ a simple stratum. For any couple of triples (E, A, B), $(E', A', B') \in \mathcal{S}(k, \sigma, \tau)$, there exists a diffeomorphism f of M_R preserving strata, and such that $f(E, A, B) = (E', A', B')$.*

Proof. If $(k, \sigma, \tau) = (k)$, $(k, \sigma, \tau) = (k, \tau)$ or $(k, \sigma, \tau) = (\tau)$ the strata are orbits, so the result is trivial.

Suppose now, that $\sigma = (\sigma_1)$ and let λ, λ' be the eigenvalues of (E, A, B) and (E', A', B') respectively. Then, because the above lemma the triple $(E, A + (\lambda' - \lambda)E, B)$ is equivalent to (E', A', B'). Hence there exist $g = (Q, P, R, F_E, F_A) \in \mathcal{G}$ such that $\alpha(g, (E, A + (\lambda' - \lambda)E, B) = (E', A', B')$. It is straightforward that the map $f(X, Y, Z) = \alpha(g, (X, (\lambda' - \lambda)X, Z))$ verifies the desired conditions. □

The constructibility condition of the strata ensures that any stratum has a Whitney regular point, so the homogeneity property over simple strata implies that all points of the strata are regular and we have the following lemma.

Lemma 5 *The stratification Σ over M_R is Whitney regular over any simple stratum.*

Theorem 2 *If $m = 1$ the Kronecker stratification is Whitney regular.*

Proof. The strata $\mathcal{S}(k)$, $\mathcal{S}(k, \tau)$, and $\mathcal{S}(\tau)$ are simple.

Let $(E, A, B) \in M_R$, $\mathcal{O}(E, A, B)$ be its orbit, and $\mathcal{S}(k, \sigma, \tau)$ be its stratum. We shall prove that Σ is Whitney regular over $\mathcal{S}(k, \sigma, \tau)$ at (E, A, B). Let Γ be the linear variety defined in §3 and we denote with the same symbol the neighborhood of (E, A, B) where the isomorphism β holds. Then, according lemma 3, it is sufficient to prove that $\Sigma \cap \Gamma$ is Whitney regular over $\mathcal{S}(k, \sigma, \tau) \cap \Gamma$ at (E, A, B).

Taking account the different expression for miniversal minimal deformation first, we suppose $B \neq 0$, and we will discuss later the case $B = 0$, Since $B \neq 0$, the triple is $\left(\begin{pmatrix} I_1 & & \\ & I_2 & \\ & & N_1 \end{pmatrix}, \begin{pmatrix} N_2 & & \\ & J & \\ & & I_3 \end{pmatrix}, \begin{pmatrix} B' \\ 0 \\ 0 \end{pmatrix}\right)$.

Now we consider the triples of matrices:

$$(E_0, A_0, B_0) = \left(\begin{pmatrix} I_1 & \\ & N_1 \end{pmatrix}, \begin{pmatrix} N_2 & \\ & I_3 \end{pmatrix}, \begin{pmatrix} B' \\ 0 \end{pmatrix}\right),$$

$$(E_1, A_1, B_1) = \left(\begin{pmatrix} I_1 & \\ & I_2 \end{pmatrix}, \begin{pmatrix} N_2 & \\ & J \end{pmatrix}, \begin{pmatrix} B' \\ 0 \end{pmatrix}\right).$$

Let Σ_0, Σ_1 be the Kronecker-stratifications of the respective spaces of triples of matrices. If we denote $\mathcal{S}_i(E_i, A_i, B_i)$ the stratum of (E_i, A_i, B_i) in Σ_i, $i = 0, 1$, we observe that $\mathcal{S}_0(E_0, A_0, B_0)$ is a simple stratum, then Σ_0 is Whitney regular over $\mathcal{S}_0(E_0, A_0, B_0) \cap \Gamma_0$ at (E_0, A_0, B_0), where Γ_0, is the linear variety defined in §2:

$$\Gamma_0 = \left\{\left(\begin{pmatrix} I_1 & \\ X_2 & X_1+N_1 \end{pmatrix}, \begin{pmatrix} N_2 & \\ & I_3 \end{pmatrix}, \begin{pmatrix} B' \\ 0 \end{pmatrix}\right)\right\} = \{(E_0(X), A_0(X), B_0(X))\}$$

(X is the parameter vector of all parameters in X_1, X_2).

In the other hand, $\mathcal{S}_1(E_1, A_1, B_1)$ is a stratum in the set of standardizable triples, It is easy to observe that the stratification of one input standardizable systems corresponds the stratification of one input standardizable systems which verifies the Whitney regular conditions (see [8] for details), So Σ_1 is Whitney regular over $\mathcal{S}_1(E_1, A_1, B_1) \cap \Gamma_1$ at (E_1, A_1, B_1), where Γ_1, is the linear variety:

$$\Gamma_1 = \left\{\left(\begin{pmatrix} I_1 & \\ & I_2 \end{pmatrix}, \begin{pmatrix} N_2 & \\ Y_1 & Y_2+J \end{pmatrix}, \begin{pmatrix} B' \\ 0 \end{pmatrix}\right)\right\} = \{(E_1(Y), A_1(Y), B_1(Y))\}.$$

(Y is the parameter vector of all parameters in Y_1, Y_2).

As above, we denote with the same symbol Γ_i the neighborhoods of (E_i, A_i, B_i) where the maps β_i are defined, so the induced stratifications $\Sigma_i \cap \Gamma_i$ $i = 0, 1$ are well defined.

Now, let us to consider the diffeomorphism

$$\varphi : \Gamma_0 \times \Gamma_1 \longrightarrow \Gamma$$

$$\varphi((E_0(X), A_0(X), B_0(X)), (E_1(Y), A_1(Y), B_1(Y))) = (E(X,Y), A(X,Y), B(X,Y)).$$

where $(E(X,Y), A(X,Y), B(X,Y))$ is

$$\left(\begin{pmatrix} I_1 & & \\ & I_2 & \\ X_2 & & X_1+N_1 \end{pmatrix}, \begin{pmatrix} N_2 & & \\ Y_1 & Y_2+J & \\ & & I_3 \end{pmatrix}, \begin{pmatrix} B' \\ 0 \\ 0 \end{pmatrix}\right).$$

We know that $\Pi = (\Sigma_0 \cap \Gamma_0) \times (\Sigma_1 \cap \Gamma_1)$ is a stratification which is Whitney regular over the stratum $(\mathcal{S}_0(E_0, A_0, B_0) \cap \Gamma_0) \times (\mathcal{S}_1(E_1, A_1, B_1) \cap \Gamma_1)$ at $((E_0, A_0, B_0), (E_1, A_1, B_1))$. Hence, because φ is a diffeomorphism to conclude the proof it is sufficient to show that φ preserve strata locally at (E, A, B). That is to say, given two points $p = (p_1, p_2)$ with

$$p_1 = \left(\begin{pmatrix} I_1 & \\ X_2 & X_1+N_1 \end{pmatrix}, \begin{pmatrix} N_2 & \\ & I_3 \end{pmatrix}, \begin{pmatrix} B' \\ 0 \end{pmatrix}\right),$$
$$p_2 = \left(\begin{pmatrix} I_1 & \\ & I_2 \end{pmatrix}, \begin{pmatrix} N_2 & \\ Y_1 & Y_2+J \end{pmatrix}, \begin{pmatrix} B' \\ 0 \end{pmatrix}\right)$$

and $q = (q_1, q_2)$ with

$$q_1 = \left(\begin{pmatrix} I_1 & \\ X_2' & X_1'+N_1 \end{pmatrix}, \begin{pmatrix} N_2 & \\ & I_3 \end{pmatrix}, \begin{pmatrix} B' \\ 0 \end{pmatrix}\right),$$
$$q_2 = \left(\begin{pmatrix} I_1 & \\ & I_2 \end{pmatrix}, \begin{pmatrix} N_2 & \\ Y_1' & Y_2'+J \end{pmatrix}, \begin{pmatrix} B' \\ 0 \end{pmatrix}\right)$$

belonging to the same stratum of Π, then the images $\varphi(p)$, $\varphi(q)$ belong to the same stratum of $\Sigma \cap \Gamma$, provided that they are sufficiently close to (E, A, B).

To prove that it suffices to see that $\varphi(p)$ and $\varphi(q)$ have the same collection of discrete invariants. We are going to proof that $r_1(\varphi(p) = r_1(\varphi(q))$ and $r_2(\varphi(p) = r_2(\varphi(q))$ analogously it can proof for the other invariant numbers. Calling $H_1 = X_1 + N_1$ and $H_2 = Y_2 + J$, $H_1' = X_1' + N_1$, $H_2' = Y_2' + J$ we have

$$r_1(p_1) = \operatorname{rank}\begin{pmatrix} I_1 & 0 \\ 0 & I_3 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} X_2 B' & 0 \\ N_2 B' & B' \end{pmatrix} =$$

$$r_1(q_1) = \operatorname{rank}\begin{pmatrix} I_1 & 0 \\ 0 & I_3 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} X_2' B' & 0 \\ N_2 B' & B' \end{pmatrix},$$

$$r_1(p_2) = \operatorname{rank}\begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} N_2 B' & B' \\ Y_1 B' & 0 \end{pmatrix} =$$

$$r_1(q_2) = \operatorname{rank}\begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} N_2 B' & B' \\ Y_1' B' & 0 \end{pmatrix}.$$

So, it is clear that

$$r_1(\varphi(p)) = n + \operatorname{rank}\begin{pmatrix} X_2B' & 0 \\ N_2B' & B' \\ Y_1B' & 0 \end{pmatrix} =$$

$$r_1(\varphi(q)) = n + \operatorname{rank}\begin{pmatrix} X_2'B' & 0 \\ N_2B' & B' \\ Y_1'B' & 0 \end{pmatrix}.$$

$$r_2(p_1) = \operatorname{rank}\begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & I_3 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} X_2B'+H_1X_2N_2B' & H_1X_2B' & 0 \\ N_2^2B' & N_2B' & B' \end{pmatrix} =$$

$$r_2(q_1) = \operatorname{rank}\begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & I_3 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} X_2'B'+H_1'X_2'N_2B' & H_1'X_2'B' & 0 \\ N_2^2B' & N_2B' & B' \end{pmatrix},$$

$$r_2(p_2) = \operatorname{rank}\begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} N_2^2B' & N_2B' & B' \\ H_2Y_1B'+Y_1N_2B' & Y_1B' & 0 \end{pmatrix} =$$

$$r_2(q_2) = \operatorname{rank}\begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} N_2^2B' & N_2B' & B' \\ H_2'Y_1'B'+Y_1'N_2B' & Y_1'B' & 0 \end{pmatrix}.$$

So, it is clear that

$$r_2(\varphi(p)) = 2n + \operatorname{rank}\begin{pmatrix} H_1X_2N_2B+X_2B' & H_1X_2B' & 0 \\ N_2^2B' & N_2B' & B' \\ H_2Y_1B'+Y_1N_2B' & Y_1B' & 0 \end{pmatrix} =$$

$$r_2(\varphi(q)) = 2n + \operatorname{rank}\begin{pmatrix} H_1X_2'N_2B'+X_2'B' & H_1'X_2'B' & 0 \\ N_2^2B' & N_2B' & B' \\ H_2'Y_1'B'+Y_1'N_2B' & Y_1'B' & 0 \end{pmatrix}.$$

For the compute of the other numbers we remark that for $\ell \leq n_1 - 1$ $\operatorname{rank}\begin{pmatrix} B' & N_2B' & \dots & N_2^{\ell}B' \end{pmatrix} = \ell + 1$ and if $n_1 - 1 < \ell$, $N_2^j = 0$ for all $j \geq \ell$.

Now, we suppose that $B = 0$. In this case the strata are $\mathcal{S}(\sigma)$ or $\mathcal{S}(\sigma,\tau)$. $\mathcal{S}(\sigma)$ corresponding to standardizable triples, so the stratification is Whitney regular over these strata. in the case $\mathcal{S}(\sigma,\tau)$, we have make the following changes in the first part of the proof $(E_0, A_0, B_0) = (N_1, I_3, 0)$ $(E_1, A_1, B_1) = (I_2, J, 0)$, $\Gamma_0 = \{(X_1 + N_1, I_3, Z_2)\} = \{p_0(X, Z_1)\}$, $\Gamma_1 = \{I_2, Y_2 + J, Z_1)\} = \{p_1(Y, Z_2)\}$ and $\varphi(p_0(X, Z_1), p_1(Y, Z_2)) = \left(\begin{pmatrix} I_2 & \\ & X_1+N_1 \end{pmatrix}, \begin{pmatrix} Y_2+J & \\ & I_3 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right)$. Taking into account that $\Sigma_i \cap \Gamma_i$ are whitney regular it suffices to prove that φ preserves strata following the first case. □

5 Bifurcation diagrams

Let U be an open subset of M_R, $\varphi : \Lambda \longrightarrow U$ a smooth family of triples of matrices .

If $\mathcal{S}(k,\sigma,\tau) \cap U$, is a stratum of $\mathcal{S} \cap U$, and φ is transverse to the induced stratification $\mathcal{S} \cap U$, then $\varphi^{-1}(\mathcal{S}_i \cap U)$ is a submanifold of Λ, with the same codimension.

$$\operatorname{codim}_\Lambda \varphi^{-1}(\mathcal{S}(k,\sigma,\tau)\cap U) = \operatorname{codim}_U(\mathcal{S}(k,\sigma,\tau)\cap U).$$

The transversality of the map φ at all strata of $\mathcal{S}\cap U$, ensures that $\bigcup \varphi^{-1}(\mathcal{S}(k,\sigma,\tau)\cap U)$ (or $\bigcup \varphi^{-1}(\mathcal{S}(k,\sigma,\tau)\cap U)$) is a stratification of Λ, which strata have the same codimension than its images in $\mathcal{S} \cap U$. This stratification is called bifurcation diagrams of φ.

We observe that, and because of Thom's transversality theorem, transverse families to the stratification may be considered generic in the following sense.

Theorem 3 (Thom) *In the space of the differentiable families $\varphi : \Lambda \longrightarrow U$, the transverse families to stratification $\mathcal{S}\cap U$ constitute a dense subset. If, in addition the stratification verifies the Whitney regularity conditions it is an open set.*

Now, we present types of possible triples of matrices in generic families with few parameters.

In the case of 3-dimensional 1-input generalized systems, we present the following example

Let (E, A, B) be a triple

$$\left(\begin{pmatrix} 5 & 8 & 3 \\ 1 & 4 & 2 \\ 1 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 8\lambda & 8+3\lambda \\ 1 & 4\lambda & 4+2\lambda \\ -1 & -\lambda & -1+2\lambda \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}\right)$$

computing the invariants given in proposition ([5]), we obtain that the continuous invariants are λ and the discrete ones $k_1 = 1$, $\nu_1 = 2$, $\nu_2 = 1$, so its canonical reduced form is $E_1 = I_3$, $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in M_{3\times 1}(\mathbb{C})$.

A generic family in a neighborhood $\mathcal{U}$ of the triple is the equivalent family of the following three-parametric $(E_1, A_1, B_1) + \{(0, Y, 0)\}$ with $Y = \begin{pmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & y_{32} & 0 \end{pmatrix}$.

The family contains the same type of triples than $(E_1, A_1, B_1 + \{(0, Y, 0)\}$, so we analyze this one, and that it contain the following types of triples

a) if $y_{32}y_{21}^2 - y_{31}^2 \neq 0$ the triple is equivalent to (E', A', B') with $E' = I_3$, $A' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in M_{3\times 1}(\mathbb{C})$.

b) if $y_{32}y_{21}^2 - y_{31}^2 = 0$. This is the Whitney umbrella surface. In this case we can find the following triples

   (a) $y_{21} = y_{31} = y_{32} = 0$, the triple is equivalent to (E', A', B'), with $E' = I_3$, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda' & 1 \\ 0 & 0 & \lambda' \end{pmatrix}$, $B' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in M_{3\times 1}(\mathbb{C})$.

   (b) $y_{21} = y_{31} = 0$, $y_{32} \neq 0$, the triple is equivalent to (E', A', B'), with $E' = I_3$, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \lambda'' \end{pmatrix}$, $B' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in M_{3\times 1}(\mathbb{C})$.

   (c) $y_{21}^2 y_{32} - y_{31}^2 = 0$, $y_{21} \neq 0$ or $y_{31} \neq 0$, the triple is equivalent to (E', A', B'), with $E' = I_n$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda' \end{pmatrix}$, $B' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in M_{3\times 1}(\mathbb{C})$.

So, in $\Sigma \cap \mathcal{U}$ there are the following four strata:

$$
\begin{aligned}
&\mathcal{S}_1 = \mathcal{S}(k),\ k = 3 \\
&\mathcal{S}_2 = \mathcal{S}(k, \sigma),\ k = 1, \sigma = (\sigma_1), \sigma_1 = 2 \\
&\mathcal{S}_3 = \mathcal{S}(k, \sigma),\ k = 1, \sigma = (\sigma_1, \sigma_2), \sigma_1 = 1, \sigma_2 = 1 \\
&\mathcal{S}_4 = \mathcal{S}(k, \sigma),\ k = 2, \sigma = (\sigma_1), \sigma_1 = 1.
\end{aligned}
$$

and the Whitney regular stratification induced in the space of parameters is,

$$
\begin{aligned}
&\varphi^{-1}\mathcal{S}_1 = \{(y_{21}, y_{31}, y_{32}) \mid y_{32}y_{21}^2 - y_{31} \neq 0\} \\
&\varphi^{-1}\mathcal{S}_2 = \{(y_{21}, y_{31}, y_{32}) \mid y_{32} = y_{21} = y_{31} = 0\} \\
&\varphi^{-1}\mathcal{S}_3 = \{(y_{21}, y_{31}, y_{32}) \mid y_{21} = y_{31} = 0, y_{32} \neq 0\} \\
&\varphi^{-1}\mathcal{S}_4 = \{(y_{21}, y_{31}, y_{32}) \mid y_{32}y_{21}^2 - y_{31} = 0, y_{21} \neq 0\}
\end{aligned}
$$

that corresponds to the Whitney umbrella regular stratification.

6 Conclusion

In this paper we prove that the stratification of one input regularizable systems is Whitney regular. Then, we can obtain precise descriptions of the bifurcation diagrams of generic families of this kind of systems.

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