The Matrix Padé Approximation in Systems of Differential Equations and Partial Differential Equations

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Abstract: In [3] we presented a technique to study the existence of rational solutions for systems of linear first-order ordinary differential equations. The method is based on a rationality characterization that involves Matrix Padé Approximants. Moreover the main ideas were only applied in the numerical resolution of a particular partial differential equation. This paper may be considered as an extension of [3], in the sense that we propose fundamental matrices directly for linear m-order ordinary differential equations without making a transformation to an equivalent system of first order. In addition, we increase its field of applications to particular solutions of the mentioned systems and to Partial Differential Equations.

Key-Words: Systems of Differential Equations, Partial Differential Equations (PDE), Matrix Padé Approximation (MPA), rational solutions, minimum degrees (m.d.)

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1 Introduction

We will use the matrix notation in Systems of Linear m-Order Ordinary Differential Equations as follows.

Considering $A_j: D\subset\mathbb{R}\to\mathbb{C}^{mxn}$ (j=1,2...m) and $Y: D\subset\mathbb{R}\to\mathbb{C}^{n+1}$, let

$$Y^{(m}(t)=A_1(t)\, Y^{(m-1}(t)+A_2(t)\, Y^{(m-2}(t)+\ldots+ A_{m-1}(t)\, Y'(t)+A_m(t)\, Y(t) \quad (1)$$

be an homogeneous system of linear m-order ordinary differential equations.

Normally, to resolve the system (1) we transform it into an equivalent one of first order, by changes of variables. We will resolve it later but not making use of that transformation.

Definition 1: F(t) is a fundamental matrix of (1) if any solution of (1) can be written as a linear combination of the columns of F(t).

Obtaining a fundamental matrix is essential to solve systems of differential equations. However there is no procedure to find it from any matrix functions $A_1(t)$, $A_2(t)...A_m(t)$. In this paper we consider that the elements of the matrices are analytic functions. It is interesting to note that we do not take into account the circle of convergence of power series, when we consider rational functions. What is important is to know their poles ([3]).

Proposition 1: If $A_1(t)$, $A_2(t)...A_m(t)$ are continuous matrix functions then (1) has a fundamental matrix of dimension n x nm.

Proof: The set of solutions for a homogeneous system of linear first-order ordinary differential equations, X'(t)=A(t) X(t), with A(t) a continuous matrix function of order n, is a linear space of dimension n (it has a fundamental matrix of dimension nxn) ([5]).

By changes of variables ($V_i(t)=Y^{(i}$, i=1,2...m-1) we transform (1) into the equivalent system of first order that follows:

$$\begin{pmatrix} Y'(t) \\ V_1'(t) \\ \vdots \\ V_{m-1}'(t) \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A_m(t) & A_{m-1}(t) & A_{m-2}(t) & \cdots & A_1(t) \end{pmatrix} \begin{pmatrix} Y(t) \\ V_1(t) \\ \vdots \\ V_{m-1}(t) \end{pmatrix} \quad (2)$$

Let $G(t)=\left(g_{ij}(t)\right)_{i,j=1}^{m}$, be a fundamental matrix for (2) such that $G(0)=I_{nm\ x\ nm}$ and $g_{ij}: D\subset\mathbb{R}\to\mathbb{C}^{nxn}$, i.e., the nm columns of G(t) constitute a base for the set of solutions of (2). Note that $g_{ji}(t)=g_{1i}^{(j-1}(t)$, i.e., the j-th row of matrices in G(t) is the (j-1)-th derivative of the first row of matrices in G(t). Considering $g_{ij}(t)=\sum_{k=0}^{\infty} g_{ijk}\, t^k$, we write the initial condition $G(0)=I_{nmxnm}$ as follows:

$$g_{1ii-1}=I_{nxn} \text{ and } g_{1ij}=0 \text{ if } j\neq i-1, \text{ for } i=1,2...m;\ j=0,1...\ m-1 \quad (3)$$

Note that in (3) we consider only coefficients of $g_{1j}(t)$, (j=1,2...m).

Let S be the set of solutions of (1), the nm columns of $F(t)\equiv(g_{11}(t)\ \ g_{12}(t)\ \ \ldots\ \ g_{1m}(t))$ -with the condition (3)- constitute a base of the linear space S. Note that F(t) is a fundamental matrix of dimension n x nm of (1). ■

Given the formal power series, $A_j(t)=\sum_{k=0}^{\infty} A_{jk}(t-t_0)^k$, $A_{jk}\in\mathbb{C}^{mxn}$ for j=1,2...m,, we calculate recursively the coefficients of the series for a fundamental matrix of

(1), $F(t)=\sum_{k=0}^{\infty} F_k (t-t_0)^k$, $F_k \in \mathbb{C}^{nxmn}$, as follows (without considering the equivalent system of first-order). Substituting

$F^{(h)}(t)=\sum_{k=0}^{\infty} (k+h)(k+h-1)...(k+1)F_{k+h}(t-t_0)^k$, h=1, 2...m, in $F^{(m)}(t)=A_1(t)F^{(m-1)}(t)+A_2(t)F^{(m-2)}(t)+ \; + A_{m-1}(t)F'(t)+A_m(t)F(t)$. Later on, we will explain more explicitly the recurrence relation for the specific examples we are going to study.

2 Matrix Padé Approximants and Rational Functions

Once the formal power series for a fundamental matrix (or for a particular solution) is obtained, it is of practical interest to get the theoretic function associated to the series. It is evident that this aim is unattainable in general and it may only be obtain in certain problems.

In this section we will consider Matrix Padé Approximation (MPA) results to determine it there is a rational solution and, if so, obtain minimum degrees - in certain sense- of the polynomials involved.

We denote as F any formal power series, with matrix coefficients as follows:

$$F(t)=\sum_{k=0}^{\infty} f_k t^k \quad f_k \in \mathbb{C}^{mxn} \quad t \in \mathbb{C} \tag{4}$$

Suppose that there exist matrix polynomials:

$$Q_h(z)=\sum_{i=0}^{h} b_i z^i \quad q_i \in \mathbb{C}^{mxn} \quad \text{para } i=0,1,..,h$$

$$N_h(z)=\sum_{i=0}^{h} n_i z^i \quad n_i \in \mathbb{C}^{mxn} \quad \text{para } i=0,1,..,h$$

$$P_g(z)=\sum_{i=0}^{g} a_i z^i \quad p_i \in \mathbb{C}^{nxn} \quad \text{para } i=0,1,..,g$$

$$D_g(z)=\sum_{i=0}^{g} d_i z^i \quad d_i \in \mathbb{C}^{mxm} \quad \text{para } i=0,1,..,g$$

where $p_0=I_{nxn}$, $d_0=I_{mxm}$, $F(t)-Q_h(t)P_g^{-1}(t)=O(t^{h+g+1})$ and $F(t)-D_g^{-1}(t)N_h(t)=O(t^{h+g+1})$, $Q_hP_g^{-1}$ is therefore said to be a right Matrix Padé Approximant (right MPA) which is denoted $^R[h/g]_F$; similarly, $D_g^{-1}N_h$ is said to be a left Matrix Padé Approximant (left MPA), which is denoted $^L[h/g]_F$. We shall use $\{^L[h/g]_F\}$ ($\{^R[h/g]_F\}$) to denote the set of all possible approximants $^L[h/g]_F$ ($^R[h/g]_F$).

As a consequence of the definition we can say that:

* $^R[h/g]_F$ exists, i.e., $\{^R[h/g]_F\}\neq\varnothing$, if and only if, the following system has a solution:

$f_{h-g+k}p_g+f_{h-g+k+1}p_{g-1}+...+f_{h+k-1}p_1=-f_{h+k}$, k=1,2...g RS(h,g)

* $^L[h/g]_F$exists, i.e., $\{^L[h/g]_F\}\neq\varnothing$, if and only if the following system can be solved:

$d_gf_{h-g+k}+d_{g-1}f_{h-g+k+1}+...+d_1f_{h+k-1}=-f_{h+k}$ k=1,2...g LS(h,g)

In both cases it is assumed that $f_i=0$, i<0. Resolving LS(h,g) and RS(h,g) we obtain the coefficients of the denominators of $^L[h/g]_F$ and $^R[h/g]_F$, respectively. The coefficients of the numerators can be obtained by $n_0=c_0$, $n_1=d_1c_0+d_0c_1...n_h=d_hc_0+...+d_0c_h$ for the left MPA and $q_0=c_0$, $q_1=c_0p_1+c_1p_0...q_h=c_0p_h+...+c_hp_0$ for the right MPA.

Note that the MPA do not necessarily have to exist. Furthermore, unlike the scalar case, if one of them does exist it is not necessarily unique [3].

If there is no need to distinguish between the left and right MPA because they are identical, we will use $[h/g]_F$ to denote the MPA of degrees (h,g) to F and $\{[h/g]_F\}$ to the set of all possible approximants $[h/g]_F$.

Various types of minimality related to the degrees of the polynomials that intervene in a rational function have been defined in literature. Taking as our starting point the idea of having a type of joint minimality for the two polynomials that intervene in a rational function, the concept left minimum degrees in this work is defined as follows.

Definition 2: We say that the degrees p and q of the matrix polynomials A_p and B_q, such that $F(t)=A_p^{-1}(t)B_q(t)$ and $A_p(0)=I$, are left minimum degrees (left m.d.), if for any other two matrix polynomials D_g and N_h of degrees g and h respectively which verify: $D_g(0)=I$ and $A_p^{-1}(t)\,B_q(t)=D_g^{-1}(t)N_h(t)$, it follows that: $h<q$ implies $g>p$ and $g<p$ implies $h>q$.

In the same way we can define right m.d., with the inverted polynomial to the right.

2.1 Some Previous Results ([3])

2.1.1 Table: Rationality and Minimality

We give the following definition associated with (4).

Definition 3: For integers i and j, i≥0 and j>0, let

$$M1(i,j)=(f_{i-j+h+k-1})_{h,k=1}^{j},$$

$$^LM4_{sr}(i,j)=(f_{i-j+h+k-1})_{h,k=1}^{j,s+r-i},$$

$$^LM5_{sr}(i,j)=(f_{i-j+h+k-1})_{h,k=1}^{j+1,s+r-i},$$

$$^RM4_{sr}(i,j)=(f_{i-j+h+k-1})_{h,k=1}^{s+r-i,j}$$

$$^RM5_{sr}(i,j)=(f_{i-j+h+k-1})_{h,k=1}^{s+r-i,j+1}.$$

By convention, if j=0 the rank of these matrices is zero for any i∈N.

Definition 4: For any nonnegative integers i, j, let T1(i,j)=rank(M1(i,j)). We will display these quantities in an infinite two-dimensional table, *Table 1*, where i and j serve to enumerate the columns and the rows respectively.

Definition 5: We define the staired block R1 to be the following subset of N^2:

R1={(i,j)∈ $\mathbb{N}^2$/rank(M1(g,h))=rank(M1(g+k,h+k)) for any k∈N, g≥i and h≥j}.

Although this set seems rather abstract, the meaning of "staired" becomes clear observing *Tables 1* in examples below.
To begin with, we can propose a method, using *Table 1*, to determine whether or not a matrix series stems from a rational function. This can be set out as follows.

2.1.1.1 Rationality
F is rational if and only if in the bottom right part of *Table 1* we can mark at least one NW-SE diagonal of infinite size where all its boxes have the same value. It is important to state that, there are several such diagonals, with their union coinciding with R1. Note that, within R1, boxes of different diagonals can have different values. Furthermore, it is assured that if $(a,b)\in R1$ then the formal power series of F and $[a/b]_F$ is the same. However, (a,b) are not necessarily left or right m.d.
In practice, the *Table 1* that we can construct is "finite". However, in some applications where a rule for the formation of the coefficients of the series F is known, certain relations between the matrices that define the elements of *Table 1* can help validate the properties to an infinite size. Otherwise, we can only say the available coefficients of F coincide with the coefficients of a rational function of certain degrees.

2.1.1.2 Minimality
The boxes (i,j) such that $T1(i,j)\in R1$, $T1(i-1,j)\notin R1$ and $T1(i,j-1)\notin R1$ are called corners of R1. In particular situations we can say that a corner of R1 corresponds with a pair of left or right m.d., for instance:
Property 1: If $(i,j)\in R1$, $(i-1,j)\notin R1$ and $T1(i,j)=jn$ then $(i-u,j-v)$ is not a pair of right m.d. for any u, v such that $1\le u\le i$ and $0\le v\le j$.
Property 2: If $(i,j)\in R1$, $(i-u,j)\notin R1$, $(i,j-v)\notin R1$, for any u, v such that $1\le u\le i$ and $1\le v\le j$, and they are not pairs of left (right) m.d., then (i,j) is a pair of left (right) m.d.

2.1.2 *Table 2*: Minimality
If F is rational, there are (and we can find) certain degrees r and s associated to two pairs of matrix polynomials which represent the function in rational form in two ways, that is to say, with the inverted polynomial of degree r, multiplied to the right or left. However, it may be that r and s are neither left nor right m.d. for representing F since there are cases in which, for instance, the function is identical to an approximant ${}^{L}[i/j]_F$ but the approximant ${}^{R}[i/j]_F$ does not exist, or viceversa; in this case, $\text{rank}(M1(i,j))\neq\text{rank}(M1(i+1,j+1))$ and thus *Table 1* would not provide information concerning the left or right approximant. For this reason, we present *Table 2* -one for the left and one for the right approximant-, whose structure reflects the possible combinations of m.d. (left and right, respectively) for representing F.
Firstly we present the following definitions.
Definition 6: Given $(s,r)\in R1$, for integer i, j, such that $0\le i\le s$ and $0\le j\le r$, let ${}^{L}T2_{sr}(i,j)=0$, if $\text{rank}({}^{L}M4_{sr}(i,j))=\text{rank}({}^{L}M5_{sr}(i,j))$, or otherwise ${}^{L}T2_{sr}(i,j)=1$. We will display these quantities in a finite two-dimensional table, *Table 2* for left approximant, which has s+1 columns and r+1 rows.
Definition 7: We denote as a staired block ${}^{L}R2_{sr}$, with $(s,r)\in R1$, the following subset of N^2:

$$ {}^{L}R2_{sr}=\{(i,j)\in \mathbb{N}^2/0\le i\le s,\ 0\le j\le r \text{ and } \text{rank}({}^{L}M4_{sr}(i,j))=\text{rank}({}^{L}M5_{sr}(i,j))\}. $$

To define *Table 2* for right approximant we must consider ${}^{R}M4_{sr}(i,j)$ and ${}^{R}M5_{sr}(i,j)$ instead of ${}^{L}M4_{sr}(i,j)$ and ${}^{L}M5_{sr}(i,j)$, respectively. We only expound the theory for the left case taking into account that for the right case it is similar.
Property 3: F is a rational function identical to ${}^{L}[q/p]_F$ where q and p are left m.d., $q\le s$ and $p\le r$, if and only if, ${}^{L}T2_{sr}(q,p)=0$, ${}^{L}T2_{sr}(q-1,p)=1$ and ${}^{L}T2_{sr}(q,p-1)=1$, that is, (q,p) is a corner of ${}^{L}R2_{sr}$.
The system corresponding to the denominator of the element of $\{{}^{L}[q/p]_F\}$ that coincides with F is:

$$ d_p f_{q-p+i}+d_{p-1} f_{q-p+i+1}+...+d_1 f_{q+i-1}=-f_{q+i} \qquad i=1,2...s+r-q $$

whose associated matrix is ${}^{L}M4_{sr}(q,p)$.
Then we will consider the results of MPA to study possible rational solutions of differential equations systems and of partial differential equations.

3 Rational Solutions for Systems of Ordinary Differential Equations
In order to obtain a clear understanding, it is important to read the following remarks relating to possible rational solutions of the system (1).
- Even if $A_1(t)$, $A_2(t)...A_m(t)$ are not rational there may exist rational solutions. For instance, the system

$$ Y'(t)=\begin{pmatrix} \frac{e^t}{2t+4} & \frac{-e^t+2}{2t+4} \\ \frac{\text{sint}}{2t+4} & \frac{-\sin t+2}{2t+4} \end{pmatrix} Y(t) $$

has a rational solution, $Y(t)=\begin{pmatrix} 2t+4 \\ 2t+4 \end{pmatrix}$.
- The fact that there exists rational solution does not imply the fundamental matrix is rational. Note that generally the linear combination of no rational functions could be a rational function, for instance, if $x(t)=\frac{\cos t}{3t-2}$ and $y(t)=\frac{-2\cos t+8}{3t-2}$ the linear combination $4x(t)+2y(t)$ is the rational function $\frac{16}{3t-2}$.
- The system (1) may have a fundamental matrix with

some rational elements and others which are not. Next we proceed to illustrate different cases.
Our aim in this section is to detect rational fundamental matrices (which are rectangular when the order of the system is greater than 1). In other case, if this is not possible, then to detect particular rational columns or particular rational solutions. If F(t) is rational then any solution of (1) is rational complying with the aims. However, when F(t) is not rational and is in analytical form it would be interesting to know if there exist any initial conditions, such that the associated particular solution to be rational. It is not easy to find the answer to this question in general cases.
We start with an example where the fundamental matrix is not rational but the system does have rational solutions.

Example 1. Let Y"(t)=C(t) Y(t) be a differential system where $C(t)=\begin{pmatrix} \frac{1}{(2-t)^2} & \frac{t^2+2}{(2-t)^3} \\ \frac{4t-8}{(t^2+2)^3} & \frac{6t^2}{(t^2+2)^2} \end{pmatrix}$. Considering $C(t)=\sum_{i=0}^{\infty} C_i t^i$, we obtain recursively the coefficients of the series for a fundamental matrix $F(t)=\sum_{i=0}^{\infty} F_i t^i$ as follows:

$$\sum_{i=0}^{\infty}(i+2)(i+1)F_{i+2}t^i = (\sum_{i=0}^{\infty}C_i t^i)(\sum_{i=0}^{\infty}F_i t^i) \quad (5)$$

Then, taking into account (3), $F_0=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $F_1=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, and considering (5) we calculate the other coefficients using the following expression:

$$F_{i+2}=(\sum_{j=0}^{i} C_j F_{i-j})/(i+2)(i+1),\ i\geq 0 \quad (6)$$

Fig. 1: ***Table 1*** **for F(t)**

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	2	2	2	2	2
2	4	4	4	4	4	4
3	6	6	6	6	6	6
4	8	8	8	8	8	8
5	10	10	10	10	10	10

Note that F(t) is not rational of orders (p,q) such that $0\leq p<5$, $0\leq q<5$. Then we have calculated *Tables 1* for each column of F(t) -four tables- and, moreover, *Tables 1* for each element of F(t) -eight tables-. In all cases, tables do not correspond to a rational function.
However, the system does have rational solutions. They are linear combinations of the columns of the fundamental matrix F(t). For instance, considering $K=\begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, we obtain the fig. 2.

Fig.2: ***Table 1*** **of function Y(t)=F(t) K**

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	3	3	3
5	5	5	5	4	3	3

This table indicates that Y can be represented as left and right approximants of the set $\{[2/3]_Y\}$. Taking into account that T1(2,3)=3 and Properties 2 and 3, (2,3) is a pair of right m.d. The representation of $^R[2/3]_Y$ is

$$Y(t)=\begin{pmatrix} 0.5+0.25t^2 \\ 0.5-0.25t \end{pmatrix}\frac{1}{1-0.5t+0.5t^2-0.25t^3}.$$

Note that we have solved RS(2,3) to obtain the coefficients of the denominator.
Although we have a representation of the solution, out of curiosity we have calculated fig. 3.

Fig. 3: ***Table 2*** **(for left approximant)**

	0	1	2
0	1	1	1
1	1	1	1
2	0	0	0
3	0	0	0

Table 2 indicates that (0,2) is a pair of left m.d. The representation of $^L[0/2]_Y$ is:

$$Y(t)=\begin{pmatrix} 1+0.5t^2 & 0 \\ 0 & 1-0.5t \end{pmatrix}^{-1}\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

Note that it is the particular solution corresponding to the initial conditions $Y_0=(0.5 \quad 0.5)^t$, $Y_1=(0.25 \quad 0)^t$. It is obvious that any solution with initial conditions $Y_0=\begin{pmatrix} c \\ c \end{pmatrix}$, $Y_1=\begin{pmatrix} 0.5c \\ 0 \end{pmatrix}$, $c\in\mathbb{C}$, is rational with degrees (0,2).
The following example could be solved with two independent equations. However, we consider it as a system to illustrate a fundamental matrix with rational columns and non-rational columns.

Example 2. Let Y"(t)=C(t) Y(t) be a differential system where $C(t)=\begin{pmatrix} \frac{2}{t^2+1} & 0 \\ 0 & \frac{6t}{t^3+2} \end{pmatrix}$.
In the same way as the last example, considering (6) we obtain the coefficients of a fundamental matrix. Its *Table 1* is in fig. 4.

Fig. 4: ***Table 1***

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	2	1	2	1	1
2	4	4	4	4	4	3
3	6	6	6	6	6	6
4	8	8	8	8	8	8
5	10	10	10	10	10	10

We can see that the fundamental matrix is not rational with orders contained in the table. However, *Tables 1* for each column of the fundamental matrix are in fig.5.

Fig. 5: ***Tables 1 for each column***

Column 1

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	1	0	1
2	1	2	2	2	2	2
3	2	3	2	3	2	3
4	3	4	4	4	4	4
5	4	5	4	5	4	5

Column 2

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	0	1	0
2	1	2	1	1	2	1
3	2	3	3	3	3	3
4	3	4	3	3	4	3
5	4	5	4	4	5	4

Column 3

0	1	2	3	4	5
0	0	0	0	0	0
1	0	1	0	0	0
2	2	2	1	0	0
3	2	3	2	1	0
4	4	4	3	2	1
5	4	5	4	3	2

Column 4

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	1	0	0
2	2	1	1	2	1	0
3	3	3	3	3	2	1
4	4	3	3	4	3	2
5	5	4	4	5	4	3

These tables indicate that the first column of the fundamental matrix is not rational; the third and fourth columns are rational (polynomial). Specifically $\begin{pmatrix} t^2+1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ t^3+2 \end{pmatrix}$.

Regarding *Table 1* for column 2, it could be the table of a rational function with degree 2 in the numerator and 3 in the denominator, but maybe the block with discontinuous line is a finite staired block ([6]). To know if the block with discontinuous line is R1 we increased the table as fig. 6.

Fig. 6

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	1	0	0	1	0	0	1
2	1	2	1	1	2	1	1	2	1	1	2
3	2	3	3	3	3	3	3	3	3	3	3
4	3	4	3	3	4	3	3	4	3	3	4
5	4	5	4	4	5	4	4	5	4	4	5
6	5	6	6	6	6	6	6	6	6	6	6
7	6	7	6	6	7	6	6	7	6	6	7
8	7	8	7	7	8	7	7	8	7	7	8
9	8	9	9	9	9	9	9	9	9	9	9
10	9	10	9	9	10	9	9	10	9	9	10

Note that, the mentioned block is a finite staired block. Therefore, the function is not rational for degrees considered in this table.

We have increased tables for columns 1, 3 and 4 and we reaffirm what we have commented about these columns.

4 Rational Solutions for Partial Differential Equations

In this section we aim to solve partial differential equations combining MPA with other known methods, in particular, Galerkin and Finite Differences.

4.1 Method of Galerkin and MPA

In general, the methods of Galerkin [4] are used in problems where there is an unknown function to be determined. Of course, partial differential equations are of this type.

In this method we take a set of basic functions and try to solve the equation, presuming that the solution is a suitable combination of these basic functions. Sometimes this solution is not consistent. Depending on the approach considered we obtain a different approximated solution.

In the following examples we combine the method of Galerkin and MPA to solve the equations.

Example 3

Let us consider the following hyperbolic equation:

$$u_t = \frac{3}{2(4+t^2)} u_x \tag{7}$$

$u(x,0) = \sin 2\pi x \qquad 0 \le x \le 1$

$u(0,t) = u(1,t) = 0 \qquad t > 0.$

We propose as solution the sum:

$$u(x,t) = \sum_{j=1}^{n} v_j(t) w_j(x) \tag{8}$$

We will choose some basic functions of x, that is, w_1, $w_2 ... w_n$. Note that for t=0, (8) only depends on w_j (j=1...n) because $v_j(0)$ (j=1...n) are constants. Due to the fact that u(x,0) = sin 2πx, we pretend that sin 2πx to be a linear combination of w_j (j=1...n). Moreover, if $w_j(0)=w_j(1)=0$ (j=1...n) the conditions u(0,t)=u(1,t)=0 (t>0) are verified with the tentative solution (8).

Substituting (8) in (7), the result is:

$$\sum_{j=1}^{n} \left(v_j'(t) w_j(x) - \frac{3}{2(4+t^2)} v_j(t) w_j'(x) \right) = 0 \tag{9}$$

As we have commented, sometimes (9) is not consistent. Then we look for approximated solutions.

Applying the interior product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ in equation (9), we obtain:

$$\sum_{j=1}^{n} \left(v_j'(t) \langle w_j, w_i \rangle - \frac{3}{2(4+t^2)} v_j(t) \langle w_j', w_i \rangle \right) = 0, \quad 1 \le i \le n \tag{10}$$

The expression (10) is a system of n homogeneous differential equations with n unknown v_j, j=1...n. It is very advantageous that $\{w_1, w_2 ... w_n\}$ to be an ortonormal system in the interval [0,1]. A suitable system such that $w_j(0)=w_j(1)=0$ (j=1...n) is

$w_j(x)=\sqrt{2}\,\text{sen}\ \pi jx$. In this case, the equation (10) in matrix form is:

$$V'(t)=A(t)\,V(t) \qquad (11)$$

where $V(t) = (v_1(t)\ \ v_2(t)\ \ ...\ \ v_n(t))^t$ and A(t) is an nxn matrix which elements are:

$$a_{ij}(t)=\frac{3}{2(4+t^2)}\left\langle w_j^{'}, w_i\right\rangle .$$

Considering the initial conditions, we have: $\sum_{j=1}^{n} v_j(0)w_j(x)=\sin 2\pi x$. Due to the fact that $\{w_1,w_2...w_n\}$ is an ortonormal system, then:

$$v_j(0)=\left\langle \sin 2\pi x, w_j\right\rangle \ \ (j=1...n) \qquad (12)$$

The expression (12) provides initial conditions for system (11).

To illustrate the procedure, suppose that n=2 in (8). Obviously, with greater n we obtain better approximation.

If n=2, then $A(t)=\frac{1}{1+\frac{t^2}{4}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Considering

$A(t)=\sum_{j=0}^{\infty} A_j t^j$, we have that $A_{2j+1}=0$,

$$A_{2j}=\begin{pmatrix}0 & \left(\frac{1}{4}\right)^j(-1)^{j+1}\\ \left(\frac{-1}{4}\right)^j & 0\end{pmatrix} \text{ for } j=0,1,2...$$

A fundamental matrix $F(t)=\sum_{j=0}^{\infty} F_j t^j$ of the system (11) such that $F_0=I$, verifies that

$$F_{k+1}=\frac{1}{k+1}\sum_{j=0}^{k} A_j F_{k-j} \qquad k\geq 0$$

Fig. 7: *Table 1* for F(t)

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	2	2	2	2	2
2	4	4	2	2	2	2
3	6	6	4	2	2	2
4	8	8	6	4	2	2
5	10	10	8	6	4	2

It indicates that $F(t)=[1/1]_F$. Solving the system LS(1,1), and calculating the numerator, F(t) can be represented as follows:

$$F(t)=\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 1/2\\ -1/2 & 0\end{pmatrix}t\right]^{-1}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}t\right]$$

or equivalently $F(t)=\frac{1}{1+\frac{t^2}{4}}\begin{pmatrix}1-\frac{t^2}{4} & -t\\ t & 1-\frac{t^2}{4}\end{pmatrix}$.

Given initial conditions, we calculate the particular solution V(t)=F(t)K, where $K\in\mathbb{R}^2$. Note that V(0)=K. Taking into account (12), $K=\begin{pmatrix}0\\ 1/\sqrt{2}\end{pmatrix}$. Therefore,

$$V(t)=\frac{1}{\sqrt{2}(1+\frac{t^2}{4})}\begin{pmatrix}-t\\ 1-\frac{t^2}{4}\end{pmatrix} \text{ and}$$

$$u(x,t)=\frac{-t\sin\pi x+(1-\frac{t^2}{4})\sin 2\pi x}{1+\frac{t^2}{4}}.$$

In the case that we are interested only in a particular solution that verifies certain initial conditions, it is not necessary to calculate a fundamental matrix. To know V(t), we suppose that $V(t)=\sum_{j=0}^{\infty} V_j t^j$, with $V_0=\begin{pmatrix}0\\ 1/\sqrt{2}\end{pmatrix}$.

Then se $V_{k+1}=\frac{1}{k+1}\sum_{j=0}^{k} A_j V_{k-j}$, k≥0.

Fig. 8: *Table 1* for V(t)

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	2	2	2
4	4	4	4	3	2	2
5	5	5	5	4	3	2

This table indicates that V can be represented as left and right approximants of the set $\{[2/2]_V\}$. Considering right approximant, *Table 2* is not necessary because in *Table 1* we can see taht (2,2) is a pair of right m.d. (taking into account Properties 1 and 2 and T1(2,2)=2). We solve the system RS(2,2) to obtain the known solution $V(t)=\begin{pmatrix}-t/\sqrt{2}\\ \frac{1}{\sqrt{2}}-\frac{t^2}{4\sqrt{2}}\end{pmatrix}\frac{1}{(1+\frac{t^2}{4})}$.

Out of curiosity we have calculated the *Table 2* for left approximant in fig. 9.

Fig. 9: *Table 2* for left approximant

	0	1	2
0	1	1	1
1	1	0	0
2	0	0	0

It indicates that (0,2) and (1,1) are two pairs of left m.d.

Let us consider the following example. In this case we need to solve a system of first order ordinary differential equations, V"(t)=A(t) V(t) where A(t) is a diagonal matrix, then each equation can be solve individually. Therefore, instead of MPA we will use scalar Padé approximation.

Example 4
Let it be the following elliptic equation:

$u_{tt}=-\dfrac{2}{\pi^2(1-t)^2}\ u_{xx}$, $t>0$, $t\neq 1$, $0\leq x\leq 1$

$u(0,t)=u(1,t)=0$; $u(x,0)=u_t(x,0)=\sin\pi x$

We propose a solution of the form $u(x,t)=\sum_{j=1}^{2} v_j(t)w_j(x)$ considering the same ortonormal system of Example 3. Substituting in the partial differential equation and dividing by $\sqrt{2}$), we obtain:

$$v_1''(t)\sin\pi x+\frac{2}{\pi^2(1-t)^2}v_1(t)(-\pi^2\sin\pi x)+$$

$$v_2''(t)\sin 2\pi x+\frac{2}{\pi^2(1-t)^2}v_2(t)(-4\pi^2\sin 2\pi x)=0$$

Then, $v_1''(t)=\dfrac{2}{(1-t)^2}v_1(t)$, $v_2''(t)=\dfrac{8}{(1-t)^2}v_2(t)$

We have two independent equations. Note that A(t) is a diagonal matrix.
The initial conditions can be deduced by:
$u(x,0)=v_1(0)w_1(x)+v_2(0)w_2(x)=\sin\pi x$, therefore,

$v_2(0)=0$ and $v_1(0)=\dfrac{1}{\sqrt{2}}$,

$u_t(x,0)=v_1'(0)w_1(x)+v_2'(0)w_2(x)=\sin\pi x$, therefore,

$v_2'(0)=0$ and $v_1'(0)=\dfrac{1}{\sqrt{2}}$

Considering the first equation $v_1''(t)=\dfrac{2}{(1-t)^2}v_1(t)$ and its conditions $v_1(0)=\dfrac{1}{\sqrt{2}}$ and $v_1'(0)=\dfrac{1}{\sqrt{2}}$, we denote the general solution as $s(t)=\sum_{i=0}^{\infty}s_i t^i$. Taking into account that $\dfrac{2}{(1-t)^2}=2\sum_{j=0}^{\infty}(j+1)t^j$ -in the circle of convergence of the series-. Substituting in the differential equation we obtain:

$$\sum_{i=0}^{\infty}(i+2)(i+1)s_{i+2}t^i=2\Big(\sum_{i=0}^{\infty}(i+1)t^i\Big)\Big(\sum_{i=0}^{\infty}s_i t^i\Big)$$

Therefore, to calculate the coefficients of the series s(t) we use the following recurrence relation:

$$s_{i+2}=2\Big(\sum_{j=0}^{i}(j+1)s_{i-j}\Big)/((i+2)(i+1)) \qquad i\geq 0.$$

Fig. 10: *Table 1* for $v_1(t)$

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	1	1	1	1	1
3	3	2	1	1	1	1
4	4	3	2	1	1	1
5	5	4	3	2	1	1

It shows that $v_1(t)=[0/1]_s$. Solving Padé equations for $[0/1]_s$, $v_1(t)=\dfrac{1/\sqrt{2}}{1-t}$. In a similar way, solving $v_2''(t)=\dfrac{8}{(1-t)^2}v_2(t)$, $v_2(0)=v_2'(0)=0$, we obtain $v_2(t)=0$. Therefore, the solution of the partial differential equation is: $u(x,t)=\dfrac{1}{1-t}\sin\pi x$.

4.2 Method of Finite Differences and MPA

We give the theory for the following hyperbolic partial differential equation (generalized wave equation):
$u_{tt}=\alpha^2(x,t)\ u_{xx}$ $\quad 0\leq x\leq 1$, $t>0$
$u(0,t)=f_1(t)$, $u(1,t)=f_2(t)$; $u(x,0)=g_1(x)$, $u_t(x,0)=g_2(x)$
In this method we choose $n\in N$, calculate $h=1/n$ and consider the points (x_i,t) where $x_i=ih$ for $i=0,1,2...n$. Note that we do not discretise variable t.
Substituting these points, for $i=1...n-1$, we obtain $u_{tt}(x_i,t)=\alpha^2(x_i,t)\ u_{xx}(x_i,t)$. If we replace second partial derivative u_{xx} with second differences we obtain:

$u_{tt}(x_i,t)=\alpha^2(x_i,t)\dfrac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{h^2}\ u_{xx}(x_i,t)$,

for $i=1...n-1$
Denoting

$$Y(t)=\begin{pmatrix}u(x_1,t)\\ u(x_2,t)\\ \vdots\\ u(x_{n-1},t)\end{pmatrix},\ b(t)=\begin{pmatrix}\dfrac{\alpha^2(x_1,t)f_1(t)}{h^2}\\ 0\\ \vdots\\ 0\\ \dfrac{\alpha^2(x_{n-1},t)f_2(t)}{h^2}\end{pmatrix} \text{ and}$$

$$A(t)=\frac{1}{h^2}\begin{pmatrix}-2\alpha^2(x_1,t) & \alpha^2(x_1,t) & 0 & \cdots & 0\\ \alpha^2(x_2,t) & -2\alpha^2(x_2,t) & \alpha^2(x_2,t) & \cdots & 0\\ 0 & \alpha^2(x_3,t) & -2\alpha^2(x_3,t) & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \alpha^2(x_{n-1},t) & -2\alpha^2(x_{n-1},t)\end{pmatrix},$$

we obtain the system of linear second-order ordinary differential equations:

$$Y''(t)=A(t)\ Y(t)+b(t) \qquad (13)$$

Note that the solution vector Y(t) is of dimension n-1.
The initial conditions for this system are:

$$Y(0)=\begin{pmatrix}g_1(x_1)\\ g_1(x_2)\\ \vdots\\ g_1(x_{n-1})\end{pmatrix},\ Y'(0)=\begin{pmatrix}g_2(x_1)\\ g_2(x_2)\\ \vdots\\ g_2(x_{n-1})\end{pmatrix}$$

Let us suppose $Y(t)=\sum_{i=0}^{\infty}Y_i(t-t_0)^i$, $A(t)=\sum_{i=0}^{\infty}A_i(t-t_0)^i$ and $b(t)=\sum_{i=0}^{\infty}b_i(t-t_0)^i$ where the coefficients of these series have suitable dimensions. Substituting in (13), the expression to calculate the coefficients of a particular solution, given the initial conditions Y_0 and Y_1, is $Y_{i+2}=(\sum_{j=0}^{i}A_jY_{i-j}+b_i)/((i+2)(i+1))$ i≥0.

It is interesting to comment that, from computational point of view, if n is large h is very small and Y_i increases considerably when i increases. To avoid errors, we consider the formal power series $\sum_{i=0}^{\infty}Y_i^*t^i$, with $Y_i^*=h^{2i}Y_i$. Note that Y(t) is rational if and only if $Y^*(t)$ is rational.

Let us study some particular examples.

Example 5

$u_{tt}=\frac{x^2}{t(1-t)^2}u_{xx}$ 0≤x≤1, t>0, t≠0, t≠1

$u(0,t)=0$ $u(1,t)=\frac{t}{1-t}$; $u(x,0.5)=x^2$ $u_t(x,0.5)=4x^2$

Considering, for instance, n=10 and t_0=0.5 (the equation has not sense in t=0) the initial conditions are:

$$Y(0.5)=\begin{pmatrix}x_1^2\\x_2^2\\\vdots\\x_9^2\end{pmatrix},\quad Y'(0.5)=4\begin{pmatrix}x_1^2\\x_2^2\\\vdots\\x_9^2\end{pmatrix}$$

$$A_i=\frac{a_i}{h^2}\begin{pmatrix}-2x_1^2 & x_1^2 & 0 & \cdots & 0\\ x_2^2 & -2x_2^2 & x_2^2 & \cdots & 0\\ 0 & x_3^2 & -2x_3^2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & x_9^2 & -2x_9^2\end{pmatrix}$$

where $a_{2i}=2^{2i+3}(i+1)$, $a_{2i+1}=2^{2i+4}(i+1)$

$$b_i=\frac{2^{i+2}(i+2)(i+1)}{h^2}\begin{pmatrix}0\\\vdots\\0\\x_9^2\end{pmatrix},\quad i\geq 0.$$

Fig. 11: *Table 1* for Y(t) is:

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	2	1	1	1	1
3	3	3	2	1	1	1
4	4	4	3	2	1	1
5	5	5	4	3	2	1

It indicates that $Y(t)=[1/1]_Y$. Solving the system RS(1,1), we obtain

$$Y(t)={}^R[1/1]_Y=\begin{pmatrix}0.01+0.02(t-0.5)\\0.04+0.08(t-0.5)\\\vdots\\0.81+1.62(t-0.5)\end{pmatrix}\frac{1}{1-2(t-0.5)}$$

which can be simplified as $Y(t)=\begin{pmatrix}0.01t\\0.04t\\\vdots\\0.81t\end{pmatrix}\frac{1}{1-t}$

Note that $u(x,t)=\frac{x^2t}{1-t}$ is the solution and that Y(t) coincides with u(x,t) discretised in x.

Example 6

Let us see now an example where left approximant is more suitable (left m.d. are smaller than right m.d.).

$u_{tt}=\frac{x^2}{(1+t)^2}u_{xx}$ t>0, 0≤x≤1

$u(0,t)=1$, $u(1,t)=\frac{1}{2+t}$; $u(x,0)=\frac{1}{1+x}$, $u_t(x,0)=-\frac{x}{(1+x)^2}$

Considering different values of n (n=5, n=10, n=50) in this example *Table 1* is highly sensitive to the threshold that we choose to decide whether and element is to be considered zero or not (Due to the limitations of finite arithmetic, certain theoretically zero elements will not be exactly zeros, which is why we have chosen a threshold so that any number with an absolute value of below it will be considered as being zero). However *Table 2* for left approximant, in most cases, shows that (0,1) is a pair of left m.d.

Note that $u(x,t)=\frac{1}{1+x(t+1)}$ is the solution and that *Table 2* indicates that $Y(t)\in\{{}^L[0/1]_Y\}$. For instance, if n=5:

$$Y(t)=\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}0.2/1.2&0&0&0\\0&0.4/1.4&0&0\\0&0&0.6/1.6&0\\0&0&0&0.8/1.8\end{pmatrix}t\right)^{-1}\begin{pmatrix}1/1.2\\1/1.4\\1/1.6\\1/1.8\end{pmatrix}$$

i.e.; $Y(t)=\begin{pmatrix}\frac{1}{1+0.2(t+1)}\\\frac{1}{1+0.4(t+1)}\\\frac{1}{1+0.6(t+1)}\\\frac{1}{1+0.8(t+1)}\end{pmatrix}$ is u(x,t) discretised in x.

5 Conclusions

In this paper we have extended results and practice of [3]. Our aim has been to highlight methods of looking for rational fundamental matrices for systems of linear m-order differential equation and particular rational solutions for these systems and for partial differential equations.

The balanced use of numerical methods, together with approximation theory is a very interesting alternative to improve the calculation of these solutions (or, alternatively, approximated solutions).

In this paper we combine Galerkin and Finite Difference methods with MPA.

Other applications were considered in [7] in connection with time series analysis and economic models.

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