An $L\omega_1\omega_1$ Axiomatization of the Linear Archimedean Continua as Merely Relational Structures

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Abstract. – We have chosen the language $L\omega_1\omega_1$ in which to formulate the axioms of two systems of the linear Archimedean continua – the point-based system, S_P, and the stretch-based system, S_I – for the following reasons: 1. It enables us to formulate all the axioms of each system in one and the same language; 2. It makes it possible to apply, without any modification, Arsenijević's two sets of rules for translating formulas of each of these systems into formulas of the other, in spite of the fact that these rules were originally formulated in a first-order language for systems that are not continuous but dense only; 3. It enables us to speak about an infinite number of elements of a continuous structure by mentioning explicitly only denumerably many of them; 4. In this way we can formulate not only Cantor's coherence condition for linear continuity but also express the large-scale and small-scale variants of the Archimedean axiom without any reference, either explicit or implicit, to a metric; 5. The models of the two axiom systems are structures that need not be relational-operational but only relational, which means that we can speak of the linear geometric continua directly and not only *via* the field of real numbers (numbers will occur as subscripts only, and they will be limited to the natural numbers).

Key-Words: Linear continuum, L_omega_1/omega_1, point-based, stretch-based axiomatization, trivial difference, Archimedean axiom

1 Introduction

1.1. The history of the problem of the continuum till Cantor

The question about the structure of the continuum was raised for the first time by eleatic philosophers Parmenides and Zeno. While Parmenides stated that what is continuous (*synechés*) is both undivided and indivisible ([11] DK 28 B **8** 5-6), Zeno, in his famous arguments against plurality, was proving:

1. that no continuous entity of a higher dimension can ever be built up out of entities of a lower dimension ([11] DK 29 B **2**);

2. that no continuum can be said to consist of elements of the same dimension ([11] DK 29 B **1**).

For instance, the addition of a point, or any number of points, would not increase the magnitude of a line segment, whereas, at the same time, no line segment can be said to be a real element of the line due to its infinite divisibility. Aristotle accepted both Zeno's conclusions, the first of them having named Zeno's Axiom ([1] *Metaph.* 1001b7), and, consequently, accepted the first Parmenides' claim, that what is continuous is undivided *per se* ([1] *Phys.* 206b14ff.), but rejected the second claim, that what is continuous is indivisible, because the indivisibility does not follow from what Zeno had proved. According to Aristotle, everything that is continuous is divisible *in indefinitum*, but it can never be divided into an infinite number of parts ([1] *Phys.* 206b7ff.)) .

Because of the indeterminateness of parts of the continuum, obtainable in one way or another through a possible division, this Aristotelian conception was later called *indefinitism*, while Kant, because of the ontological priority of the continuum as such over its possible parts, called space and time *composita idealia* ([17], p. 304).

This Aristotelian doctrine was the received view of the continuum till the end of the nineteenth century. Its only alternatives were Epicurus' atomism and the middle-age infinitesimalism. However, though physicists sometimes endorsed atomism, it was never seriously accepted as an analysis of the continuum, while mathematicians used infinitesimals only as 'useful fictions', as Leibniz put it ([18], pp. 91-95)..

Cantor was the first who offered the analysis of the *continuum qua compositum reale*, claiming that he finally resolved 'the great struggle' among the followers of Aristotle and Epicurus, who 'either leave ultimate elements of matter totally indeterminate, or […] assume them to be so-called *atoms* of *very small*, yet *not entirely disappearing space-time contents*' ([9], p 275). According to Cantor's analysis, the entities of a higher dimension may be said to consist of the entities of a lower dimension. Actually, Cantor accepted Zeno's axiom that the line cannot be built up of points if this is to be done step-by-step, but claimed that this impossibility was based on the fact that the linear continuum is not a well-ordered but only ordered set of points. If, however, a non-denumerable number of points (whose cardinal number is greater than $\aleph_0$) are put together, they can make up a linear continuum, given that following two conditions are met: 1. the set should be *perfekt* (perfect) and 2. it should be *zusammenhängend* (coherent) (see [9], p. 190).. The first condition had been known since the beginning of the analysis of the continuum: the continuum is *dense* because each of its points is an accumulation point of an infinite number of points. The second condition, however, is one of the Cantor's greatest discoveries: each accumulation of an infinite number of points must have the accumulation point that is an element of the basic set of points itself.

1.2. Development after Cantor

Cantor's theory of continuum has been enormously influential: logicians have formalized it, mathematicians have accepted it as a basis for Standard Analysis, and physicists have not quantized space and time, in spite of the fact that they acknowledged the existence of the quantum of action. Even in Non-Standard Analysis, the Cantorian structure is the basic continuum, extended later through the introduction of infinitesimals via rejection of the unrestricted applicability of the Archimedean axiom. However, there are still two groups of Cantor's opponents: Intuitionists and Neo-Aristotelians, whose theories will be of our concern, the Neo-Aristotelian one in particular.

Intuitionists reject the Cantorian concept of the actual infinity, and use only the Aristotelian concept of the dynamic (or potential) infinite. So they speak about denumerable finite number of *constructed objects* that may only get greater and greater unboundedly but never becomes actually infinite (cf. [7], 270ff.). This treatment enables them to to prove various statements about the objects introduced in such a way whenever they have a *recursive control* of what they speak about and *inductive way* to prove a theorem, but they do not allow us to speak of infinite *sets* or *classes* as 'finished entities' ([7], p. 433) but only of "*spreads*" of entities "*in statu nascendi*" ([26], p. 52), whose number may be always greater and greater. For instance, we may speak of natural numbers as a *species*, without restricting our discourse to a finite number of them, and can also prove, by using mathematical induction, that any of these numbers must be odd or even, but we mustn't refer explicitly to "the set of all the natural numbers (whose cardinal number is $\aleph_0$)".

Contrary to Intuitionists, Neo-Aristotelians do accept the concept of actual infinity, but reject Cantorian points as basic elements of the continuous structures. In the last four decades (cf. [2], [4] - [6], [8], [10], [13], [15], [16], [19], [22] - [25]), they have elaborated a conception of the linear continuum whose elements are Aristotelian stretches, which may not only precede each other but also be included into each other, abut each other, and overlap with each other. It is important to notice that stretches, in their analysis, are not only potential parts of the continuum and that both Cantor's conditions (to be perfect and coherent) can be modified as to be applicable to the stretch-based analysis of the continuum.

2 What we intend to do

2.1 The 'great struggle' between Cantorians and Neo-Aristotelians and a peace-making strategy

In contrast to what Cantor did, who resolved the 'great struggle' between Aristotelians and Epicurians by building up a completely new theory of the continuum, one of the main purposes of this article is to argue that the 'great struggle' about continuum between Cantorians and Neo-Aristotelians is "much ado about nothing", since the Cantorian point-based system – S_P – and the Neo-Aristotelian stretch-based system – S_I – can be shown to be both syntactically and semantically only trivially different.

Intuitively, the underlying idea is that stretches can unequivocally be introduced within S_P as intervals between two distinct points while points can unequivocally be introduced within S_I as abutment places of two stretches (or two equivalence classes of stretches). This possibility suggests a possible "systematic connection" between the two systems ([4], p. 84; cf. also [6]), so that, if we can manage to formulate the *structure preserving translation rules*, which would map unequivocally each formula of S_P into a formula of S_I and each formula of S_I into a formula of S_P, we might claim that S_P and S_I are only trivially different given that the two systems are complete and that all the theorems are always translated into theorems and non-theorems into non-theorems.

2.2 The removal of two objections to the peace-making strategy

One could object that, however intuitively acceptable can be to define stretches in S_I as intervals between distinct points, and points in S_P as the abutment places of abutting stretches, there must be a striking discrepancy between the entities of the corresponding models of S_P and S_I, since intervals can be open, half-open, and closed, while stretches are neither open, nor half-open, nor closed. However, this discrepancy can be easily compensated by letting *stretches* stand for the *closed* intervals *in contrast* to *sets* of an infinite number of stretches having either greatest lower or least upper bounds or both, which represent *half-open* and *open* intervals, respectively.

The second objection seems to be more serious. According to Quine's famous slogan "To be assumed as an entity is to be reckoned as the value of a variable" ([20], p. 13), two formal theories are *not* trivially different if there is no model in which their variables range over the elements of one and the same basic set, and in the case of S_P and S_I their variables can never do this. The solution to the problem is to use Arsenijević's definition of the *generalized* concepts of syntactically and semantically trivial differences between formal theories (cf. [2]). Namely, in the case in which there is no model in which variables of some two complete formal systems range over the elements of one and the same basic set, it can still happen that there are two Felix Bernstein's functions (cf. [9], p. 450) (one of them mapping each variable of one of the systems into a variable *tuple* of the other, the other mapping each variable of the latter system into a variable *tuple* of the former), which enable us to formulate *two* structure preserving translation rules of all the formulas of one of the systems *into* (not onto) a subset of all the formulas of the other. If, in addition, theorems are always translated into theorems and non-theorems into non-theorems, the two systems are syntactically trivially different in the generalized sense. *Semantically*, in the case of the *intended models* of S_P and S_I, the two systems are trivially different because when speaking about points we cannot avoid *automatically saying something unequivocal* about stretches, and *vice versa*.

2.3 The role of the infinitary language $L\omega_1\omega_1$

Contrary to the perfectness condition, which is formulable in first-order languages (see axiom 8 of S_P and axiom 8 of S_I in **3.1** and **3.2** below), the second Cantor's condition for the linear continuity (*zusammenhängend sein*) cannot be expressed in any first-order language, since we have to state something about *any accumulation of an infinite number of elements* of the basic set. This condition is therefore normally formulated in a second-order language, in which variables range not only over individuals but also over sets, properties, relations, etc.

However, the coherence condition, both for S_P as well for S_I, can also be expressed in the language $L\omega_1\omega_1$, which differs from the first order language only through the fact that formulas are allowed to contain denumerable sequences of quantifiers as well as conjunctions and disjunctions with denumerably many conjuncts and disjuncts, respectively. This language is the weakest possible one in which all the axioms for S_P and S_I can be formulated, and we choose this language in which to axiomatize the two systems because it has more advantages in comparison to the second order language, particularly in view of our purposes.

By using $L\omega_1\omega_1$, we avoid unnecessary ontological commitments in general. In particular, the fact that all the variables will be directly interpretable as ranging over individuals enables us to introduce all the axioms by referring explicitly to a denumerable number of them only, which should be acceptable even to Intuitionists.

The big problem of using the translation rules formulated by Arsenijević (in [2]) for comparing S_P and S_I, in the case in which some axioms are formulated in the second-order languages, lies in the fact that those rules have been tailored to first-order languages, because they were originally intended for the comparison of the systems that are only dense and not continuous. However, if we axiomatize S_P and S_I in $L\omega_1\omega_1$, Arsenijević's rules become applicable without any modification.

It will turn out that the Archimedean axiom, whose large-scale version and small-scale version were originally formulated as metrical statements, can be formulated, by using $L\omega_1\omega_1$, purely topologically, without the use of multiplication and division operations. Actually, we shall be able to treat the set of axioms of any of the two systems of the linear Archimedean continuum in a purely Hilbertian way, as implicitly defining any relational structure that satisfies them, and so to express truths about *any* linear continuum *directly*, by deriving theorems, and *not* via the field of real numbers as a *particular* continuous structure. Numbers will occur as variable subscripts only, and they will be limited to the natural numbers.

Last but not least, ontologically speaking, we normally think of reality as objects standing in certain relations. It is therefore much more natural to express facts about the continuum as merely relational (cf. [14]).

3 Comparison between S_P and S_I

3.1 Axiomatization of the Point-Based System

Let, in the intended model of S_P, the individual variables $\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \beta_1, \beta_2, \ldots, \beta_i, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots, \delta_1, \delta_2, \ldots, \delta_i, \ldots, \ldots$ range over a set of null-dimensional points, and let the relation symbols $\equiv$, $<$, and $>$ be interpreted as the identity, precedence, and succession relations respectively. Let the elementary *wffs* of S_P be $\alpha_m \equiv \alpha_n$, $\alpha_m < \alpha_n$, and $\alpha_m > \alpha_n$,
where $\alpha_m > \alpha_n \Leftrightarrow_{\text{def.}} \alpha_n < \alpha_m$. Finally, let the axiom schemes of S_P be the following twelve formulas, which we shall refer to as (A_P1), (A_P2),…, (A_P12):

1. $(\alpha_n)\neg\alpha_n < \alpha_n$
2. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l < \alpha_m \wedge \alpha_m < \alpha_n \Rightarrow \alpha_l < \alpha_n)$
3. $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \vee \alpha_n < \alpha_m \vee \alpha_m \equiv \alpha_n)$
4. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l \equiv \alpha_m \wedge \alpha_l < \alpha_n \Rightarrow \alpha_m < \alpha_n)$
5. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l \equiv \alpha_m \wedge \alpha_n < \alpha_l \Rightarrow \alpha_n < \alpha_m)$
6. $(\alpha_m)(\exists\alpha_n)\alpha_m < \alpha_n$
7. $(\alpha_m)(\exists\alpha_n)\alpha_n < \alpha_m$
8. $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \Rightarrow (\exists\alpha_l)(\alpha_m < \alpha_l \wedge \alpha_l < \alpha_n))$
9. $(\alpha_1)(\alpha_2)\ldots(\alpha_i)\ldots ((\exists\beta_1)(\wedge_{1\le i<\omega}\, \alpha_i < \beta_1) \Rightarrow$
 $\Rightarrow (\exists\gamma_1)(\wedge_{1\le i<\omega}\, \alpha_i < \gamma_1 \wedge$
 $\wedge \neg(\exists\delta_1)(\wedge_{1\le i<\omega}\, \alpha_i < \delta_1 \wedge \delta_1 < \gamma_1)))$
10. $(\alpha_1)(\alpha_2)\ldots(\alpha_i)\ldots ((\exists\beta_1)(\wedge_{1\le i<\omega}\, \alpha_i > \beta_1) \Rightarrow$
 $\Rightarrow (\exists\gamma_1)(\wedge_{1\le i<\omega}\, \alpha_i > \gamma_1 \wedge$
 $\wedge \neg(\exists\delta_1)(\wedge_{1\le i<\omega}\, \alpha_i > \delta_1 \wedge \delta_1 > \gamma_1)))$
11. $(\exists\alpha_1)(\exists\alpha_2)\ldots(\exists\alpha_n)\ldots(\alpha_2 < \alpha_1 \wedge$
 $\wedge \wedge_{1\le i<\omega}\alpha_{2i--1} < \alpha_{2i+1} \wedge \wedge_{1\le i<\omega}\alpha_{2i+2} < \alpha_{2i} \wedge$
 $\wedge (\beta) \wedge_{1\le i<\omega}(\alpha_i < \beta \wedge \beta < \alpha_{i+2} \Rightarrow$
 $\Rightarrow \wedge_{1\le k<\omega} \neg\beta\equiv\alpha_k) \wedge (\gamma)\vee_{1\le i,\, j<\omega}(\alpha_i < \gamma \wedge \gamma < \alpha_j))$
12. $(\exists\alpha_1)\ldots(\exists\alpha_n)\ldots((\beta)\vee_{1\le i,j<\omega}(\alpha_i < \beta \wedge \beta < \alpha_j) \wedge$
 $\wedge(\gamma)(\delta)(\gamma<\delta \Rightarrow \vee_{1\le i<\omega} (\gamma<\alpha_k \wedge \alpha_k<\delta)))$

3.2 Axiomatization of the Stretch-Based System

Let, in the intended model of S_I, the individual variables $a_1, a_2, \ldots, a_i, \ldots, b_1, b_2, \ldots, b_i, \ldots, c_1, c_2, \ldots, c_i, \ldots, d_1, d_2, \ldots, d_i, \ldots, \ldots$ range over one-

dimensional stretches, and let the relation symbols $=$, $\prec$, $\succ$, $\{$, $\}$, $\cap$, and $\Subset$, be interpreted as the identity, precedence, succession, abutment, overlapping, and inclusion relations respectively. Let the elementary *wffs* be $a_m = a_n$, $a_m \prec a_n$, $a_m \succ a_n$, $a_m \{ a_n$, $a_m \} a_n$, $a_m \cap a_n$, and $a_m \Subset a_n$,
where

$a_m \succ a_n \Leftrightarrow_{\text{def.}} a_n \prec a_m$ and $a_m \} a_n \Leftrightarrow_{\text{def.}} a_n \{ a_m$,

$a_m \{ a_n \Leftrightarrow_{\text{def.}} a_m \prec a_n \wedge \neg(\exists a_l)(a_m \prec a_l \wedge a_l \prec a_n)$,

$a_m \cap a_n \Leftrightarrow_{\text{def.}} (\exists a_l)(\exists a_k)(a_l \prec a_n \wedge \neg a_l \prec a_m \wedge a_m \prec$
$\prec a_k \wedge \neg a_n \prec a_k)$,

$a_m \Subset a_n \Leftrightarrow_{\text{def.}} \neg a_m = a_n \wedge (a_l)(a_l \cap a_m \Rightarrow a_l \cap a_n)$.

Finally, let axiom schemes of S_I be the following twelve formulas, which we shall refer to as (A_I1), (A_I2),…, (A_I12):

1. $(a_n)\neg a_n \prec a_n$
2. $(a_k)(a_l)(a_m)(a_n)(a_k \prec a_m \wedge a_l \prec a_n \Rightarrow a_k \prec a_n \vee a_l \prec a_m)$
3. $(a_m)(a_n)(a_m \prec a_n \Rightarrow a_m \{ a_n \vee (\exists a_l)(a_m \{ a_l \wedge a_l \{ a_n))$
4. $(a_k)(a_l)(a_m)(a_n)(a_k \{ a_m \wedge a_k \{ a_n \wedge a_l \{ a_m \Rightarrow a_l \{ a_n)$
5. $(a_k)(a_l)(a_m)(a_n)(a_k \{ a_l \wedge a_l \{ a_n \wedge a_k \{ a_m \wedge a_m \{ a_n \Rightarrow$
$\Rightarrow a_l = a_m)$
6. $(a_m)(\exists a_n)\ a_m \prec a_n$
7. $(a_m)(\exists a_n)\ a_n \prec a_m$
8. $(a_m)(\exists a_n)\ a_n \Subset a_m$
9. $(a_1)(a_2)\ldots(a_i)\ldots((\exists u)(\wedge_{1\le i<\omega}\ a_i \prec u) \Rightarrow$
$\Rightarrow (\exists v)\ (\wedge_{1\le i<\omega}\ a_i \prec v \wedge \neg(\exists w)\ (\wedge_{1\le i<\omega} a_i \prec w \wedge$
$\wedge\ w \prec v)))$
10. $(a_1)(a_2)\ldots(a_i)\ldots((\exists u)(\wedge_{1\le i<\omega}\ a_i \succ u) \Rightarrow$
$\Rightarrow (\exists v)(\wedge_{1\le i<\omega} a_i \succ v \wedge \neg(\exists w)(\wedge_{1\le i<\omega}\ a_i \succ w \wedge$
$\wedge\ w \succ v)))$
11. $(\exists a_1)(\exists a_2)\ldots(\exists a_n)\ldots$
$\ldots(a_2 \{ a_1 \wedge \wedge_{1\le i<\omega} a_{2i-1} \{ a_{2i+1} \wedge \wedge_{1\le i<\omega} a_{2i+2} \{ a_{2i} \wedge$
$\wedge\ (b) \vee_{1\le i,\, j<\omega} (a_i \prec b \wedge b \prec a_j))$
12. $(\exists a_1)(\exists a_2)\ldots(\exists a_n)\ldots((b)(\vee_{1\le i<\omega} b = a_i \Rightarrow$
$\Rightarrow\ (\vee_{1\le j<\omega} b \{ a_j \wedge \vee_{1\le k<\omega} a_k \{ b)) \wedge$
$\wedge\ (c)(\vee_{1\le i<\omega} c = a_i \Rightarrow \vee_{1\le j<\omega} a_j \Subset c) \wedge$
$\wedge\ (d) \vee_{1\le i,\, j<\omega}(a_i \prec d \wedge d \prec a_j) \wedge$
$\wedge\ (e) \vee_{1\le i,\, j<\omega}(a_i \cap e \wedge e \cap a_j))$

3.3 Comments on some Axioms

The interpretation of the first eight axioms of both systems needs no special comments. They implicitly define dense, unbounded, and linearly ordered structures. However, the rest of the axioms need some comments.

Ad (A_P9) and (A_P10), and (A_I9) and (A_I10). - While the first Cantor's condition for the linear continuity is met by a axiom (A_P8), the second condition (the coherence condition) is met, for the whole class of isomorphic models, only by two axioms, (A_P9) and (A_P10), which state the existence of the least upper and the greatest lower bound, respectively. It might be of interest to note why it is so. Namely, we need both (A_P9) and (A_P10) in order to make the class of all the models for S_P isomorphic. Let us suppose that, though the elements of the intended model of S_P are points, they are, instead (as [in 10]), the sets of numbers of closed intervals between any two numbers a and b such that $a \in Q$ and $b \in R$, and $<$ is interpreted as "is a proper subset of". Then, the relational structure $\langle \{[a, b] \mid a \in Q, b \in R\}, \subset \rangle$ satisfies the set of axioms (A_P1),…, (A_P9) but the coherence condition is not met. Let us take, for instance, the set of intervals $[a_1, b]$, $[a_2, b]$,…, $[a_n, b]$,… such that a_1 is a number smaller than b and any a_{n+1} is smaller than a_n, and where π is the accumulation point of the set of numbers a_1, a_2,…, a_n,… . There is no greatest lower bound for this set of intervals, in spite of the fact that the least upper bound always exists. – A similar example can be constructed for showing that we need both (A_I9) and (A_I10).

Ad (A_P11) and (A_I11). The intended meaning of the large-scale variant of the Archimedean axiom can be expressed by choosing a denumerable set of discrete points (in S_P) or abutting stretches (in S_I) distributed over the whole continuum and claiming that for any element of the structure there are two distinct elements (points or stretches) from the given sets such that one of them lies on one side and the other on the other side of the given element (point or stretch). As a consequence, a theorem (whose stretch-based version will be proved below) stating the compactness property of the corresponding structure exhibits the intended meaning of the Archimedean axiom in its most obvious form.

Ad (A_P12) and (A_I12). The two systems of the linear continuum in which numbers are neither mentioned nor used (except as variable subscripts) are insensitive to a distinction between Archimedean and non-Archimedean structures, which both belong to the class of their models (cf. [12]). Since there is no metric, obtainable either geometrically *via* the equality relation holding between stretches or arithmetically through the operations of multiplication and division, the large-scale and the small-scale variant of the Archimedean axiom must be formulated purely topologically by mentioning denumerably many of points and stretches only. This constitutes an important novelty of our approach.

For precluding infinitesimals in S_P, we have to state that it is possible to choose a denumerable set of dense points that covers the continuum in such a way that for any two points there is a point from the chosen set that lies between them. In S_I, we have to state that there are no stretches, like monads in the Robinsonian non-standard field $*R$ (cf. [21], p. 57), which are impenetrable, from both sides, by some two members of a chosen denumerable set of abutting and dense stretches.

3.4 The Triviality of the Difference between S_P and S_I

In order to show that the two axiom systems, S_P and S_I, are only trivially different in the sense defined in [2], we shall first cite two sets of *translation rules*.

Let f be a function $f : \alpha_n \longrightarrow \langle a_{2n-1}, a_{2n}\rangle$ (n = 1, 2,…) mapping variables of S_P into ordered pairs of variables of S_P, and let C_1-C_5 be the following translation rules providing a 1-1 translation of all the *wffs* of S_P into a subset of the *wffs* of S_I (where $=^C$ means "is to be translated according to syntactic constraints C as"):

C_1: $\alpha_n \equiv \alpha_m =^C a_{2n-1} \wr a_{2n} \wedge a_{2m-1} \wr a_{2m} \wedge a_{2n-1} \wr a_{2m}$,

C_2: $\alpha_n < \alpha_m =^C a_{2n-1} \wr a_n \wedge a_{2m-1} \wr a_{2m} \wedge$
$\wedge\, a_{2n-1} \prec a_{2m} \wedge \neg a_{2n-1} \wr a_{2m}$,

C_3: $\neg F_P =^C \neg C(F_P)$, where F_P is a *wff* of S_P translated according to C_1-C_5 into *wff* $C(F_P)$ of S_I,

C_4: $F_P' \heartsuit F_P'' =^C C(F_P') \heartsuit C(F_P'')$, where ♥ stands for ⇒ or ∧ or ∨ or ⇔, and F_P' and F_P'' stands for two *wffs* of S_P translated according to C_1-C_5 into two *wffs* of S_I,$C(F_P')$ and $C(F_P'')$ respectively,

C_5: $(\alpha_n)\Omega(\alpha_n) =^C (a_{2n-1})(a_{2n})((a_{2n-1} \wr a_{2n}) \Rightarrow$
$\Rightarrow \Omega^*(a_{2n-1}, a_{2n}))$
and
$(\exists\alpha_n)\Omega(\alpha_n) =^C (\exists a_{2n-1})(\exists a_{2n})((a_{2n-1} \wr a_{2n}) \wedge$
$\wedge\, \Omega^*(a_{2n-1}, a_{2n}))$,
where $\Omega(\alpha_n)$ is a formula of S_P translated into formula $\Omega^*(a_{2n-1}, a_{2n})$ of S_P according to C_1-C_5.

Let f^* be a function $f^* : a_n \longrightarrow \langle \alpha_{2n-1}, \alpha_{2n}\rangle$ (n = 1, 2,…) mapping variables of S_I into ordered pairs of variables of S_P, and let C^*_1-C^*_5 be the following translation rules providing a 1-1 translation of all the *wffs* of S_I into a subset of the *wffs* of S_P (where $=^{C^*}$ is to be understood analogously to $=^C$):

C^*_1: $a_n = a_m =^{C^*} \alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \alpha_{2n-1} \equiv$
$\equiv \alpha_{2m-1} \wedge \alpha_{2n} \equiv \alpha_{2m}$,

C^*_2: $a_n \prec a_m =^{C^*} \alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge$
$\wedge\, \neg\alpha_{2m-1} < \alpha_{2n}$,

C^*_3: $\neg F_I =^{C^*} \neg C^*(F_I)$, where F_I is a *wff* of S_I translated according to C^*_1-C^*_5 into *wff* $C(F_I)$ of S_P,

C^*_4: $F_I' \heartsuit F_I'' =^{C^*} C^*(F_I') \heartsuit C^*(F_I'')$, where ♥ stands for ⇒ or ∧ or ∨ or ⇔, and F_I' and F_I'' stands for two *wffs* of S_I translated according to C^*_1-C^*_5 into two *wffs* of S_P, $C^*(F_I')$ and $C^*(F_I'')$ respectively,

C^*_5: $(a_n)\Phi(a_n) =^{C^*} (\alpha_{2n-1})(\alpha_{2n})((\alpha_{2n-1} < \alpha_{2n}) \Rightarrow$
$\Rightarrow \Phi^*(\alpha_{2n-1}, \alpha_{2n}))$
and
$(\exists a_n)\Phi(a_n) =^{C^*} (\exists\alpha_{2n-1})(\exists\alpha_{2n})((\alpha_{2n-1} < \alpha_{2n}) \wedge$
$\wedge\, \Phi^*(\alpha_{2n-1}, \alpha_{2n}))$,
where $\Phi(a_n)$ is a formula of S_I translated into formula $\Phi^*(\alpha_{2n-1}, \alpha_{2n})$ of S_P

according to C^*_1-C^*_5.

In [2], Arsenijević has shown that by using C_1-C_5 and C^*_1-C^*_5 for translating (A_P1),…, (A_P8) into S_I and (A_I1),…, (A_I8) into S_P, respectively, we always get theorems. Now, the same holds for the translations of (A_P9),…, (A_P12) into S_I and (A_I9),…, (A_I12) into S_P. Let us prove within S_I the translation of (A_P9), which will be (after an appropriate shortening of the resulting formula) denoted by (A_P9)*.

(A_P9)*

$$(a_1)(a_2)\ldots(a_i)\ldots\ (\wedge_{1\le i<\omega}\, a_{2i-1} \lmoustache a_{2i} \ \Rightarrow$$
$$\Rightarrow ((\exists b_1)(\exists b_2)(\, b_1 \lmoustache b_2 \wedge (\wedge_{1\le i<\omega}\, a_i \prec b_2)) \Rightarrow$$
$$\Rightarrow (\exists c_1)(\exists c_2)(c_1 \lmoustache c_2 \wedge (\wedge_{1\le i<\omega}\, a_i \prec c_2) \ \wedge$$
$$\wedge \neg(\exists d_1)\neg(\exists d_2)(d_1 \lmoustache d_2 \wedge ((\wedge_{1\le i<\omega}\, a_i \prec d_2) \wedge$$
$$\wedge\, d_1 \prec c_2 \wedge \neg\, d_1 \lmoustache c_2))))$$

Proof for (A_P9)*

Let us assume both $\wedge_{1\le i<\omega}\, a_{2i-1} \lmoustache a_{2i}$ and $(\exists b_1)(\exists b_2)(b_1 \lmoustache b_2 \ \wedge (\wedge_{1\le i<\omega}\, a_i \prec b_2))$, which are the two antecedents of (A_P9)*. Now, since for any i ($1\le i<\omega$), $a_i \prec b_2$, it follows directly from (A_I9) that there is v such that $a_i \prec v$ and, for no w, both $a_i \prec w$ and $w \prec v$.

Let us now assume, contrary to the statement of the consequent of (A_P9)*, that for any two c_1, c_2 such that $c_1 \lmoustache c_2$ and for any i ($1\le i<\omega$) $a_i \prec c_2$, there are always d_1 and d_2 such that $d_1 \lmoustache d_2$ and for any i ($1\le i<\omega$) $a_i \prec d_2$, so that $d_1 \prec c_2$ and $\neg\, d_1 \lmoustache c_2$. But then, if we take c_2 to be just v from the consequent of (A_I9) (and c_1 any interval such that $c_1 \lmoustache c_2$), the assumption that for any i ($1\le i<\omega$)$a_i \prec c_2$ but $d_1 \prec c_2$ and $\neg\, d_1 \lmoustache c_2$ contradicts the choice of c_2, since if $c_2 = v$, then, according to (A_I9), for any d_1 and d_2 such that $d_1 \lmoustache d_2$ and for any i ($1\le i<\omega$)$a_i \prec d_2$, it cannot be that $d_1 \prec c_2$ and $\neg d_1 \lmoustache c_2$. (Q.E.D.)

4 Application

Let us, finally, prove two theorems in S_I that are of interest for different reasons. The first of them makes clear what is the trick of our formulation of the large-scale version of the Archimedean axiom *via* a chosen denumerable set of abutting stretches distributed over the both sides of the continuum: it is sufficient to have effective control over the continuum by a denumerable number of its discrete elements for making any of its elements surpassable in a finite number of steps, which means that the essence of the Archimedean axiom is topological, having nothing to do with a presupposed metric and depending on no arithmetical operation. The second theorem is a variant of Bolzano-Weierstrass' statement, which turns out to be not only a consequence of the small-scale variant of the Archimedean axiom but also not to be provable without it.

The S_I formulation of the Theorem stating the compactness property for stretches:

$$(c)(d)(\ c \prec d) \Rightarrow$$
$$\Rightarrow (\exists e_1)(\exists e_2)\ldots(\exists e_m)((e_1 \lmoustache e_2 \wedge \lmoustache \ldots \lmoustache e_m) \wedge$$
$$\wedge\, (\exists f)(\exists g)(f \lmoustache e_1 \wedge f \prec c \wedge \neg f \lmoustache c \wedge e_{m+1} \lmoustache g \wedge d \prec g \wedge \neg d \lmoustache g))$$

Proof.

Let us choose those i and m, for which a_i and a_{i+m} mentioned in (A_I11) are just those members of the set a_1, a_2,...,a_n... for which it holds that $a_i \prec c$ and $d \prec a_{i+m+1}$. Let us take then e_1, e_2,…e_m to be just a_{i+1}, a_{i+2},…a_{i+m}. Now, if we take f to be a_i and g to be a_{i+m+1}, we get directly that the statement of the theorem is true.

A stretch-based variant of the Bolzano-Weierstrass Theorem:

$$(c)(d)(h_1)(h_2)\ldots(h_i)\ldots(c \prec d \wedge \neg c \lmoustache d \wedge c \lmoustache h_1 \wedge$$
$$\wedge\ \wedge_{1\le i<\omega}\, h_i \lmoustache h_{i+1} \wedge (\exists e)(e \prec d \wedge \wedge_{1\le i<\omega}\, h_i \prec e) \Rightarrow$$
$$\Rightarrow (\exists b_1)(\exists b_2)\ldots(\exists b_i)\ldots(d \succ b_1 \wedge \wedge_{1\le i<\omega} b_i \rmoustache b_{i+1} \wedge$$
$$\wedge (\exists f)(\exists g\,)(f \lmoustache g \wedge f \lmoustache (b_i)_{\,1\le i<\omega} \ \wedge g \rmoustache (h_i)_{\,1\le i<\omega})$$

where

$$f \lmoustache (b_i)_{\,1\le i<\omega} \ \Leftrightarrow_{\text{def.}} \wedge_{1\le i<\omega} (f \prec b_i) \wedge \neg(\exists v)(f \prec v \ \wedge$$
$$\wedge \wedge_{1\le i<\omega} (v \prec b_i))$$

and

$$g \rmoustache (h_i)_{\,1\le i<\omega} \ \Leftrightarrow_{\text{def.}} \wedge_{1\le i<\omega}(g \succ h_i) \wedge \neg(\exists w)(g \succ w \ \wedge$$
$$\wedge \wedge_{\,1\le i<\omega} (w \succ h_i))$$

Proof.

Since the set of stretches $a_1, a_2,\ldots, a_i,\ldots$ from (A_I11) is dense and it holds for each of its members that it abuts some member of the set while some other member abuts it, we can take $b_1, b_2,\ldots, b_i,\ldots$ to be those $a_{j,1}, a_{j,2},\ldots, a_{j,i},\ldots$, respectively, for which the condition $d \succ a_{j1} \wedge \wedge \bigwedge_{1\leq i<\omega} a_{j,i} \wr a_{j,i+1}$ is met. Now, if f is the greatest lower bound of the set $a_{j,1}, a_{j,2},\ldots, a_{j,i},\ldots$, the statement of the theorem is true. But, let us suppose that, contrary to the statement of the theorem, f is not the greatest lower bound for any set $a_{k1}, a_{k2},\ldots, a_{ki},\ldots$ which is a subset of $a_1, a_2,\ldots, a_i,\ldots$ and which lies within e. This would mean, however, that there is some stretch w that is penetrable by no member of the set $a_1, a_2,\ldots, a_i,\ldots$, which directly contradicts the statement of (A_I12).

5 Conclusion

After formulating in $L\omega_1\omega_1$ the axioms of the Cantorian and the Aristotelian systems of the linear Archimedean continuum, we have shown how, by using appropriate translation rules, the axiom of the point-based system (A_P9), which states the existence of the lowest upper bound, can be proved as a theorem in the stretch-based system. In a similar way, it can be shown that after translating (A_P10), (A_P11), and (A_P12) into S_I, and (A_I9), (A_I10), (A_I11), and (A_I12) into S_P, we also get theorems of S_I and S_P, respectively. This means that S_P and S_I are only trivially different according to Arsenijević's definition given in [3]. In section 4, we have proved, by using the stretch-based system, two important theorems of classical analysis. These proofs strongly suggest that other classical theorems concerning the linear Archimedean continuum can also be formulated as being about merely relational structures and proved on the basis of the cited axioms without the use of the algebraic relational-operational structure of real numbers, which presents a prospect for further investigations.

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